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Generalized fuzzy c-distance and a common fixed point theorem in fuzzy cone metric spaces

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Abstract. In this paper, an idea of generalized fuzzy c-distance in fuzzy cone metric space is introduced. A common fixed point theorem is established for a pair of self mappings in fuzzy cone metric spaces by using the concept of generalized fuzzy c-distance.

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1. Introduction

The idea of cone metric space is very recent and is introduced by H. Long-Guang et. al [11] which is a generalization of metric space. In cone metric space, authors replace the real numbers by an ordering real Banach space. The study of common fixed point theorems for mappings satisfying certain contractive conditions is now a vigorous research activity. Many authors developed more results on common fixed point theorems for different types of contraction mappings (for references please see [1, 3, 5, 6, 7, 9]).

In [2], the idea of fuzzy cone metric space is introduced and study some fixed point theorems. Recently Shenghua Wang et. al [14] have been developed a distance called c-distance on cone metric space and by using this concept some fixed point results have been established.

There is an advantage to use c-distance to establish common fixed point theorems, since it is not required that contraction mappings be weakly compatible.

Following the idea of c-distance introduced by Shenghua Wang et. al [14], in [4], an idea of fuzzy c-distance in fuzzy cone metric space is introduced and by using this concept, some fixed point theorems are proved. On the other hand Sushanta Kumar Mahato et. al [13] introduce a new concept of generalized c-distance in cone metric space and study some fixed point results in such spaces.

In this paper, following the idea of generalized c-distance introduced in [13], concept of generalized fuzzy c-distance in fuzzy cone metric space is given. It is seen that every fuzzy c-distance is a generalized fuzzy c-distance of any order $j \in N$ but converse is not true. By using this concept, a common fixed point theorem is established for a pair of self-mappings in fuzzy cone metric spaces.
Generally to establish common fixed point theorems in fuzzy metric spaces, the contraction mappings should be weakly compatible but by using this new concept (generalized fuzzy c-distance), it is possible to establish common fixed point theorems in fuzzy cone metric spaces without using weakly compatible mappings.

The organization of the paper is as follows:
Section 2, comprises some preliminary results which are used in this paper.
An idea of generalized fuzzy c-distance in fuzzy cone metric space is introduced in Section 3. In Section 4, a common fixed point theorem is established.

2. Preliminaries

A fuzzy real number is a mapping $x: R \to [0\ ,\ 1]$ over the set R of all reals.
A fuzzy real number x is convex if $x(t) \geq min\ (x(s)\ ,\ x(r))$ where $s \leq t \leq r$.
α-level set of a fuzzy real number x is defined by $\{t \in R : x(t) \geq \alpha\}$ where $\alpha \in (0,1]$.
If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha\ ,\ b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha\ ,\ b_\alpha]$ means the interval $(-\infty\ ,\ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy real number x is called non-negative if $x(t) = 0,\ \forall t < 0$.
Each real number r is considered as a fuzzy real number denoted by $\bar{r}$ and defined by
$\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
Kaleva [10] (Felbin [8]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0\ ,\ 1]$ where $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0\ ,\ 1]$.

According to Mizumoto and Tanaka [12], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

$$(x \oplus y)(t) = Sup_{s \in R} min\ \{x(s)\ ,\ y(t-s)\},\quad t \in R$$
$$(x \ominus y)(t) = Sup_{s \in R} min\ \{x(s)\ ,\ y(s-t)\},\quad t \in R$$
$$(x \odot y)(t) = Sup_{s \in R, s \neq \underline{0}} min\ \{x(s)\ ,\ y(\tfrac{t}{s})\},\quad t \in R$$

Proposition 2.1 ([12]). *Let* $\eta\ ,\ \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$, $\alpha \in (0\ ,\ 1]$. *Then*

$$[\eta \bigoplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha\ ,\ b^1_\alpha + b^2_\alpha]$$
$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha\ ,\ b^1_\alpha - a^2_\alpha]$$
$$[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha\ ,\ b^1_\alpha b^2_\alpha]$$

Definition 2.2 ([10])**.** A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a_\alpha^n = a_\alpha$ and $\lim_{n\to\infty} b_\alpha^n = b_\alpha$ where $[\eta_n]_\alpha = [a_\alpha^n, b_\alpha^n]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.

