Annals of Fuzzy Mathematics and Informatics
Volume 10, No. 3, (September 2015), pp. 423–431
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy rough topological entropy structure spaces

S. Padmapriya, M. K. Uma, E. Roja

Received 22 June 2014; Revised 8 December 2014; Accepted 23 March 2015

Abstract.
The purpose of this paper is to introduce the concepts of fuzzy rough topological entropy sets, fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn spaces, contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous functions, fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ normal spaces and fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal spaces are introduced and studied. Some interesting properties are also discussed.

2010 AMS Classification: 54A40-03E72

Keywords: Fuzzy rough topological entropy set, Fuzzy rough $\mathscr{T}\mathscr{E}$ structure space, Fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 space, Fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space, Fuzzy rough $\mathscr{T}\mathscr{E}$ normal space, Fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal space.

Corresponding Author: S. Padmapriya (priyasathi17@gmail.com)

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [8]. Fuzzy sets have applications in many fields such as information [6] and control [7]. The theory of fuzzy topological spaces was introduced and developed by Chang [1]. Nanda and Majumdar [2] studied the concept of fuzzy rough sets. Mondal and Samanta [3] introduced and developed the definitions of image of fuzzy rough sets and inverse image of fuzzy rough sets. Pawlak [5] introduced the definition of rough sets. The concepts of fuzzy rough topological dynamical system, fuzzy rough topological entropy were introduced by S. Padmapriya, M. K. Uma and E. Roja [4]. In this chapter, the concepts of fuzzy rough topological entropy sets, fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn spaces, contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous functions, fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ normal spaces and fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal spaces are introduced and studied. Some interesting properties are also discussed.

2. Preliminaries

Definition 2.1 ([3])**.** Let U be a set and $\mathscr{B}$ be a Boolean subalgebra of the Boolean algebra of all subsets of U. Let L be a lattice. Let X be a rough set. Then $X = (X_L, X_U) \in \mathscr{B}^2$ with $X_L \subset X_U$.

A fuzzy rough set $A = (A_L, A_U)$in X is characterized by a pair of maps $\mu_{A_L} : X_L \to L$ and $\mu_{A_U} : X_U \to L$ with the property that $\mu_{A_L}(x) \leq \mu_{A_U}(x)$ for all $x \in X_U$.

Note 2.2 ([3])**.** In particular, L could be the closed interval $[0, 1]$.

Definition 2.3 ([3])**.** For any two fuzzy rough sets $A = (A_L, A_U)$ and $B = (B_L, B_U)$ in X we define

(i) $A = B$ iff

$$\mu_{A_L}(x) = \mu_{B_L}(x) \text{ for every } x \in X_L \text{ and}$$

$$\mu_{A_U}(x) = \mu_{B_U}(x) \text{ for every } x \in X_U.$$

(ii) $A \subseteq B$ iff

$$\mu_{A_L}(x) \leq \mu_{B_L}(x) \text{ for every } x \in X_L \text{ and}$$

$$\mu_{A_U}(x) \leq \mu_{B_U}(x) \text{ for every } x \in X_U.$$

(iii) $C = A \cup B$ iff

$$\mu_{C_L}(x) = max\{\mu_{A_L}(x), \mu_{B_L}(x)\} \text{ for all } x \in X_L,$$

$$\mu_{C_U}(x) = max\{\mu_{A_U}(x), \mu_{B_U}(x)\} \text{ for all } x \in X_U.$$

(iv) $D = A \cap B$ iff

$$\mu_{D_L}(x) = min\{\mu_{A_L}(x), \mu_{B_L}(x)\} \text{ for all } x \in X_L,$$

$$\mu_{D_U}(x) = min\{\mu_{A_U}(x), \mu_{B_U}(x)\} \text{ for all } x \in X_U.$$

More generally, if L is complete lattice, then for any index set I, if $\{A_i : i \in I\}$ is a family of fuzzy rough sets we have $E = \cup_i A_i$ iff

$$\mu_{E_L}(x) = sup_{i \in I} \mu_{A_{L_i}}(x) \text{ for every } x \in X_L$$

$$\text{and } \mu_{E_U}(x) = sup_{i \in I} \mu_{A_{U_i}}(x) \text{ for every } x \in X_U.$$

