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On somewhat fuzzy δ-continuous functions

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Abstract. In this paper the concepts of somewhat fuzzy δ-continuous functions, somewhat fuzzy δ-open functions between fuzzy topological spaces are introduced and studied. Besides giving characterizations of these functions, several interesting properties of these functions are studied. The concepts of fuzzy δ-resolvable spaces and fuzzy δ-irresolvable spaces are also introduced and studied in this paper.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh in his classical paper [15] in the year 1965. Thereafter the paper of C. L. Chang [2] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The notion of continuity is of fundamental importance in almost all branches of Mathematics. Hence it is of considerable significance from applications view point, to formulate and study new variants of fuzzy continuity.

K. K. Azad [1] introduced the concept of fuzzy regular open sets and fuzzy regular closed sets in fuzzy topological spaces. Z. Petricevic [9] introduced the concept of fuzzy δ-open sets and fuzzy δ-closed sets in fuzzy topological spaces. The concept of δ-continuous functions in classical topology was introduced and studied by T. Noiri in [8]. The notion of fuzzy δ-continuous functions between fuzzy topological spaces has been introduced by Supriti Saha [11]. In classical topology, the class of somewhat continuous functions was introduced and studied by Karl. R. Gentry and Hughes B. Hoyle [6]. Later, the concept of "somewhat" in classical topology has been extended to fuzzy topological spaces. Somewhat fuzzy continuous functions, somewhat fuzzy open functions on fuzzy topological spaces were introduced and studied by G. Thangaraj and G. Balasubramanian in [13]. In this paper, the concepts of somewhat fuzzy δ-continuous functions, somewhat fuzzy δ-open functions are introduced and studied. Besides giving characterizations of these functions, several interesting properties of these functions are studied. E. Hewitt [5] introduced the concepts of resolvability and irresolvability in topological spaces. The concepts of δ-resolvability and fuzzy δ-irresolvability in fuzzy setting are introduced and studied in this paper. Some results concerning functions that preserve the fuzzy δ-resolvable and fuzzy δ-irresolvable spaces in the context of images and pre images are also obtained.

2. Preliminaries

By a fuzzy topological space we shall mean a non-empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X, T).

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define :

(i) $\lambda \vee \mu : X \to [0,1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$.
(ii) $\lambda \wedge \mu : X \to [0,1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.
(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

More generally, for a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i(\lambda_i)$ and intersection $\delta = \wedge_i(\lambda_i)$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define the interior $int(\lambda)$ and the closure $cl(\lambda)$ of λ as follows :

(i) $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$,
(ii) $cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([1]). *For a fuzzy set λ of a fuzzy topological space (X, T),*

(i) $1 - cl(\lambda) = int(1 - \lambda)$
(ii) $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.4 ([2]). Let (X, T) and (Y, S) be any two fuzzy topological spaces. Let f be a function from the fuzzy topological space (X, T) to the fuzzy topological space (Y, S). Let λ be a fuzzy set in (Y, S). The inverse image of λ under f written as $f^{-1}(\lambda)$ is the fuzzy set in (X, T) defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$, for all $x \in X$. Also the image of λ in (X, T) under f written as $f(\lambda)$ is the fuzzy set in (Y, S) defined by

$$f(\lambda)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda(x), & \text{if } \mathrm{f}^{-1}(\mathrm{y}) \text{ is non} - \text{empty} \\ 0, & \text{if otherwise} \end{cases} \quad \text{each } y \in Y.$$

Definition 2.5 ([2]). Let $f : (X, T) \to (Y, S)$ be a mapping. For fuzzy sets λ and μ of (X, T) and (Y, S) respectively, the following statements hold.
(1)$ff^{-1}(\mu) \leq \mu$;
(2)$f^{-1}f(\lambda) \geq \lambda$;

(3)$f(1-\lambda) \geq 1 - f(\lambda)$;
(4)$f^{-1}(1-\mu) = 1 - f^{-1}(\mu)$;
(5) If f is injective, then $f^{-1}f(\lambda) = \lambda$;
(6) If f is surjective, then $ff^{-1}(\mu) = \mu$;
(7) If f is bijective, then $f(1-\lambda) = 1 - f(\lambda)$.

Lemma 2.6 ([1])**.** *Let $f : (X,T) \to (Y,S)$ be a mapping and $\{\lambda_\alpha\}$ be a family of fuzzy sets of Y. Then*

(a) $f^{-1}(\cup_\alpha \lambda_j) = \cup_\alpha f^{-1}(\lambda_j)$,
(b) $f^{-1}(\cap_\alpha \lambda_j) = \cap_\alpha f^{-1}(\lambda_j)$.

Lemma 2.7 ([3])**.** *Let $f : (X,T) \to (Y,S)$ be a mapping and $\{A_j\},\ j \in J$ be a family of fuzzy sets of X. Then*

(a) $f(\cup_{j\in J} A_j) = \cup_{j\in J} f(A_j)$
(b) $f(\cap_{j\in J} A_j) \leq \cap_{j\in J} f(A_j)$.

Lemma 2.8 ([1])**.** *Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. If λ is a fuzzy set of X and μ is a fuzzy set of Y, then $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Definition 2.9 ([1])**.** A fuzzy set λ of a fuzzy space X, is called (i) a fuzzy regular open set of X if $int\big(cl(\lambda)\big) = \lambda$ and (ii) a fuzzy regular closed set of X if $cl\big(int(\lambda)\big) = \lambda$.

Definition 2.10 ([1])**.** A fuzzy set λ of a fuzzy space X is fuzzy regular open if and only if $1-\lambda$ is fuzzy regular closed.

Remark 2.11. It is clear that every fuzzy regular open (closed) set is a fuzzy open (closed) set. The converse need not be true. Also the union (intersection) of any two fuzzy regular open (closed) sets need not be a fuzzy regular open (closed) set.