Note 2.3 ([10])**.** If $\eta,\ \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.4 ([10])**.** For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.5 ([8])**.** Let X be a vector space over R.
Let $||\ ||\ : X \to R^*(I)$ and let the mappings
$L, U\ : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfy

$$L(0\ ,\ 0) = \ 0 \text{ and } U(1\ ,\ 1) = \ 1.$$

Write
$[||x||]_\alpha \ = \ [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ for $x \in X,\ 0\ < \alpha\ \leq 1$ and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0\ \in (0\ ,\ 1]$ independent of x such that for all $\alpha\ \leq\ \alpha_0$,

(A) $||x||_\alpha^2\ <\ \infty$
(B) $\inf||x||_\alpha^1\ >\ 0$.

The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if

(i) $||x||\ =\ \bar{0}$ if and only if $x\ =\ \underline{0}$;
(ii)$||rx||\ =\ |r|||x||,\ x \in X,\ r \in R$;
(iii) for all $x, y\ \in X$,
(a) whenever $s\ \leq\ ||x||_1^1,\ t \leq ||y||_1^1$ and $s + t\ \leq ||x + y||_1^1$,
$||x + y||(s + t)\ \geq\ L(||x||(s)\ ,\ ||y||(t))$,
(b) whenever $s\ \geq\ ||x||_1^1,\ t \geq ||y||_1^1$ and $s + t\ \geq ||x + y||_1^1$,
$||x + y||(s + t)\ \leq\ U(||x||(s)\ ,\ ||y||(t))$

Remark 2.6 ([8])**.** Felbin proved that,
if $L\ = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 1.1 is equivalent to

$$||x + y||\ \preceq ||x||\ \bigoplus ||y||.$$

Further $||\ ||_\alpha^i;\ i = 1, 2$ are crisp norms on X for each $\alpha \in (0\ ,\ 1]$.

Definition 2.7 ([2])**.** Let $(E, ||\ ||)$ be a fuzzy real Banach space where $||\ || : E \to R^*(I)$.
Denote the range of $||\ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.8 ([2])**.** A member $\eta \in A \subset R^*(I)$ is said to be an interior point if $\exists r > 0$ such that
$S(\eta, r) = \{\delta \in R^*(I)\ : \eta \ominus \delta \prec \bar{r}\} \subset A$.
Set of all interior points of A is called interior of A.

Definition 2.9 ([2])**.** A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.10 ([2]). A subset P of $E^*(I)$ is called a fuzzy cone if
(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;
(ii) $a, b \in R,\ a, b \geq 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.

Note 2.11. If $\eta \in P$ then $\ominus\eta \in P \quad \Rightarrow \eta = \bar{0}$.
For, suppose $[\eta]_\alpha = [\eta^1_\alpha\ ,\ \eta^2_\alpha],\ \alpha \in (0, 1]$.
Since $\eta \in P \subset E^*(I)$, we have $\eta^1_\alpha,\ \eta^2_\alpha \geq 0\ \forall\alpha \in (0, 1]$.
Now $[\ominus\eta]_\alpha = [-\eta^2_\alpha\ ,\ -\eta^1_\alpha],\ \alpha \in (0, 1]$.
If $\eta \neq \bar{0}$, then $\eta^1_\alpha,\ \eta^2_\alpha > 0\ \forall\alpha \in (0, 1]$.
i.e. $-\eta^2_\alpha\ \leq\ -\eta^1_\alpha < 0\ \forall\alpha \in (0, 1]$.
This implies that $\ominus\eta$ does not belong to P. Hence $\eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
with $\bar{0} \leq ||x|| \leq ||y||$ implies $||x|| \preceq K||y||$. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that $||x_1|| \leq ||x_2|| \leq .... \leq ||x_n|| \leq .... \leq ||y||$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to \bar{0}$ as $n \to \infty$.
Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent.

In the following we always assume that E is a fuzzy real Banach space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.12 ([2]). Let X be a nonempty set. Suppose the mapping
$d:\ X \times X \to E^*(I)$ satisfies
(Fd1) $\bar{0} \leq d(x, y)\ \forall x, y \in X$ and $d(x, y) = \bar{0}$ iff $x = y$;
(Fd2) $d(x, y) = d(y, x)\ \ \forall x, y \in X$;
(Fd3) $d(x, y) \leq d(x, z) \oplus d(z, y)\ \ \forall x, y, z \in X$.
Then d is called a fuzzy cone metric and (X, d) is called a fuzzy cone metric space.

Definition 2.13 ([2]). Let (X, d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << ||c||$ there is a positive integer N such that for all $n > N,\ d(x_n, x) << ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

Definition 2.14 ([2]). Let (X, d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall m, n > N,\ d(x_n, x_m) << ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.15 ([2]). Let (X, d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Definition 2.16 ([1]). Let f and g be self mappings defined on a set X. If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition 2.17. Let X be any nonempty set and $f, g\ : X \to X$ be mappings. Pair (f, g) is called weakly compatible if for $x \in X,\ fx = gx$ implies $fgx = gfx$.