Similarly $F = \cap_i A_i$ iff

$$\mu_{F_L}(x) = inf_{i \in I} \mu_{A_{L_i}}(x) \text{ for every } x \in X_L$$

$$\text{and } \mu_{F_U}(x) = inf_{i \in I} \mu_{A_{U_i}}(x) \text{ for every } x \in X_U.$$

We define the complement A' of A by the ordered pair (A'_L, A'_U) of membership functions where

$$\mu_{A'_L}(x) = 1 - \mu_{A_U}(x) \text{ for all } x \in X_L$$
$$\text{and } \mu_{A'_U}(x) = 1 - \mu_{A_L}(x) \text{ for all } x \in X_U.$$

Definition 2.4 ([2])**.** The null fuzzy rough set and whole fuzzy rough set in X are defined by $\tilde{0} = (0_L, 0_U)$ and $\tilde{1} = (1_L, 1_U)$.

Definition 2.5 ([2])**.** Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$.

Let $A = (A_L, A_U)$ be a fuzzy rough set in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L), Y_U = f(X_U)$. The image of A under f, denoted by $f(A) = (f(A_L), f(A_U)$ and is defined by

$$f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\} \text{ for every } y \in Y_L, \text{ and}$$
$$f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\} \text{ for every } y \in Y_U.$$

Definition 2.6 ([2])**.** Let $B = (B_L, B_U)$ be a fuzzy rough set in Y where $Y = (Y_L, Y_U) \in \mathcal{B}_1'$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}_1^2$, where $X_L = f^{-1}(Y_L), X_U = f^{-1}(Y_U)$. Then the inverse image of B under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ and is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)) \text{ for every } x \in X_L \text{ and}$$
$$f^{-1}(B_U)(x) = B_U(f(x)) \text{ for every } x \in X_U.$$

Definition 2.7 ([4])**.** A fuzzy rough topology (in short, FRT) is a family τ of fuzzy rough sets in $X = (X_L, X_U)$ satisfying the following axioms:

(i) $\tilde{0}, \tilde{1} \in \tau$,
(ii) $A \cap B \in \tau$ for any $A, B \in \tau$,
(iii) $\cup A_i \in \tau$ for any arbitrary family $\{A_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a fuzzy rough topological space (in short, FRTS) and any fuzzy rough set in τ is known as fuzzy rough open set (in short, FROS) in X.

Definition 2.8 ([4])**.** Let (X, τ) and (Y, σ) be any two fuzzy rough topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is said to be a fuzzy rough continuous function if $f^{-1}(A)$ is a fuzzy rough open set in (X, τ), for each fuzzy rough open set A in (Y, σ).

Definition 2.9 ([4])**.** Let (X, f) be a fuzzy rough topological dynamical system. Let $\mathfrak{S}$ be a fuzzy rough open cover of X and $\mathcal{C}$ be a non-empty fuzzy rough compact set of X such that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $\mathcal{P}_{\mathcal{C}}(\mathfrak{S})$ be the smallest cardinality of all subcovers (for $\mathcal{C}$) of $\mathfrak{S}$, that is,

$$\mathcal{P}_{\mathcal{C}}(\mathfrak{S}) = min\{|\mathfrak{L}| : \mathfrak{L} \subset \mathfrak{S}, \mathcal{C} \subseteq \cup_{\mathfrak{L} \in \mathfrak{S}} \mathfrak{L}\}$$

Since $\mathcal{C}$ is fuzzy rough compact, $\mathcal{P}_{\mathcal{C}}(\mathfrak{S})$ is a positive integer. Let $\mathcal{R}_{\mathcal{C}}(\ \mathfrak{S}\ ) = log\mathcal{P}_{\mathcal{C}}(\mathfrak{S})$.