Theorem 2.12 ([1])**.** *(a) The closure of a fuzzy open set is a fuzzy regular closed set and (b) the interior of a fuzzy closed set is a fuzzy regular open set.*

Theorem 2.13 ([1])**.** *(a) The intersection of two fuzzy regular open sets is a fuzzy regular open set and (b) the union of two fuzzy regular closed sets is a fuzzy regular closed set.*

Definition 2.14 ([10])**.** Two fuzzy sets μ and γ of X are said to be disjoint if they do not intersect at any point of X. That is, $\mu(x) + \gamma(x) \leq 1$, for all $x \in X$.

3. Fuzzy δ-open sets

Definition 3.1 ([9])**.** A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy δ-open set if $\lambda = \vee_i(\lambda_i)$, where (λ_i) is a fuzzy regular open set for each i.

A fuzzy set λ in a fuzzy topological space (X,T) is a fuzzy δ-closed set if $1-\lambda$ is a fuzzy δ-open set in (X,T).

Definition 3.2 ([9])**.** Let (X,T) be a fuzzy topological space and λ be any fuzzy set in (X,T). We define the δ-interior $int_\delta(\lambda)$ and the δ-closure $cl_\delta(\lambda)$ of λ as follows:

(i) $int_\delta(\lambda) = \vee\{\mu/\mu \leq \lambda,\ \mu \text{ is fuzzy } \delta-\text{open in X}\}$,
(ii) $cl_\delta(\lambda) = \wedge\{\mu/\lambda \leq \mu,\ \mu \text{ is fuzzy } \delta-\text{closed in X}\}$.

Lemma 3.3 ([12]). *For a fuzzy set λ of a fuzzy topological space (X, T),*
(i). $1 - int_\delta(\lambda) = cl_\delta(1-\lambda)$ *(ii).* $1 - cl_\delta(\lambda) = int_\delta(1-\lambda)$.

Theorem 3.4 ([12]). *For any fuzzy set λ in a fuzzy topological space (X, T), $cl_\delta(\lambda) = \wedge\{cl(\mu)/\lambda \leq cl(\mu),\ \mu \in T\}$.*

Theorem 3.5 ([12]). *The finite union of fuzzy δ-closed sets is also fuzzy δ-closed. That is, if $\lambda = cl_\delta(\lambda)$ and $\mu = cl_\delta(\mu)$, then $\lambda \vee \mu = cl_\delta(\lambda \vee \mu)$.*

Properties 3.6. Let (X, T) be a fuzzy topological space. Then

(1) $cl_\delta(0) = 0$ and $cl_\delta(1) = 1$.
(2) If $\lambda \leq \mu$, then $cl_\delta(\lambda) \leq cl_\delta(\mu)$, for fuzzy sets λ and μ in (X, T).
(3) A fuzzy set λ is fuzzy δ-closed if and only if $\lambda = cl_\delta(\lambda)$ [4].
(4) A fuzzy set λ is fuzzy δ-open if and only if $\lambda = int_\delta(\lambda)$ [12].
(5) $cl(\lambda) \leq cl_\delta(\lambda)$, for a fuzzy set λ in (X, T) [7].
(6) $int_\delta(\lambda) \leq int(\lambda)$, for a fuzzy set λ in (X, T) [12].
(7) $cl_\delta(cl_\delta(\lambda)) = cl_\delta(\lambda)$, for a fuzzy set λ in (X, T) [12].

Remark 3.7. It is clear that any fuzzy regular open set is fuzzy δ-open and any fuzzy δ-open set is fuzzy open in a fuzzy topological space. Furthermore, if a fuzzy set λ is fuzzy semi-open in a fuzzy topological space X, then $cl(\lambda) = cl_\delta(\lambda)$ [12].

Theorem 3.8 ([12]). *Let f be a surjection from a fuzzy topological space $(X, \tau X)$ onto a fuzzy topological space $(Y, \tau Y)$. Then the following implications hold.*

(a) *f is fuzzy δ-continuous.*
(b) *$f([\lambda]_\delta) \leq [f(\lambda)]_\delta$, for every fuzzy set λ in X.*
(c) *$[f^{-1}(\mu)]_\delta \leq f^{-1}([\mu]_\delta)$, for every fuzzy set μ in X.*
(d) *For every fuzzy δ-closed set μ in Y, $f^{-1}(\mu)$ is fuzzy δ-closed in X.*
(e) *For every fuzzy δ-open set μ in Y, $f^{-1}(\mu)$ is fuzzy δ-open in X.*

Remark 3.9. The fuzzy continuity and the fuzzy δ-continuity are independent notions.

Definition 3.10 ([14]). Let (X, T) and (Y, S) be any two fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called a somewhat fuzzy nearly continuous function if $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$, there exists a non-zero fuzzy open set μ of (X, T) such that $\mu \leq cl[f^{-1}(\lambda)]$. That is, $f : (X, T) \to (Y, S)$ is a somewhat fuzzy nearly continuous function if $intclf^{-1}(\lambda) \neq 0$, for any non-zero fuzzy open set λ in (Y, S).

Definition 3.11 ([14]). Let (X, T) and (Y, S) be any two fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called a somewhat fuzzy nearly open function if for all $\lambda \in T$ and $f(\lambda) \neq 0$, there exists a non-zero fuzzy open set μ of (Y, S) such that $\mu \leq cl[f(\lambda)]$. That is, a function $f : (X, T) \to (Y, S)$ is a somewhat fuzzy nearly open function if $intclf(\lambda) \neq 0$, for every non-zero fuzzy open set λ of (X, T).

4. Somewhat fuzzy δ-continuous functions

Definition 4.1. A function $f : (X, T) \to (Y, S)$ from a fuzzy topological space (X, T) into another fuzzy topological space (Y, S) is called somewhat fuzzy δ-continuous if $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$ implies that there exist a fuzzy δ-open set μ in (X, T) such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$. That is, $int_\delta[f^{-1}(\lambda)] \neq 0$.