Proposition 2.18 ([1])**.** *Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence $w = f(x) = g(x)$, then w is the unique common fixed point of f and g.*

Definition 2.19 ([4])**.** Let $(X\ ,\ d)$ be a fuzzy cone metric space. Then the mapping $Q :\ X \times X \to E^*(I)$ is called a c-fuzzy distance on X if the following conditions hold:
(Q1) $\bar{0} \le Q(x, y)\ \ \forall x, y \in X$;
(Q2) $Q(x, z) \le Q(x, y) \oplus Q(y, z)\ \ \forall x, y, z \in X$;
(Q3) $\forall x \in X$, if $Q(x, y_n) \le \eta$ for some $\eta = \eta(x) \in P,\ n \ge 1$,
then $Q(x, y) \le \eta$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(Q4) $\forall c \in E$ with $\bar{0} \ll ||c||,\ \exists e \in E$ with $\bar{0} \ll ||e||$ such that $Q(z, x) \ll ||e||$ and $Q(z, y) \ll ||e||$ imply $d(x, y) \ll ||c||$.

3. Generalized fuzzy c-distance on fuzzy cone metric space

Definition 3.1. Let $(X, d, \le)$ be a fuzzy cone metric space and $j \in N$ (set of natural numbers). Then the mapping
$Q :\ X \times X \to E^*(I)$ is called a generalized fuzzy c-distance of order j on X if the following conditions hold:
(GQ1) $\bar{0} \le Q(x, y)\ \ \forall x, y \in X$;
(GQ2) $Q(x, z) \le \sum_{i=0}^{j} Q(x_i, x_{i+1})\ \ \forall x, z \in X$ and for all distinct points $x_i \in X,\ i \in \{1, 2, 3, ...\}$ each of them different from $x(= x_0)$ and $z(= x_{j+1})$;
(GQ3) $\forall x \in X$, if $Q(x, y_n) \le \eta$ for some $\eta = \eta(x) \in P,\ n \ge 1$,
then $Q(x, y) \le \eta$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(GQ4) $\forall c \in E$ with $\bar{0} \ll ||c||,\ \exists e \in E$ with $\bar{0} \ll ||e||$ such that $Q(z, x) \ll ||e||$ and $Q(z, y) \ll ||e||$ imply $d(x, y) \ll ||c||$.

Remark 3.2. Every fuzzy c-distance is a generalized fuzzy c-distance of order 1. In fact, every fuzzy c-distance may also be considered as a generalized fuzzy c-distance of any order $j \in N$.
But the converse does not hold and is justified by the following example.

Example 3.3. Let $E = R$ (set of real numbers).
Define $||\ || :\ E \to R^*(I)$ by

$$||x||(t) = \begin{cases} \frac{t}{|x|} & \text{if } 0 \le t \le |x|, x \ne \theta \\ 1 & \text{if } t = |x| = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $[||x||]_\alpha = \ [\alpha|x|, |x|]\ \ \forall \alpha \in (0, 1]$.
It is easy to verify that (i) $||x|| = \bar{0}$ iff $x = \theta$ (ii) $||rx|| = |r|||x||$ and
(iii) $||x + y|| \preceq ||x|| \oplus ||y||\ \ \forall x, y \in E$.
Then $(E, ||\ ||)$ is a fuzzy normed linear space (Felbin's sense for L= min and U = max).