Proposition 2.10 ([4])**.** *Let (X, f) be a fuzzy rough compact topological dynamical system. Let $\mathfrak{S}$ be a fuzzy rough open cover of X and $\mathcal{C}$ be a non-empty fuzzy rough compact set of X such that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then $lim_{n\to\infty} \frac{1}{n}\mathcal{R}_{\mathcal{C}}(\cup\{f^{-i}(\mathfrak{S}) : i = 0, 1, .., n-1\})$ exists.*

3. Fuzzy rough topological entropy structure spaces

In this section, the concepts of fuzzy rough topological entropy sets, fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ open sets, fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 spaces, fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn spaces, contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous functions and fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff spaces are introduced and studied. Some interesting properties are also discussed.

Notation 3.1. $\mathscr{I}^X$ denotes the collection of all fuzzy rough sets in X.

Definition 3.2 ([4])**.** Let (X, τ) be a fuzzy rough topological space and $f : (X, \tau) \to (X, \tau)$ be a fuzzy rough continuous function. Then the pair (X, f) is called a fuzzy rough topological dynamical system. If X is fuzzy rough compact, then (X, f) is called a fuzzy rough compact topological dynamical system.

Example 3.3. Let $X = \{a, b\}$ be a non-empty set and $\mathfrak{B}$ a Boolean subalgebra of the Boolean algebra of all subsets of X. Let X be a rough set. Then $X = (X_L, X_U) \in \mathfrak{B}^2$ with $X_L \subset X_U$, where $X_L = \{a\}$ and $X_U = \{a, b\}$. Let $A = ((\frac{a}{0.2}), (\frac{a}{0.3}, \frac{b}{0.3}))$ and $B = ((\frac{a}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}))$ be fuzzy rough sets of X. Then the family $\tau = \{\tilde{0}, \tilde{1}, A, B\}$ is a fuzzy rough topology on X. Clearly, (X, τ) is a fuzzy rough topological space. Let $f : (X, \tau) \to (X, \tau)$ be an identity function. Clearly, $f : (X, \tau) \to (X, \tau)$ is a fuzzy rough continuous function. Then (X, f) is a fuzzy rough topological dynamical system.

Definition 3.4 ([4])**.** Let (X, f) be a fuzzy rough topological dynamical system. Let $D(X, f)$ be the set $\{\mathcal{C} \in \mathscr{I}^X : \mathcal{C}$ is fuzzy rough compact such that $f(\mathcal{C}) \subseteq \mathcal{C}\}$ and $\mathfrak{S}$ be a fuzzy rough open cover of (X, τ). Then for $\mathcal{C} \in D(X, f)$,

$$Ent^*(f, \mathfrak{S}, \mathcal{C}) = lim_{n\to\infty} \frac{1}{n} \mathcal{R}_{\mathcal{C}}(\cup\{f^{-i}(\mathfrak{S}) : i = 0, 1, .., n-1\})$$

is called the fuzzy rough topological entropy of f on $\mathcal{C}$ relative to $\mathfrak{S}$

Definition 3.5. Let (X, f) be a fuzzy rough topological dynamical system. For every fuzzy rough continuous function $f : (X, \tau) \to (X, \tau)$ the set $\mathcal{A}$ is defined by $\mathcal{A} = \{Ent^*(f, \mathfrak{S}, \mathcal{C})/\mathcal{C} \in D(X, f)$ and $Ent^*(f, \mathfrak{S}, \mathcal{C}) \in [0, 1]\}$. Clearly $\mathcal{A}$ is a subset of $[0, 1]$. This $\mathcal{A}$ is called as fuzzy rough topological entropy set (in short, fuzzy rough $\mathscr{T}\mathscr{E}$ set).

Notation 3.6. $\mathscr{F}\mathscr{R}\mathscr{E}^X$ denotes the collection of all fuzzy rough $\mathscr{T}\mathscr{E}$ sets in X.

Definition 3.7. A fuzzy rough topological entropy structure (in short, fuzzy rough $\mathscr{T}\mathscr{E}$ structure) on a non-empty set X is a family $\mathbb{T}$ of fuzzy rough topological entropy sets in X satisfying the following axioms:

(i) $\varnothing, [0, 1] \in \mathbb{T}$,
(ii) $\mathcal{A}_1 \cap \mathcal{A}_2 \in \mathbb{T}$ for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{T}$,
(iii) $\cup\mathcal{A}_i \in \mathbb{T}$ for any arbitrary family $\{\mathcal{A}_i | i \in J\} \subseteq \mathbb{T}$.