Proposition 4.2. *If $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S), then f is a somewhat fuzzy continuous function.*

Proof. Let λ be a non-zero fuzzy open set in (Y, S) such that $f^{-1}(\lambda) \neq 0$. Since f is a somewhat fuzzy δ-continuous function from (X, T) into (Y, S), there exist a fuzzy δ-open set μ in (X, T) such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$. That is, $int_\delta[f^{-1}(\lambda)] \neq 0$. But $int_\delta[f^{-1}(\lambda)] \leq int[f^{-1}(\lambda)]$, implies that $int[f^{-1}(\lambda)] \neq 0$. Therefore f is a somewhat fuzzy continuous function. □

Remark 4.3. The implications contained in the following diagram are true and the reverse implications need not be true.

$$\begin{tikzcd}
\text{Somewhat Fuzzy } \delta\text{-Continuity} \arrow[d] \\
\text{Somewhat Fuzzy Continuity} \arrow[d] \\
\text{Somewhat Fuzzy Nearly Continuity}
\end{tikzcd}$$

Proposition 4.4. *If $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S) and $g : (Y, S) \to (Z, W)$ is a fuzzy continuous function from (Y, S) into a fuzzy topological space (Z, W), then $g \circ f : (X, T) \to (Z, W)$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W).*

Proof. Let λ be a non-zero fuzzy open set in (Z, W). Since g is a fuzzy continuous function from (Y, S) into (Z, W), $g^{-1}(\lambda)$ is a fuzzy open set in (Y, S). Since f is a somewhat fuzzy δ-continuous function from (X, T) into (Y, S) and $g^{-1}(\lambda) \in S$ and $g^{-1}(\lambda) \neq 0$, there exists a non-zero fuzzy δ-open set μ of (X, T) such that $\mu \leq f^{-1}(g^{-1}(\lambda))$. That is, $\mu \leq (g \circ f)^{-1}(\lambda)$. Hence $int_\delta(g \circ f)^{-1}(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W). □

Proposition 4.5. *If $f : (X, T) \to (Y, S)$ is a fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S) and $g : (Y, S) \to (Z, W)$ is a somewhat fuzzy δ-continuous function from (Y, S) into a fuzzy topological space (Z, W), then $g \circ f : (X, T) \to (Z, W)$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W).*

Proof. Let λ be a non-zero fuzzy open set in (Z, W). Since g is a somewhat fuzzy δ-continuous function from (Y, S) into (Z, W), there exists a non-zero fuzzy δ-open set μ of (Y, S) such that $\mu \leq g^{-1}(\lambda)$. Then, $f^{-1}(\mu) \leq f^{-1}(g^{-1}(\lambda))$. That is, $f^{-1}(\mu) \leq (g \circ f)^{-1}(\lambda)$. Again since f is a fuzzy δ-continuous function from (X, T) into (Y, S) and μ is a fuzzy δ-open set in (Y, S), $f^{-1}(\mu)$ is fuzzy δ-open in (X, T).

Hence $int_\delta(g \circ f)^{-1}(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W). □

Proposition 4.6. *If $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S) and $g : (Y, S) \to (Z, W)$ is a somewhat fuzzy δ-continuous function from (Y, S) into a fuzzy topological space (Z, W), then $g \circ f : (X, T) \to (Z, W)$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W).*

Proof. Let λ be a non-zero fuzzy open set in (Z, W). Since g is a somewhat fuzzy δ-continuous function from (Y, S) into (Z, W), there exists a non-zero fuzzy δ-open set μ of (Y, S) such that $\mu \leq g^{-1}(\lambda)$. Then, $f^{-1}(\mu) \leq f^{-1}(g^{-1}(\lambda))$. That is, $f^{-1}(\mu) \leq (g \circ f)^{-1}(\lambda))$. Since any fuzzy δ-open set is fuzzy open in a fuzzy topological space, μ is a fuzzy open set in (Y, S). Again since f is a fuzzy somewhat fuzzy δ-continuous function from (X, T) into (Y, S) and μ is a fuzzy open set in (Y, S), there exists a fuzzy δ-open set η in (X, T) such that $\eta \leq f^{-1}(\mu)$. This implies that $\eta \leq f^{-1}(\mu) \leq (g \circ f)^{-1}(\lambda)$. Hence $int_\delta(g \circ f)^{-1}(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-continuous function from (X, T) into (Z, W). □

Proposition 4.7. *If $f : (X, T) \to (Y, S)$ is a function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S) and the graph function $g : X \to X \times Y$ of f is somewhat fuzzy δ-continuous, then f is somewhat fuzzy δ-continuous.*

Proof. Let λ be a non-zero fuzzy open set in (Y, S). Then $1 \times \lambda$ is a fuzzy open set in $X \times Y$. Since g is a somewhat fuzzy δ-continuous from X into $X \times Y$ we have $int_\delta[g^{-1}(1 \times \lambda)] \neq 0$. But $f^{-1}(\lambda) = 1 \wedge f^{-1}(\lambda) = g^{-1}(1 \times \lambda)$. This implies that $int_\delta[f^{-1}(\lambda)] \neq 0$. Hence f is a somewhat fuzzy δ-continuous function from (X, T) into (Y, S). □

Definition 4.8. A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy δ-dense if there exists no fuzzy δ-closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl_\delta(\lambda) = 1$.

Proposition 4.9. *If f is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S), then, $f^{-1}(1 - \lambda)$ is not a fuzzy δ-dense set in (X, T), for a fuzzy open set λ in (Y, S).*

Proof. Let f be a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S). Then, for each $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$, there exists a fuzzy δ-open set μ in (X, T) such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$. That is, $int_\delta[f^{-1}(\lambda)] \neq 0$. Now $1 - int_\delta[f^{-1}(\lambda)] \neq 1$. This implies that $cl_\delta[1 - f^{-1}(\lambda)] \neq 1$ and hence $cl_\delta[f^{-1}(1 - \lambda)] \neq 1$. Therefore, $f^{-1}(1 - \lambda)$ is not a fuzzy δ-dense set in (X, T). □

Proposition 4.10. *If f is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) into a fuzzy topological space (Y, S), then $f^{-1}(1 - \lambda)$ is not a fuzzy dense set in (X, T), for a fuzzy open set λ in (Y, S).*