Let $\{x_n\}$ be a Cauchy sequence in $(E, || \, ||)$. So $\lim_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.
$\Rightarrow \lim_{m,n\to\infty} ||x_n - x_m||^1_\alpha = \lim_{m,n\to\infty} \alpha|x_n - x_m| = 0 \ \forall \alpha \in (0,1]$
$\Rightarrow \lim_{m,n\to\infty} |x_n - x_m| = 0$
$\Rightarrow \{x_n\}$ is a Cauchy sequence in $(E, | \, |)$.
Since $(E, | \, |)$ is complete, $\exists x \in E$ such that $\lim_{n\to\infty} |x_n - x| = 0$
Thus $\lim_{n\to\infty} ||x_n - x|| = \bar{0}$.
So $(E, || \, ||)$ is a real complete fuzzy normed linear space.
Define $P = \{\eta \in E^*(I) : \ \eta \succeq \bar{0}\}$.
(i) P is fuzzy closed.
For, consider a sequence $\{\delta_n\}$ in P such that $\lim_{n\to\infty} \delta_n = \delta$.
i.e. $\lim_{n\to\infty} \delta^i_{n,\alpha} = \delta^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$.
Now, $\delta_n \succeq \bar{0} \ \forall n$
$\Rightarrow \delta^i_{n,\alpha} \geq 0$ for $i = 1, 2$ and $\alpha \in (0, 1]$.
$\Rightarrow \lim_{n\to\infty} \delta^i_{n,\alpha} \geq 0$ for $i = 1, 2$ and $\alpha \in (0, 1]$.
$\Rightarrow \delta \succeq \bar{0}$.
So $\delta \in P$ and hence P is fuzzy closed.
(ii) It is obvious that for $a, b \in R$; $a, b \geq 0$ and $\eta, \delta \in P$ implies $a\eta \oplus b\delta \in P$.
Thus P is a fuzzy cone in E.
Now choose the ordering of E w.r.t. P as $\preceq$ and suppose $X = \{x_1, x_2, x_3, x_4\} \subset R$ and define
$d : \ X \times X \to E^*(I)$ by $d(x, y) = ||x - y|| \ \forall x, y \in X$. Then (X, d) is a fuzzy cone metric space.
Let $Q : X \times X \to E^*(I)$ by $Q(x_1, x_2) = Q(x_2, x_1) = \bar{5}$.
$Q(x_1, x_3) = Q(x_3, x_1) = Q(x_2, x_3) = Q(x_3, x_2) = \bar{2}$.
$Q(x_1, x_4) = Q(x_4, x_1) = Q(x_2, x_4) = Q(x_4, x_2) = Q(x_3, x_4) = Q(x_4, x_3) = \bar{1}$.
$Q(x, x) = \bar{0} \ \forall x \in X$.
Then Q satisfies the condition (GQ2) for $j = 2$.
The conditions (GQ1) and (GQ3) are obvious.
For (GQ4), take any $c \in E$ with $||c|| \succeq \bar{0}$ and put $||e|| = \frac{\bar{1}}{2}$.
Now $Q(z, x) \ll ||e||$ and $Q(x, y) \ll ||e||$
$\Rightarrow \frac{\bar{1}}{2} \ominus Q(z, x) \in intP$ and $\frac{\bar{1}}{2} \ominus Q(x, y) \in intP$
$\Rightarrow \frac{1}{2} - Q^1_\alpha(z, x) > 0$ and $\frac{1}{2} - Q^2_\alpha(z, x) > 0 \ \forall \alpha \in (0, 1]$
$\Rightarrow Q^1_\alpha(z, x) = Q^2_\alpha(z, x) = 0 \ \forall \alpha \in (0, 1] \ \Rightarrow z = x$.
Similarly $x = y$. Hence $x = y = z$.
So $d(x, y) = \bar{0} \ll ||c||$. Thus (GQ4) holds.
Thus Q is a generalized fuzzy c-distance of order 2 on X.
Now $Q^1_\alpha(x_1, x_2) = 5 > Q^1_\alpha(x_1, x_3) + Q^1_\alpha(x_3, x_2) = 2 + 2 = 4$.
$Q^2_\alpha(x_1, x_2) = 5 > Q^2_\alpha(x_1, x_3) + Q^2_\alpha(x_3, x_2) = 2 + 2 = 4$.
So $Q(x_1, x_2) \preceq Q(x_1, x_3) \oplus Q(x_3, x_2) = 4$ does not hold.
Hence Q is not a fuzzy c-distance on X.

4. Common fixed point theorem

In this Section a common fixed point theorem is established by using generalized fuzzy c-distance and the Theorem is justified by an example.

Theorem 4.1. *Let (X, d) be a fuzzy cone metric space, P be a normal cone with normal constant K and Q be a generalized fuzzy c-distance of order j on X. Suppose the mappings $f, g: X \to X$ satisfy*

$$Q(fx, fy) \le a_1 Q(gx, gy) \oplus a_2 Q(gx, fx) \oplus a_3 Q(gy, fy) \tag{4.1.1}$$

where $x, y \in X$ and $a_1, a_2, a_3 \in [0, 1)$ satisfying $a_1 + a_2 + a_3 < 1$ and further suppose that for some $\alpha_0 \in (0, 1]$,

$$\bigwedge_{x \in X} \{Q^1_{\alpha_0}(gx, y) + Q^1_{\alpha_0}(fx, y) + Q^1_{\alpha_0}(gx, fx)\} > 0 \tag{4.1.2}$$

$\forall y \in X$ where y is not a point of coincidence of f and g. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subset g(X)$, $\exists x_1 \in X$ such that $fx_0 = gx_1$. Proceeding in this way, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1}$, $n = 0, 1, 2, ...$