In this case the ordered pair $(X, \mathbb{T})$ is called a fuzzy rough topological entropy structure space (in short, fuzzy rough $\mathscr{T}\mathscr{E}$ structure space) and every member of $\mathbb{T}$ is known as a fuzzy rough topological entropy open set (in short, fuzzy rough $\mathscr{T}\mathscr{E}$ open set) in $(X, \mathbb{T})$.

Definition 3.8. The complement $\mathcal{E}'$ of a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{E}$ in a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is called a fuzzy rough topological entropy closed set (in short, fuzzy rough $\mathscr{T}\mathscr{E}$ closed set) in $(X, \mathbb{T})$.

Definition 3.9. Let $(X, \mathbb{T})$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space and $\mathcal{A}$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ set in X. Then the fuzzy rough $\mathscr{T}\mathscr{E}$ interior of $\mathcal{A}$ is denoted by $FR\mathscr{T}\mathscr{E}int(\mathcal{A})$ and is defined by

$FR\mathscr{T}\mathscr{E}int(\mathcal{A}) = \cup\{\mathcal{B} : \mathcal{B}$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ open set in $(X, \mathbb{T})$ and $\mathcal{B} \subseteq \mathcal{A}\}$.

Definition 3.10. Let $(X, \mathbb{T})$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space and $\mathcal{B}$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ set in X. Then the fuzzy rough $\mathscr{T}\mathscr{E}$ closure of $\mathcal{B}$ is denoted by $FR\mathscr{T}\mathscr{E}cl(\mathcal{B})$ and is defined by

$FR\mathscr{T}\mathscr{E}cl(\mathcal{B}) = \cap\{\mathcal{A} : \mathcal{A}$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed set in $(X, \mathbb{T})$ and $\mathcal{B} \subseteq \mathcal{A}\}$.

Definition 3.11. Let $(X, \mathbb{T})$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space. For any $\mathcal{A} \in \mathscr{F}\mathscr{R}\mathscr{E}^X$,

(i) $\mathcal{A}$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed set if and only if $FR\mathscr{T}\mathscr{E}cl(\mathcal{A}) = \mathcal{A}$.
(ii) $\mathcal{A}$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ open set if and only if $FR\mathscr{T}\mathscr{E}int(\mathcal{A}) = \mathcal{A}$.

Definition 3.12. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ T_1 if for each pair of distinct points x and y in $[0, 1]$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{A}$ and $\mathcal{B}$ in $(X, \mathbb{T})$ containing x and y respectively, such that $y \notin \mathcal{A}$ and $x \notin \mathcal{B}$

Definition 3.13. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 if for each pair of distinct points x and y in $[0, 1]$, there exist disjoint fuzzy rough $\mathscr{T}\mathscr{E}$ open sets A and B in $(X, \mathbb{T})$ such that $x \in \mathcal{A}$, $y \in \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \varnothing$.

Definition 3.14. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn if for each pair of distinct points x and y in $[0, 1]$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{A}$ and $\mathcal{B}$ in $(X, \mathbb{T})$ containing x and y respectively, such that $FR\mathscr{T}\mathscr{E}cl(\mathcal{A}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{B}) = \varnothing$.

Definition 3.15. Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. A function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is called a contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous function if $f^{-1}(\mathcal{A})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed set in $(X, \mathbb{T})$, for every fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{A}$ in $(Y, \mathbb{S})$.

Definition 3.16. Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. A function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is said to be a fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous function if for each point $x \in [0, 1]$ and each fuzzy rough $\mathscr{T}\mathscr{E}$ closed set $\mathcal{B}$ in $(Y, \mathbb{S})$ with $f(x) \in \mathcal{B}$, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{A}$ in $(X, \mathbb{T})$ such that $x \in \mathcal{A}$ and $f(\mathcal{A}) \subset FR\mathscr{T}\mathscr{E}cl(\mathcal{B})$.