Proof. Let λ be a non-zero fuzzy open set in (Y, S). Since f is a somewhat fuzzy δ-continuous function from (X, T) into (Y, S), by proposition 4.9, $f^{-1}(1 - \lambda)$ is not a fuzzy δ-dense set in (X, T). Now $cl(\mu) \leq cl_\delta(\mu)$, for a fuzzy set μ in (X, T), implies that $cl(f^{-1}(1-\lambda)) \leq cl_\delta(f^{-1}(1-\lambda))$.Then, $cl(f^{-1}(1-\lambda)) \neq 1$.[Otherwise $cl(f^{-1}(1-\lambda)) = 1$, will imply that $cl_\delta(f^{-1}(1-\lambda)) = 1$, a contradiction]. Therefore, $f^{-1}(1-\lambda)$ is not a fuzzy dense set in (X,T). □

Proposition 4.11. *Let (X,T) and (Y,S) be any two fuzzy topological spaces. Let $f : (X,T) \to (Y,S)$ be a one-to-one and onto function. Then the following are equivalent:*

(1) *f is somewhat fuzzy δ-continuous.*
(2) *If λ is a fuzzy closed set of (Y,S) such that $f^{-1}(\lambda) \neq 1$, then there exists a fuzzy δ-closed set $\mu \neq 1$ of (X,T) such that $\mu \geq f^{-1}(\lambda)$.*
(3) *If λ is a fuzzy δ-dense set in (X,T), then $f(\lambda)$ is a fuzzy δ-dense set in (Y,S).*

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$. Let f be a somewhat fuzzy δ-continuous function from (X,T) into (Y,S) and λ is a fuzzy closed set in (Y,S) such that $f^{-1}(\lambda) \neq 1$. Clearly $1-\lambda \in S$ and $f^{-1}(1-\lambda) = 1 - f^{-1}(\lambda) \neq 0$ (since $f^{-1}(\lambda) \neq 1$). By (1), there exists a fuzzy δ-open set γ in (X,T) such that $\gamma \leq f^{-1}(1-\lambda)$. Then $\gamma \leq 1 - f^{-1}(\lambda)$ and hence $f^{-1}(\lambda) \leq 1-\gamma$. Clearly $1-\gamma$ is a fuzzy δ-closed set in (X,T) and by taking $1-\gamma = \mu$, we find that (1) $\Longrightarrow$ (2) is proved.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$. Let λ be a fuzzy δ-dense set in (X,T) and suppose that $f(\lambda)$ is not a fuzzy δ-dense set in (Y,S). Then there exists a non-zero fuzzy δ-closed set γ of (Y,S) such that $f(\lambda) < \gamma < 1 \longrightarrow (A)$. Since $\gamma < 1$, $f^{-1}(\gamma) \neq 1$ and so by (2), there exists a fuzzy δ-closed set $\mu \neq 1$ of (X,T) such that $\mu \geq f^{-1}(\gamma)$. Then, $\mu \geq f^{-1}(\gamma) \geq f^{-1}f(\lambda) \geq \lambda$. [from(A)]. That is, there exists a fuzzy δ-closed set μ in (X,T) such that $\mu > \lambda$, which is a contradiction to the assumption on λ. Therefore (2) $\Longrightarrow$ (3) is proved.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$. Let $\lambda(\neq 0) \in S$ and $f^{-1}(\lambda) \neq 0$. Suppose that there exists no fuzzy δ-open set μ in (X,T) such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$. That is, $int_\delta[f^{-1}(\lambda)] = 0$. Then, $1 - int_\delta[f^{-1}(\lambda)] = 1$. This will imply that $cl_\delta[1 - f^{-1}(\lambda)] = 1$. Then, by (3), $f[1-f^{-1}(\lambda)]$ will be a fuzzy δ-dense set in (Y,S). That is, $cl_\delta(f[1-f^{-1}(\lambda)]) = 1$. But $f[1-f^{-1}(\lambda)] = f[f^{-1}(1-\lambda)] \leq 1-\lambda < 1$. Now $f[1-f^{-1}(\lambda)] \leq 1-\lambda$, implies that $cl_\delta(f[1-f^{-1}(\lambda)]) \leq cl_\delta(1-\lambda) < cl_\delta(1) = 1$ [$cl(1) \leq cl_\delta(1)$, implies that $1 \leq cl_\delta(1)$]and then we will have $1 < 1$, a contradiction. Hence we must have $int_\delta[f^{-1}(\lambda)] \neq 0$. Therefore f is a somewhat fuzzy δ-continuous function from (X,T) into (Y,S). □

Proposition 4.12. *Let (X,T) and (Y,S) be any two fuzzy topological spaces. If the function $f : (X,T) \to (Y,S)$ is a somewhat fuzzy δ-continuous, 1-1 and onto function and if $int_\delta(\lambda) = 0$, for any non-zero fuzzy set λ in (X,T), then $int_\delta[f(\lambda)] = 0$ in (Y,S).*

Proof. Let λ be a non-zero fuzzy set in (X,T) such that $int_\delta(\lambda) = 0$. Now $1 - int_\delta(\lambda) = 1$, implies that $cl_\delta(1-\lambda) = 1$. Since f is a somewhat fuzzy δ-continuous function from (X,T) into (Y,S) and $1-\lambda$ is a fuzzy δ-dense set in (X,T), by proposition 4.11, $f(1-\lambda)$ is a fuzzy δ-dense set in (Y,S). That is, $cl_\delta[f(1-\lambda)] = 1$. Since f is a 1-1 and onto function, $f(1-\lambda) = 1 - f(\lambda)$. Then we have $cl_\delta[1-f(\lambda)] = 1$ and hence $int_\delta[f(\lambda)] = 0$ in (Y,S). □

Definition 4.13. A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy δ-nowhere dense if there exists no non-zero fuzzy δ-open set μ in (X, T) such that $\mu < cl_\delta(\lambda)$. That is, $int_\delta\big(cl_\delta(\lambda)\big) = 0$.