Now by using (4.1.1), we have,

$Q(gx_n, gx_{n+1}) = Q(fx_{n-1}, fx_n) \le a_1 Q(gx_{n-1}, gx_n) \oplus a_2 Q(gx_{n-1}, fx_{n-1}) \oplus a_3 Q(gx_n, fx_n)$

$\le a_1 Q(gx_{n-1}, gx_n) \oplus a_2 Q(gx_{n-1}, gx_n) \oplus a_3 Q(gx_n, gx_{n+1})$

So

$$Q(gx_n, gx_{n+1}) \le r Q(gx_{n-1}, gx_n) \tag{4.1.3}$$

where $r = \frac{a_1 + a_2}{1 - a_3} < 1$.

By repeated application of (4.1.3) we obtain,

$$Q(gx_n, gx_{n+1}) \le r^n Q(gx_0, gx_1). \tag{4.1.4}$$

We may assume that $gx_n \ne gx_m$ for all distinct $m, n \in \{0, 1, 2, ...\}$.

For, if $gx_n = gx_m$ for some $m, n \in \{0, 1, 2, ...\}$, $m \ne n$ then assuming $m > n$ and we may write

$$gx_n = gx_{n+k} \text{ where } k = m - n \ge 1. \tag{4.1.5}$$

Put $y = gx_n$. Then

$Q(y, gx_{n+1}) = Q(gx_n, gx_{n+1}) = Q(gx_{n+k}, gx_{n+1}) = Q(fx_{n+k-1}, fx_n)$

$\le a_1 Q(gx_{n+k-1}, gx_n) \oplus a_2 Q(gx_{n+k-1}, fx_{n+k-1}) \oplus a_3 Q(gx_n, fx_n)$

$= a_1 Q(gx_{n+k-1}, gx_{n+k}) \oplus a_2 Q(gx_{n+k-1}, gx_{n+k}) \oplus a_3 Q(y, gx_{n+1})$

So

$$Q(y, gx_{n+1}) \le r\ Q(gx_{n+k-1}, gx_{n+k}). \tag{4.1.6}$$

By repeating the relation (4.1.3), we obtain from (4.1.6)

$Q(y, gx_{n+1}) \le r^k Q(y, gx_{n+1})$.

Since $0 \le r < 1$, it follows that $Q(y, gx_{n+1}) = \bar{0}$

and thus

$$Q(gx_n, gx_{n+1}) = \bar{0}. \tag{4.1.7}$$

Now using (4.1.1), (4.1.3) and (4.1.7) we obtain,
$Q(y,y) = Q(gx_n, gx_n) = Q(gx_{n+k}, gx_{n+k}) = Q(fx_{n+k-1}, fx_{n+k-1})$
$\leq a_1 Q(gx_{n+k-1}, gx_{n+k-1}) \oplus a_2 Q(gx_{n+k-1}, fx_{n+k-1}) \oplus a_3 Q(gx_{n+k-1}, fx_{n+k-1})$
$= a_1 Q(gx_{n+k-1}, gx_{n+k-1}) \oplus (a_2 + a_3) Q(gx_{n+k-1}, gx_{n+k})$
$\leq a_1 Q(gx_{n+k-1}, gx_{n+k-1}) \oplus (a_2 + a_3) r^{k-1} Q(gx_n, gx_{n+1})$
$\Rightarrow Q(y,y) \leq a_1 Q(gx_{n+k-1}, gx_{n+k-1})$ by (4.1.7).
Similarly $Q(gx_{n+k-1}, gx_{n+k-1}) \leq a_1 Q(gx_{n+k-2}, gx_{n+k-2})$.
Proceeding in this way, we obtain after k-th step,

$$Q(gx_{n+1}, gx_{n+1}) \leq a_1 Q(gx_n, gx_n). \tag{4.1.8}$$

Thus

$$Q(y,y) \leq a_1 Q(gx_{n+k-1}, gx_{n+k-1}) \leq ..... \leq a_1^k Q(gx_n, gx_n) = a_1^k Q(y,y) \tag{4.1.9}$$