Proposition 3.17. *Let $(X, \mathbb{T})$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ structure space. Suppose that for each pair of distinct points x_1 and x_2 in $[0, 1]$, there exists a function f of $(X, \mathbb{T})$ into a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space $(Y, \mathbb{S})$ such that $f(x_1) \neq f(x_2)$. Moreover, let f be contra fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous at x_1 and x_2. Then $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 space.*

Proof. Let x_1 and x_2 be any two distinct points in $[0, 1]$. Then suppose that there exist an fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space $(Y, \mathbb{S})$ and a function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ such that $f(x_1) \neq f(x_2)$ and f is fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous at x_1 and x_2. Let $w = f(x_1)$ and $z = f(x_2)$. Then $w \neq z$. Since Y is fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{U}$ and $\mathcal{V}$ containing w and z, respectively such that $FR\mathscr{T}\mathscr{E}cl(\mathcal{U}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) = \varnothing$. Since f is fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous at x_1 and x_2, then there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{A}$ and $\mathcal{B}$ containing x_1

and x_2, respectively such that $f(\mathcal{A}) \subset FR\mathscr{T}\mathscr{E}cl(\mathcal{U})$ and $f(\mathcal{B}) \subset FR\mathscr{T}\mathscr{E}cl(\mathcal{V})$. Since $FR\mathscr{T}\mathscr{E}cl(\mathcal{U}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) = \varnothing$, $\mathcal{A} \cap \mathcal{B} = \varnothing$. Hence, $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 space. □

Definition 3.18. Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. A function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is said to be a fuzzy rough $\mathscr{T}\mathscr{E}$ open function if $f(\mathcal{A})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ open set in $(Y, \mathbb{S})$, for every fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{A}$ in $(X, \mathbb{T})$.

Proposition 3.19. *Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. If $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ open function and $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space, then $(Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space.*

Proof. Let $(X, \mathbb{T})$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space. For each pair of distinct points x and y in $[0, 1]$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{A}$ and $\mathcal{B}$ in X such that $FR\mathscr{T}\mathscr{E}cl(\mathcal{A}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{B}) = \varnothing$. Since f is a fuzzy rough $\mathscr{T}\mathscr{E}$ open function, $f(\mathcal{A})$ and $f(\mathcal{B})$ are fuzzy rough $\mathscr{T}\mathscr{E}$ open sets in $(Y, \mathbb{S})$. Since $FR\mathscr{T}\mathscr{E}cl(\ \mathcal{A}\ ) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{B}) = \varnothing$, $FR\mathscr{T}\mathscr{E}cl(f(\mathcal{A})) \cap FR\mathscr{T}\mathscr{E}cl(f(\mathcal{B})) = \varnothing$. Hence, $(Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space. □

Proposition 3.20. *Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. If $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous injection and $(Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space, then $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 space.*

Proof. For each pair of distinct points x_1 and x_2 in $[0, 1]$ and f is a fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous function of $(X, \mathbb{T})$ into a fuzzy rough $\mathscr{T}\mathscr{E}$ Urysohn space $(Y, \mathbb{S})$ such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by Proposition 3.17, $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ T_2 space. □

Definition 3.21. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ regular if for each $x \in [0, 1]$ and each fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{B}$ in $(X, \mathbb{T})$ containing x, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{A}$ in $(X, \mathbb{T})$ containing x such that $FR\mathscr{T}\mathscr{E}cl(\mathcal{A}) \subset \mathcal{B}$.

Definition 3.22. Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. A function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is said to be a fuzzy rough $\mathscr{T}\mathscr{E}$ continuous function if for each $x \in [0, 1]$ and each fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{B}$ in $(Y, \mathbb{S})$ with $f(x) \in \mathcal{B}$, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{A}$ in $(X, \mathbb{T})$ such that $x \in \mathcal{A}$ and $f(\mathcal{A}) \subset \mathcal{B}$.