Proposition 4.14. *Let (X, T) and (Y, S) be any two fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-continuous, 1-1 and onto function and if λ is a fuzzy δ-nowhere dense set in (X, T), then $int_\delta f[cl_\delta(\lambda)] = 0$ in (Y, S).*

Proof. Let $\lambda(\neq 0)$ be a fuzzy δ-nowhere dense set in (X, T). Then, $int_\delta\big(cl_\delta(\lambda)\big) = 0$. Then, by proposition 4.12, $int_\delta f[cl_\delta(\lambda)] = 0$ in (Y, S). □

Proposition 4.15. *Let (X, T) and (Y, S) be any two fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-continuous, 1-1 and onto function and if λ is a fuzzy δ-nowhere dense set in (X, T), then $int_\delta f[cl(\lambda)] = 0$ in (Y, S).*

Proof. Let $\lambda(\neq 0)$ be a fuzzy δ-nowhere dense set in (X, T). Then, $int_\delta\big(cl_\delta(\lambda)\big) = 0$. Then, by proposition 4.12, $int_\delta f[cl_\delta(\lambda)] = 0$ in (Y, S). Now $cl(\lambda) \leq cl_\delta(\lambda)$, implies that $f[cl(\lambda)] \leq f[cl_\delta(\lambda)]$ and hence we have $int_\delta f[cl(\lambda)] \leq int_\delta f[cl_\delta(\lambda)]$. Therefore $int_\delta f[cl(\lambda)] = 0$ in (Y, S). □

Proposition 4.16. *Let X, X_1 and X_2 be fuzzy topological spaces and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be any fuzzy continuous function. If $f : X \to X_1 \times X_2$ is a somewhat fuzzy δ-continuous function, then $p_i \circ f$ is also a somewhat fuzzy δ-continuous function for $i = 1, 2$.*

Proof. For any non-zero open fuzzy set λ of X_i, we have $(p_i \circ f)^{-1}(\lambda) = f^{-1}(p_i^{-1}(\lambda))$. Now $p_i^{-1}(\lambda) \neq 0$ (since $\lambda \neq 0$). Since p_i is fuzzy continuous, $p_i^{-1}(\lambda)$ is fuzzy open and since f is a somewhat fuzzy δ-continuous function, there exist a fuzzy δ-open set μ in (X, T) such that $\mu \neq 0$ of X and $\mu \leq f^{-1}(p_i^{-1}(\lambda))$. That is, $\mu \leq (p_i \circ f)^{-1}(\lambda)$. Hence $p_i \circ f$ is a somewhat fuzzy δ-continuous function for $i = 1, 2$. □

Definition 4.17. A fuzzy topological space (X, T) is called a fuzzy δ-D\space if every non-zero fuzzy open set λ of X, is fuzzy δ-dense in X.

Proposition 4.18. *If f is a somewhat fuzzy δ-continuous function from a fuzzy topological space (X, T) onto a fuzzy topological space (Y, S) and if (X, T) is a fuzzy δ-D\space, then (Y, S) is a fuzzy δ-D\space.*

Proof. Let λ be a non-zero fuzzy open set in (Y, S). We wish to prove that λ is a fuzzy δ-dense set in (Y, S). Suppose not. Then there exists a fuzzy δ-closed set μ in (Y, S) such that $\lambda < \mu < 1$ and $f^{-1}(\lambda) < f^{-1}(\mu) < f^{-1}(1) = 1$. Since $\lambda \neq 0$, $f^{-1}(\lambda) \neq 0$ and since f is somewhat fuzzy δ-continuous, there exists a fuzzy δ-open set γ in (X, T) such that $\gamma \leq f^{-1}(\lambda)$ and $\gamma \leq f^{-1}(\lambda) < f^{-1}(\mu) < cl_\delta[f^{-1}(\mu)] < 1$. That is, $\gamma < cl_\delta[f^{-1}(\mu)] < 1$. This contradicts the fact that (X, T) is a fuzzy δ-D\space, in which the fuzzy δ-open set γ in (X, T) must be a fuzzy δ-dense in (X, T) and it proves that (Y, S) is a fuzzy δ-D\space. □

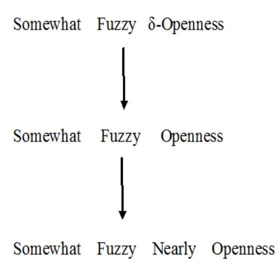
5. Somewhat fuzzy δ-open functions

Definition 5.1. A function $f : (X,T) \to (Y,S)$ from a fuzzy topological space (X,T) into another fuzzy topological space (Y,S) is called somewhat fuzzy δ-open if $\lambda \in T$ and $\lambda \neq 0$ implies that there exists a fuzzy δ-open set η in (Y,S) such that $\eta \neq 0$ and $\eta \leq f(\lambda)$. That is, $int_\delta[f(\lambda)] \neq 0$.

Proposition 5.2. *If $f : (X,T) \to (Y,S)$ is a somewhat fuzzy δ-open function from a fuzzy topological space (X,T) into a fuzzy topological space (Y,S), then f is a somewhat fuzzy open function.*

Proof. Let λ be a non-zero fuzzy open set in (X,T) such that $f(\lambda) \neq 0$. Since f is a somewhat fuzzy δ-open function from (X,T) into (Y,S), there exist a fuzzy δ-open set μ in (Y,S) such that $\mu \neq 0$ and $\mu \leq f(\lambda)$. That is, $int_\delta[f(\lambda)] \neq 0$. But $int_\delta[f(\lambda)] \leq int[f(\lambda)]$, implies that $int[f(\lambda)] \neq 0$. Therefore f is a somewhat fuzzy open function. □

Remark 5.3. The implications contained in the following diagram are true and the reverse implications need not be true.