Since $k < 1$, we have $Q(y,y) = \bar{0}$.
Thus we get $Q(gx_{n+1}, y) = \bar{0}$ (from (4.1.7)) and $Q(y,y) = \bar{0}$.
Now by (Q4), we have $d(gx_{n+1}, y) = \bar{0}$. Hence $gx_{n+1} = y$.
Thus $fx_n = y = gx_n$.
It follows that y is a point of coincidence of f and g, which contradicts the hypothesis.
So we may assume that $gx_n \neq gx_m$ for all distinct $m, n \in \{0, 1, 2, ...\}$.
Let $m, n \in N$ with $m > n$. Taking $m = n + p,\ p = 1, 2, 3, ...$ and using (4.1.1) and (4.1.4) we have,
$Q(gx_n, gx_m) = Q(fx_{n-1}, fx_{m-1})$
$\leq a_1 Q(gx_{n-1}, gx_{m-1}) \oplus a_2 Q(gx_{n-1}, fx_{n-1}) \oplus a_3 Q(gx_{m-1}, fx_{m-1})$
$= a_1 Q(gx_{n-1}, gx_{m-1}) \oplus a_2 Q(gx_{n-1}, gx_n) \oplus a_3 Q(gx_{m-1}, gx_m)$
$\leq a_1 Q(gx_{n-1}, gx_{m-1}) \oplus a_2 r^{n-1} (gx_0, gx_1) \oplus a_3 r^{m-1} Q(gx_0, gx_1)$
$\leq a_1 Q(gx_{n-1}, gx_{m-1}) \oplus (a_2 + a_3) r^{n-1} Q(gx_0, gx_1)$ (since $r^{m-1} \leq r^{n-1}$).
Continuing in this way, we obtain after nth step,

$$\begin{aligned} Q(gx_n, gx_m) &\leq a_1^n Q(gx_0, gx_p) \oplus (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + ... + a_1^{n-1}] Q(gx_0, gx_1) \\ &= a_1^n Q(gx_0, gx_p) \oplus \beta_n Q(gx_0, gx_1). \end{aligned} \tag{4.1.10}$$

where $\beta_n = (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + ... + a_1^{n-1}]$.
We now show that

$$Q(gx_0, gx_p) \leq \frac{1}{1 - a_1^j}(\frac{1}{1-r} + \beta_j) M \tag{4.1.11}$$

where $M = Q(gx_0, gx_1) \oplus Q(gx_0, gx_2) \oplus ... \oplus Q(gx_0, gx_j) \in P$.
If $p \leq j$ then
$Q(gx_0, gx_1) \leq (1 + \beta_j) Q(gx_0, gx_p)$ (since $\beta_j \geq 0\ \forall j$)
$\leq [(1 + r + r^2 + ...) + \beta_j] Q(gx_0, gx_p)$
$= (\frac{1}{1-r} + \beta_j) Q(gx_0, gx_p)$
$\leq (1 + a_1^j + (a_1^j)^2 + ...)(\frac{1}{1-r} + \beta_j) Q(gx_0, gx_p)$
$\leq \frac{1}{1-a_1^j}(\frac{1}{1-r} + \beta_j) M.$
If $p > j$ then $\exists s \in N$ such that $p = sj + t$ where $0 \leq t < j,\ t \in N$.

If $t = 0$ then by using (4.1.4) and (4.1.10), we have

$$\begin{aligned} Q(gx_0, gx_p) &\leq Q(gx_0, gx_1) \oplus Q(gx_1, gx_2) \oplus .... \oplus Q(gx_{j-1}, gx_j) \oplus Q(gx_j, gx_p) \\ &\leq Q(gx_0, gx_1) \oplus rQ(gx_0, gx_1) \oplus .... \oplus r^{j-1}Q(gx_0, gx_1) \\ &\quad \oplus a_1^j Q(gx_0, gx_{p-j}) \oplus \beta_j Q(gx_0, gx_1) \\ &= (\sum_{\gamma=0}^{j-1} r^\gamma + \beta_j) Q(gx_0, gx_1) \oplus a_1^j Q(gx_0, gx_{p-j}) \end{aligned} \tag{4.1.12}$$

By repeated application of (4.1.12), we obtain after (s-1)th step,

$$Q(gx_0, gx_p) \leq [1+a_1^j+(a_1^j)^2+....+(a_1^j)^{s-2}](\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_1)\oplus(a_1^j)^{s-1}Q(gx_0, gx_j)$$

$$\leq [1+a_1^j+(a_1^j)^2+....+(a_1^j)^{s-2}](\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_1)\oplus(a_1^j)^{s-1}(\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_j)$$

$$\leq [1 + a_1^j + (a_1^j)^2 + ... + (a_1^j)^{s-1}](\sum_{\gamma=0}^{j-1} r^\gamma + \beta_j)M$$