Proposition 3.23. *Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. If $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous function and $(Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ regular space, then f is a fuzzy rough $\mathscr{T}\mathscr{E}$ continuous function.*

Proof. Let x be an arbitrary point of $[0, 1]$ and $\mathcal{U}$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ open set containing $f(x)$. Since $(Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ regular space, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{W}$ in Y containing $f(x)$ such that $FR\mathscr{T}\mathscr{E}cl(\mathcal{W}) \subset \mathcal{U}$. Since f is a fuzzy rough $\mathscr{T}\mathscr{E}^*$ continuous function, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{V}$ in X such that $f(\mathcal{V}) \subset FR\mathscr{T}\mathscr{E}cl(\mathcal{W})$. Then,

$$f(\mathcal{V}) \subset FR\mathscr{T}\mathscr{E}cl(\mathcal{W}) \subset \mathcal{U}.$$

This implies that $f(\mathcal{V}) \subset \mathcal{U}$.

Hence, f is a fuzzy rough $\mathscr{T}\mathscr{E}$ continuous function. □

Definition 3.24. A fuzzy rough $\mathscr{T}\mathscr{E}$ set $\mathcal{A}$ of a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is called
(i) a fuzzy rough $\mathscr{T}\mathscr{E}$ regular open set of $(X, \mathbb{T})$ if $FR\mathscr{T}\mathscr{E}int(FR\mathscr{T}\mathscr{E}cl(\mathcal{A})) = \mathcal{A}$, and
(ii) a fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed set of $(X, \mathbb{T})$ if $FR\mathscr{T}\mathscr{E}cl(FR\mathscr{T}\mathscr{E}int(\mathcal{A})) = \mathcal{A}$.

Definition 3.25. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff if for any two distinct points x_1 and x_2 in $[0, 1]$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed sets $\mathcal{A}$ and $\mathcal{B}$ in $(X, \mathbb{T})$ such that $x_1 \in \mathcal{A}$, $x_2 \notin \mathcal{A}$, $x_1 \notin \mathcal{B}$ and $x_2 \in \mathcal{B}$.

Proposition 3.26. *Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. If $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is a contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous injection and $(Y, \mathbb{S})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff, then $(X, \mathbb{T})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ T_1.*

Proof. Suppose that $(Y, \mathbb{S})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ weakly Hausdorff. For any two distinct points y_1 and y_2 in $[0, 1]$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed sets $\mathcal{A}$ and $\mathcal{B}$ in $(Y, \mathbb{S})$ such that $f(y_1) \in \mathcal{A}$, $f(y_2) \notin \mathcal{A}$, $f(y_1) \notin \mathcal{B}$ and $f(y_2) \in \mathcal{B}$. Thus there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Since f is a contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous function, $f^{-1}(\mathcal{A})$ and $f^{-1}(\mathcal{B})$ are fuzzy rough $\mathscr{T}\mathscr{E}$ open sets in $(X, \mathbb{T})$ such that $x_1 \in f^{-1}(\mathcal{A})$, $x_2 \notin f^{-1}(\mathcal{A})$, $x_1 \notin f^{-1}(\mathcal{B})$ and $x_2 \in f^{-1}(\mathcal{B})$. This shows that $(X, \mathbb{T})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ T_1. □

4. Properties of fuzzy rough $\mathscr{T}\mathscr{E}$ normal spaces and fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal spaces

In this section, the concepts of fuzzy rough $\mathscr{T}\mathscr{E}$ normal spaces and fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal spaces are introduced and some of their properties are studied.

Definition 4.1. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is said to be fuzzy rough $\mathscr{T}\mathscr{E}$ normal if each pair of non-empty disjoint fuzzy rough $\mathscr{T}\mathscr{E}$ closed sets in $(X, \mathbb{T})$ can be separated by disjoint fuzzy rough $\mathscr{T}\mathscr{E}$ open sets in $(X, \mathbb{T})$.

Definition 4.2. Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. A function $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is said to be a fuzzy rough $\mathscr{T}\mathscr{E}$ closed function if $f(\mathcal{A})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed set in $(Y, \mathbb{S})$, for every fuzzy rough $\mathscr{T}\mathscr{E}$ closed set $\mathcal{A}$ of $(X, \mathbb{T})$.