Proposition 5.4. *If $f : (X,T) \to (Y,S)$ is a somewhat fuzzy open function from a fuzzy topological space (X,T) into a fuzzy topological space (Y,S) and $g : (Y,S) \to (Z,W)$ is a fuzzy δ-open function from (Y,S) into a fuzzy topological space (Z,W), then $g \circ f : (X,T) \to (Z,W)$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W).*

Proof. Let λ be a non-zero fuzzy open set in (X,T). Since f is a fuzzy open function from (X,T) into (Y,S), $f(\lambda)$ is a fuzzy open set in (Y,S). Since g is a somewhat fuzzy δ-open function from (Y,S) into (Z,W) and $f(\lambda) \in S$ and $f(\lambda) \neq 0$, there exists a non-zero fuzzy δ-open set μ of (Z,W) such that $\mu \leq g(f(\lambda))$. That is, $\mu \leq (g \circ f)(\lambda)$. Hence $int_\delta(g \circ f)(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W). □

Proposition 5.5. *If $f : (X,T) \to (Y,S)$ is a somewhat fuzzy δ-open function from a fuzzy topological space (X,T) into a fuzzy topological space (Y,S) and $g : (Y,S) \to (Z,W)$ is a somewhat fuzzy δ-open function from (Y,S) into a fuzzy topological space (Z,W), then $g \circ f : (X,T) \to (Z,W)$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W).*

Proof. Let λ be a non-zero fuzzy open set in (X,T). Since f is a somewhat fuzzy δ-open function from (X,T) into (Y,S), there exists a non-zero fuzzy δ-open set μ of (Y,S) such that $\mu \leq f(\lambda)$. Then, $g(\mu) \leq g(f(\lambda))$. That is, $g(\mu) \leq (g \circ f)(\lambda)$. Since any fuzzy δ-open set is fuzzy open in a fuzzy topological space, μ is a fuzzy open set in (Y,S). Since g is a somewhat fuzzy δ-open function from (Y,S) into (Z,W) and $\mu \in S$ and $g(\mu) \neq 0$, there exists a non-zero fuzzy δ-open set η of (Z,W) such that $\eta \leq g(\mu)$. This implies that $\eta \leq g(\mu) \leq (g \circ f)(\lambda)$. Hence $int_\delta(g \circ f)(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W). □

Definition 5.6. ([12]) Let $f : (X,T) \to (Y,S)$ be a fuzzy mapping.

(1). f is said to be fuzzy δ-open if for each fuzzy δ-open set A in X, $f(A)$ is fuzzy δ-open in Y.

(2). f is said to be fuzzy δ-closed if for each fuzzy δ-closed set B in X, $f(B)$ is fuzzy δ-closed in Y.

Proposition 5.7. *If $f : (X,T) \to (Y,S)$ is a somewhat δ-open function from a fuzzy topological space (X,T) into a fuzzy topological space (Y,S) and $g : (Y,S) \to (Z,W)$ is a fuzzy δ-open function from (Y,S) into a fuzzy topological space (Z,W), then $g \circ f : (X,T) \to (Z,W)$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W).*

Proof. Let λ be a non-zero fuzzy open set in (Z,W). Since f is a somewhat fuzzy δ-open function from (X,T) into (Y,S), there exists a non-zero fuzzy δ-open set μ of (Y,S) such that $\mu \leq f(\lambda)$. Then $g(\mu) \leq g(f(\lambda))$. That is, $g(\mu) \leq (g\circ f)(\lambda)$. Since g is a fuzzy δ-open function from(Y,S) into (Z,W) and μ is a fuzzy δ-open set in (Y,S), $g(\mu)$ is fuzzy δ-open in (X,T). Hence $int_\delta(g \circ f)(\lambda) \neq 0$. Therefore, $g \circ f$ is a somewhat fuzzy δ-open function from (X,T) into (Z,W). □

Proposition 5.8. *Let $f : (X,T) \to (Y,S)$ be a function from a fuzzy topological space (X,T) into another fuzzy topological space (Y,S). Then the following conditions are equivalent:*

(1). *f is somewhat fuzzy δ-open.*

(2). *If λ is a fuzzy δ-dense set in (Y,S), then $f^{-1}(\lambda)$ is a fuzzy δ-dense set in (X,T).*

Proof. **(1)** $\Longrightarrow$ **(2)**. Assume (1). Let λ be a fuzzy δ-dense set in (Y,S) and suppose that $f^{-1}(\lambda)$ is not a fuzzy δ-dense set in (X,T). Then there exists a non-zero fuzzy δ-closed set γ of (X,T) such that $f^{-1}(\lambda) < \gamma < 1$. Then, $1-\gamma < 1 - f^{-1}(\lambda) = f^{-1}(1-\lambda)$. Now $1-\gamma$ is a fuzzy δ-open set in (X,T). Since any fuzzy δ-open set is fuzzy open in a fuzzy topological space, $1-\gamma$ is a fuzzy open set in (X,T). Since $\gamma < 1$, $1-\lambda \neq 0$. Since f is a somewhat fuzzy δ-open function, there exists a fuzzy δ-open set $\mu \neq 0$ in (Y,S) such that $\mu \leq f(1-\gamma)$ and hence $\mu \leq f\big(f^{-1}(1-\lambda)\big) \leq 1-\lambda$. That is, $\lambda < 1-\mu < 1$ and $1-\mu$ is a fuzzy δ-closed set in (Y,S), implies that λ is not a fuzzy δ-dense set in (X,T), which is a contradiction to the assumption on λ. Therefore (1) $\Longrightarrow$ (2) is proved.

(2) $\Longrightarrow$ **(1)**. Let $\lambda(\neq 0) \in T$ and $f(\lambda) \neq 0$. Suppose that there exists no fuzzy δ-open set μ in (Y,S) such that $\mu \neq 0$ and $\mu \leq f(\lambda)$. That is, $int_\delta[f(\lambda)] = 0$. Then, $1 - int_\delta[f(\lambda)] = 1$. This will imply that $cl_\delta[1 - f(\lambda)] = 1$. Now $f^{-1}[1-f(\lambda)] =$

$1 - f^{-1}[f(\lambda)] \leq 1 - \lambda < 1$ (*since* $\lambda \neq 0$). That is, $f^{-1}[1 - f(\lambda)] < 1$. Then $cl_\delta\Big(f^{-1}\big(1 - f(\lambda)\big)\Big) < cl_\delta(1) = 1$. This will imply that $cl_\delta\Big(f^{-1}\big(1 - f(\lambda)\big)\Big) \neq 1$, a contradiction. Hence we must have $int_\delta[f(\lambda)] \neq 0$. Therefore f is a somewhat fuzzy δ-open function from (X, T) into (Y, S). □

Proposition 5.9. *Let (X, T) and (Y, S) be any two fuzzy topological spaces and $f : (X, T) \to (Y, S)$ be a one-to-one and onto function. Then the following are equivalent:*