$\leq \frac{1}{1-a_1^j}(\frac{1}{1-r} + \beta_j)M.$

If $t \neq 0$ then,

$$\begin{aligned} Q(gx_0, gx_p) &\leq Q(gx_0, gx_1) \oplus Q(gx_1, gx_2) \oplus ..... \oplus Q(gx_{j-1}, gx_j) \oplus Q(gx_j, gx_p) \\ &\leq (\sum_{\gamma=0}^{j-1} r^\gamma + \beta_j)Q(gx_0, gx_1) \oplus a_1^j Q(gx_0, gx_{p-j}). \end{aligned} \tag{4.1.13}$$

By repeated application of (4.1.13), we obtain after sth step

$$Q(gx_0, gx_p) \leq [1+a_1^j+(a_1^j)^2+....+(a_1^j)^{s-1}](\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_1)\oplus(a_1^j)^s Q(gx_0, gx_t)$$

$$\leq [1+a_1^j+(a_1^j)^2+....+(a_1^j)^{s-1}](\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_1)\oplus(a_1^j)^s(\sum_{\gamma=0}^{j-1} r^\gamma+\beta_j)Q(gx_0, gx_t)$$

$$\leq [1 + a_1^j + (a_1^j)^2 + ... + (a_1^j)^s](\sum_{\gamma=0}^{j-1} r^\gamma + \beta_j)M.$$

i.e. $Q(gx_0, gx_p) \leq \frac{1}{1-a_1^j}(\frac{1}{1-r} + \beta_j)M.$

Thus for the case $p > j$ we have,

$$Q(gx_0, gx_p) \leq \frac{1}{1 - a_1^j}(\frac{1}{1 - r} + \beta_j)M \tag{4.1.14}$$

Now from (4.1.10), it follows that $\forall m, n \in N$ with $m > n$,

$Q(gx_n, gx_m) \leq \frac{a_1^n}{1-a_1^j}(\frac{1}{1-r} + \beta_j)M \oplus \beta_n Q(gx_0, gx_1)$, i.e.,

$$Q(gx_n, gx_m) \leq b_n M \tag{4.1.15}$$

where $b_n = \frac{a_1^n}{1-a_1^j}(\frac{1}{1-r} + \beta_j)$ and $b_n \to 0$ as $n \to \infty$.

Now $M = Q(gx_0, gx_1) \oplus ..... \oplus Q(gx_0, gx_j) \in P$.

Thus $\exists z \in E$ such that $M = ||z|| \in P \subset E^*(I)$.
So from (4.1.15), we get

$$Q(gx_n, gx_m) \leq b_n||z||. \tag{4.1.16}$$

Since P is a normal cone with normal constant K we have $Q(gx_n, gx_m) \preceq Kb_n||z||$
$\Rightarrow Q^i_\alpha(gx_n, gx_m) \leq Kb_n||z||^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$ and for $m > n,\ m, n \in N$
$\Rightarrow \lim_{m,n\to\infty} Q^i_\alpha(gx_n, gx_m) = 0$ for $i = 1, 2$ and $\alpha \in (0, 1]$
$\Rightarrow \lim_{m,n\to\infty} Q(gx_n, gx_m) = \bar{0}$.
It follows that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, $\exists u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$.
By (4.1.16) and (Q3), we have

$$Q(gx_n, u) \leq b_n||z||. \tag{4.1.17}$$

From (4.1.17), since P is normal we get

$$\begin{aligned} &Q(gx_n, u) \preceq Kb_n||z|| \\ \Rightarrow &Q^i_\alpha(gx_n, u) \leq Kb_n||z||^i_\alpha \text{ for } i = 1, 2 \text{ and } \alpha \in (0, 1] \end{aligned} \tag{4.1.18}$$

Suppose u is not a point of coincidence of f and g. Then by hypothesis and (4.1.18) we have for some $\alpha_0 \in (0, 1]$,
$0 < inf\{Q^1_{\alpha_0}(gx, u) + Q^1_{\alpha_0}(fx, u) + Q^1_{\alpha_0}(gx, fx) :\ x \in X\}$
$\leq inf\{Q^1_{\alpha_0}(gx_n, u) + Q^1_{\alpha_0}(fx_n, u) + Q^1_{\alpha_0}(gx_n, fx_n) :\ n \in N\}$
$= inf\{Q^1_{\alpha_0}(gx_n, u) + Q^1_{\alpha_0}(gx_{n+1}, u) + Q^1_{\alpha_0}(gx_n, gx_{n+1}) :\ n \in N\}$
$\leq inf\{kb_n||z||^1_{\alpha_0} + kb_{n+1}||z||^1_{\alpha_0} + r^n KQ(gx_0, gx_1) :\ n \in N\}$
$= 0$ $(b_n \to 0, r^n \to 0$ as $n \to \infty)$, which is a contradiction.
Thus u is a point of coincidence of f and g. So $\exists z \in X$ such that $fz = gz = u$.
For uniqueness, suppose $\exists w (\neq u) \in X$ such that $fx = gx = w$ for some $x \in X$.
Then $Q(u, u) = Q(fz, fz) \leq a_1Q(gz, gz) \oplus a_2Q(gz, fz) \oplus a_3Q(gz, fz) = (a_1 + a_2 + a_3)Q(u, u)$.
Since $a_1 + a_2 + a_3 < 1$, it follows that $Q(u, u) = \bar{0}$.
By similar argument as above we have $Q(w, w) = \bar{0}$.
Now $Q(u, w) = Q(fz, fx) \leq a_1Q(gz, gx) \oplus a_2Q(gz, fz) \oplus a_3Q(gx, fx)$
$\leq a_1Q(u, w) \oplus a_2Q(u, u) \oplus a_3Q(w, w) \ = \ a_1Q(u, w)$
It follows that $Q(u, w) = \bar{0}$ (since $a_1 < 1$).
Now we have $Q(u, w) = \bar{0}$ and $Q(u, u) = \bar{0}$.
Thus for any $e \in E$ with $\bar{0} \ll ||e||$ we get $Q(u, w) \ll ||e||$ and $Q(u, u) \ll ||e||$ and hence by (Q4) we get $d(w, u) \ll ||e||$. Since $e \in E$ is arbitrary we have $d(w, u) = 0$.
Hence $w = u$. Thus f and g have a unique point of coincidence in X.
If f and g are weakly compatible, then by Proposition 2.18, f and g have a unique common fixed point. □

The above Theorem 4.1 is justified by the following example.

Example 4.2. Consider the Example 3.3. Take $X = \{x_1,\ x_2,\ x_3,\ x_4\}$.
We define $f, g\ :\ X \to X$ by $fx = x_3\ \forall x \in X$,
$g(x) = x_3$ for $x \in \{x_1, x_3, x_4\}$ and $g(x) = x_4$ for $x = x_2$.
We see that x_3 is the unique point of coincidence of f and g.
Now we show that $\forall y \in X,\ y \neq x_3$,

Inf $\{Q^1_{\alpha_0}(gx,y)+Q^1_{\alpha_0}(fx,y)+Q^1_{\alpha_0}(gx,fx):\ x\in X\}>0$.
For, take $x=x_1$ then for $y\neq x_3$ we have,
Inf $\{Q^1_{\alpha_0}(gx_1,y)+Q^1_{\alpha_0}(fx_1,y)+Q^1_{\alpha_0}(gx_1,fx_1)\}$
$=$ Inf $\{Q^1_{\alpha_0}(x_3,y)+Q^1_{\alpha_0}(x_3,y)+Q^1_{\alpha_0}(x_3,x_3)\}\ =1+1+0=2>0$.
Similarly for $x=x_2,\ x_4$ and for $y\neq x_3$ we have
Inf $\{Q^1_{\alpha_0}(gx,y)+Q^1_{\alpha_0}(fx,y)+Q^1_{\alpha_0}(gx,fx)\}>0$.
Hence $\forall y\in X,\ y\neq x_3$, Inf $\{Q^1_{\alpha_0}(gx,y)+Q^1_{\alpha_0}(fx,y)+Q^1_{\alpha_0}(gx,fx):\ x\in X\}>0$.
Since $\forall x,y\in X,\ \ Q(fx,fy)=Q(x_3,x_3)=\bar{0}$, we have,
$Q(fx,fy)\leq a_1Q(gx,gy)\oplus a_2Q(gx,fx)\oplus a_3Q(gy,fy)$ where $a_1,a_2,a_3\in[0,1)$ with $a_1+a_2+a_3<\ 1$ holds $\forall x,y\in X$.
Thus all the conditions of the Theorem 4.1 are hold. So x_3 is the unique common fixed point of f and g in X.

5. Conclusion

Following the concept of fuzzy c-distance in cone metric space introduced by the present author, in this paper, an idea of generalized fuzzy c-distance in fuzzy cone metric space is introduced. It is seen that every fuzzy c-distance is a generalized fuzzy c-distance but converse is not true.
There is an advantage to use generalized fuzzy c-distance to establish common fixed point theorems in fuzzy cone metric spaces, since it is not required that contraction mappings be weakly compatible.
Since fixed point theorems are used to the existence theorems for solutions of the integral equations, I think that the results of this paper will be used to the existence theorems for solutions of the fuzzy integral equations.

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