(1). *f is somewhat fuzzy δ-open.*
(2). *If λ is a fuzzy closed set of (X, T) such that $f(\lambda) \neq 1$, then there exists fuzzy δ-closed $\mu \neq 1$ of (X, T) such that $\mu > f(\lambda)$.*

Proof. **(1)** $\Longrightarrow$ **(2)**. Let f be a somewhat fuzzy δ-open function from (X, T) into (Y, S) and λ is a fuzzy closed set in (X, T) such that $f(\lambda) \neq 1$. Clearly $1 - \lambda \in S$ and $f(1 - \lambda) = 1 - f(\lambda) \neq 0$ (since f is one-to-one and onto). By (1), there exists a fuzzy δ-open set γ in (Y, S) such that $\gamma \leq f(1 - \lambda)$. Then $\gamma \leq 1 - f(\lambda)$ and hence $f(\lambda) \leq 1 - \gamma$. Clearly $1 - \gamma$ is a fuzzy δ-closed set in (X, T) and by taking $1 - \gamma = \mu$, we find that (1) $\Longrightarrow$ (2) is proved.

(2) $\Longrightarrow$ **(1)**. Let λ be a non-zero fuzzy open set in (X, T) such that $f(\lambda) \neq 0$. Then, $1 - \lambda$ is a fuzzy closed set in (X, T) such that $f(1 - \lambda) = 1 - f(\lambda) \neq 1$ (Since f is one-to-one and onto, $f(1 - \lambda) = 1 - f(\lambda)$). By hypothesis, there exists a fuzzy δ-closed set $\mu \neq 1$ of (X, T) such that $\mu > f(1 - \lambda)$. Then $\mu > 1 - f(\lambda)$. Hence $1 - \mu < f(\lambda)$, where $1 - \mu$ is a fuzzy δ-open set in (Y, S). Therefore f is a somewhat fuzzy δ-open function and (2) $\Longrightarrow$ (1) is proved. □

Proposition 5.10. *Let (X, T) and (Y, S) be any two fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is a somewhat fuzzy δ-open function an if $int_\delta(\lambda) = 0$, for any non-zero fuzzy set λ in (Y, S), then $int_\delta[f^{-1}(\lambda)] = 0$ in (X, T).*

Proof. Let $\lambda(\neq 0)$ be a non-zero fuzzy set in (Y, S) such that $int_\delta(\lambda) = 0$. Now $1 - int_\delta(\lambda) = 1$, implies that $cl_\delta(1 - \lambda) = 1$. Since f is a somewhat fuzzy δ-open function from (X, T) into (Y, S) and $1 - \lambda$ is a fuzzy δ-dense set in (Y, S), by proposition 5.8, $f^{-1}(1 - \lambda)$ is a fuzzy δ-dense set in (X, T). That is, $cl_\delta[f^{-1}(1 - \lambda)] = 1$. Since $f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda)$, we have $cl_\delta[1 - f^{-1}(\lambda)] = 1$ and hence $int_\delta[f^{-1}(\lambda)] = 0$ in (X, T). □

6. Fuzzy δ-resolvable and fuzzy δ-irresolvable spaces

Definition 6.1. A fuzzy topological space (X, T) is called a fuzzy δ-resolvable space if there exists a non-zero fuzzy δ-dense set λ in (X, T) such that $cl_\delta(1 - \lambda) = 1$. Otherwise (X, T) is called a fuzzy δ-irresolvable space.

Proposition 6.2. *If a fuzzy topological space (X, T) has a pair of fuzzy δ-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$, then (X, T) is a fuzzy δ-resolvable space.*

Proof. Let the fuzzy topological space (X, T) has a pair of fuzzy δ-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$. We wish to prove that (X, T) is a fuzzy δ-resolvable space. Assume the contrary. Then, for all fuzzy δ-dense sets λ_i in (X, T) we have $cl_\delta(1 - \lambda_i) \neq 1$. In particular $cl_\delta(1 - \lambda_2) \neq 1$, implies that there exists a non-zero fuzzy δ-closed set μ in (X,T) such that $1-\lambda_2 < \mu < 1$. Then, $\lambda_1 \leq 1-\lambda_2$, implies that $\lambda_1 \leq 1-\lambda_2 < \mu < 1$ and hence $\lambda_1 < \mu < 1$. This will imply that $cl_\delta(\lambda_1) \neq 1$, a contradiction. Therefore, (X,T) is a fuzzy δ-resolvable space. □

Proposition 6.3. *If $\vee_{i=1}^{n}(\lambda_i) = 1$, where $int_\delta(\lambda_i) = 0$ in a fuzzy topological space (X,T), then (X,T) is a fuzzy δ-resolvable space.*

Proof. Now $\vee_{i=1}^{n}(\lambda_i) = 1$, implies that $1 - \vee_{i=1}^{n}(\lambda_i) = 0$. Then, $\wedge_{i=1}^{n}(1-\lambda_i) = 0$. Then there must be atleast two non-zero disjoint fuzzy sets $1-\lambda_i$ and $1-\lambda_j$ in (X,T). Hence we have $(1-\lambda_i)+(1-\lambda_j) \leq 1$. This implies that $(1-\lambda_j) \leq \lambda_i$. Then $cl_\delta(1-\lambda_j) \leq cl_\delta(\lambda_i)$. But $int_\delta(\lambda_j) = 0$, implies that $cl_\delta(1-\lambda_j) = 1 - int_\delta(\lambda_j) = 1 - 0 = 1$. Then $1 \leq cl_\delta(\lambda_i)$. That is, $cl_\delta(\lambda_i) = 1$. Also $int_\delta(\lambda_i) = 0$, implies that $cl_\delta(1-\lambda_i) = 1$. Therefore (X,T) has a fuzzy δ-dense set λ_i such that $cl_\delta(1-\lambda_i) = 1$. Therefore (X,T) is a fuzzy δ-resolvable space. □

Proposition 6.4. *A fuzzy topological space (X,T) is a fuzzy δ-irresolvable space if and only if $int_\delta(\lambda) \neq 0$, for each fuzzy δ-dense set λ in (X,T).*

Proof. Let (X,T) be a fuzzy δ-irresolvable space. Then, for each fuzzy δ-dense set λ in (X,T), we have $cl_\delta(1-\lambda) \neq 1$ and hence $1 - int_\delta(\lambda) = cl_\delta(1-\lambda) \neq 1$. Thus, $int_\delta(\lambda) \neq 0$, for each fuzzy δ-dense set λ in (X,T).

Conversely, let $int_\delta(\lambda) \neq 0$, for each fuzzy δ-dense set λ in (X,T). Suppose that (X,T) is a fuzzy λ resolvable space. Then, there exists a non-zero fuzzy δ-dense set λ in (X,T) such that $cl_\delta(1-\lambda) = 1$ and hence $1 - int_\delta(\lambda) = cl_\delta(1-\lambda) = 1$, implies that $int_\delta(\lambda) = 0$, a contradiction to the hypothesis. Therefore, (X,T) is a fuzzy δ-irresolvable space. □

7. Functions and fuzzy δ-resolvable spaces, fuzzy δ-irresolvable spaces

Proposition 7.1. *If the function $f : (X,T) \to (Y,S)$ from a fuzzy topological space (X,T) onto another fuzzy topological space (Y,S) is a somewhat fuzzy δ-continuous and 1-1 function and if (X,T) is a fuzzy δ-resolvable space, then (Y,S) is a fuzzy δ-resolvable space.*

Proof. Let f be a somewhat fuzzy δ-continuous function from a fuzzy δ-resolvable space (X,T) onto a fuzzy topological space (Y,S). Since (X,T) is a fuzzy δ-resolvable space, by proposition 6.3, $\vee_{i=1}^{n}(\lambda_i) = 1$, where $int_\delta(\lambda_i) = 0$ in (X,T). Now $f(\vee_{i=1}^{n}(\lambda_i)) = f(1) = 1$. (since f is onto). Then, by lemma 2.7, $\vee_{i=1}^{n} f(\lambda_i) = 1$. Since f is a somewhat fuzzy δ-continuous and 1 -1 function from (X,T) onto (Y,S) and $int_\delta(\lambda_i) = 0$ in (X,T), by proposition 4.12, $int_\delta[f(\lambda_i)] = 0$ in (Y,S). Thus we have $\vee_{i=1}^{n} f(\lambda_i) = 1$, where, $int_\delta[f(\lambda_i)] = 0$ in (Y,S). Then, by proposition 6.3, (Y,S) is a fuzzy δ-resolvable space. □

Proposition 7.2. *If the function $f : (X,T) \to (Y,S)$ from a fuzzy topological space (X,T) into another fuzzy topological space (Y,S) is a somewhat fuzzy δ-open function and if (X,T) is a fuzzy δ-irresolvable space, then $(Y,S)$is a fuzzy δ-irresolvable space.*

Proof. Let $\lambda \neq 0$ be an arbitrary fuzzy set in (Y,S) such that $cl_\delta(\lambda) = 1$. We claim that $int_\delta(\lambda) \neq 0$. Assume the contrary. That is, $int_\delta(\lambda) = 0$. Since f is a somewhat fuzzy δ-open function from (X,T) into (Y,S), we have, by proposition

5.10, $int_\delta[f^{-1}(\lambda)] = 0$ in (X,T) and by proposition 5.8, $f^{-1}(\lambda)$ is a fuzzy δ-dense set in (X,T), for a fuzzy δ-dense set λ in (Y,S). Thus we have $int_\delta[f^{-1}(\lambda)] = 0$ for a fuzzy δ-dense set $f^{-1}(\lambda)$ in (X,T). But this is a contradiction to (X,T), being a fuzzy δ-irresolvable space in which $int_\delta(\mu) \neq 0$, for each fuzzy δ-dense set μ in (X,T) (by proposition 6.4). Hence our assumption that $int_\delta(\lambda) = 0$, for a fuzzy δ-dense set λ in (Y,S), does not hold. Hence we must have that $int_\delta(\lambda) \neq 0$, for each fuzzy δ-dense set λ in (Y,S). Therefore $(Y,S)$is a fuzzy δ-irresolvable space. □

Proposition 7.3. *If the function $f : (X,T) \to (Y,S)$ from a fuzzy topological space (X,T) onto another fuzzy topological space (Y,S) is a somewhat fuzzy δ-continuous and 1-1 function and if (Y,S) is a fuzzy δ-irresolvable space, then $(X,T)$is a fuzzy δ-irresolvable space.*

Proof. Let $\lambda \neq 0$ be an arbitrary fuzzy set in (X,T) such that $cl_\delta(\lambda) = 1$. We claim that $int_\delta(\lambda) \neq 0$. Assume the contrary. That is, $int_\delta(\lambda) = 0$. Since f is a somewhat fuzzy δ-continuous and 1-1 function from (X,T) into (Y,S), we have, by proposition 4.12, $int_\delta[f(\lambda)] = 0$ in (Y,S) and by proposition 4.11, $f(\lambda)$ is a fuzzy δ-dense set in (Y,S), for a fuzzy δ-dense set λ in (X,T). Thus, we have $int_\delta[f(\lambda)] = 0$ for a fuzzy δ-dense set $f(\lambda)$ in (Y,S). But this is a contradiction to (Y,S), being a fuzzy δ-irresolvable space in which $int_\delta(\mu) \neq 0$, for each fuzzy δ-dense set μ in (Y,S) (by proposition 6.4). Hence our assumption that $int_\delta(\lambda) = 0$, for a fuzzy δ-dense set λ in (X,T), does not hold. Hence we must have that $int_\delta(\lambda) \neq 0$, for each fuzzy δ-dense set λ in (X,T). Therefore (X,T) is a fuzzy δ-irresolvable space. □

8. Conclusions

The concepts of somewhat fuzzy δ-continuous functions, somewhat fuzzy δ-open functions between fuzzy topological spaces are introduced and studied. Also fuzzy δ -resolvability and fuzzy δ-irresolvability of fuzzy topological spaces are introduced and studied. Some results concerning functions that preserve the fuzzy δ-resolvable spaces and fuzzy δ-irresolvable spaces in the context of images and pre-images are obtained in this paper.

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