Proposition 4.3. *Let $(X, \mathbb{T})$ and $(Y, \mathbb{S})$ be any two fuzzy rough $\mathscr{T}\mathscr{E}$ structure spaces. If $f : (X, \mathbb{T}) \to (Y, \mathbb{S})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed function, contra fuzzy rough $\mathscr{T}\mathscr{E}$ continuous injection and $(Y, \mathbb{S})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ normal, then $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ normal space.*

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be disjoint fuzzy rough $\mathscr{T}\mathscr{E}$ closed sets in $(X, \mathbb{T})$. Since f is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed function and injective, $f(\mathcal{U}_1)$ and $f(\mathcal{U}_2)$ are fuzzy rough $\mathscr{T}\mathscr{E}$ closed sets in $(Y, \mathbb{S})$. Since $(Y, \mathbb{S})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ normal, $f(\mathcal{U}_1)$ and $f(\mathcal{U}_2)$ are separated by disjoint fuzzy rough $\mathscr{T}\mathscr{E}$ closed sets $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Hence, $\mathcal{U}_i \subset f^{-1}(\mathcal{V}_i)$, $f^{-1}(\mathcal{V}_i)$ are fuzzy rough $\mathscr{T}\mathscr{E}$ open sets of $(X, \mathbb{T})$ for $i = 1, 2$ and $f^{-1}(\mathcal{V}_2) \cap f^{-1}(\mathcal{V}_1) = \varnothing$ and thus, $(X, \mathbb{T})$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ normal space. □

Definition 4.4. A fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal if and only if for every fuzzy rough $\mathscr{T}\mathscr{E}$ closed set $\mathcal{E}$ and for any fuzzy rough $\mathscr{T}\mathscr{E}$ regular open set $\mathcal{W}$ containing $\mathcal{E}$, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{V}$ such that $\mathcal{E} \subseteq \mathcal{V} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \subseteq \mathcal{W}$.

Proposition 4.5. *Let $\mathcal{E}$ be any fuzzy rough set in a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space $(X, \mathbb{T})$. Then*

(i) $(FR\mathscr{T}\mathscr{E}cl(\mathcal{E}))' = FR\mathscr{T}\mathscr{E}int(\mathcal{E}')$.
(ii) $FR\mathscr{T}\mathscr{E}cl(\mathcal{E}') = (FR\mathscr{T}\mathscr{E}int(\mathcal{E}))'$.

Proof. (i) Let $\mathcal{E}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ set in $(X, \mathbb{T})$. Thus,

$$(FR\mathscr{T}\mathscr{E}cl(\mathcal{E}))' = (\cap\{\mathcal{B} : \mathcal{B} \in \mathbb{T} \text{ and } \mathcal{B} \supseteq \mathcal{E}\})'$$

$$\begin{aligned} &= \cup\{\mathcal{B}' : \mathcal{B}' \in \mathbb{T} \text{ and } \mathcal{B}' \subseteq \mathcal{E}'\} \\ &= \cup\{\mathcal{G} : \mathcal{G} \in \mathbb{T} \text{ and } \mathcal{G} \subseteq \mathcal{E}'\} \\ &= FR\mathscr{T}\mathscr{E}int(\mathcal{E}'). \end{aligned}$$

This implies that $(FR\mathscr{T}\mathscr{E}Scl(\mathcal{E}))' = FR\mathscr{T}\mathscr{E}int(\mathcal{E}')$.
(ii) The proof is similar to that of (i). □

Proposition 4.6. *Let $(X, \mathbb{T})$ be a fuzzy rough $\mathscr{T}\mathscr{E}$ structure space. Then the following are equivalent:*

(i) *$(X, \mathbb{T})$ is fuzzy rough $\mathscr{T}\mathscr{E}$ almost normal.*
(ii) *For every fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed set $\mathcal{E}$ and every fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{W}$ containing $\mathcal{E}$, there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{V}$ such that $\mathcal{E} \subseteq \mathcal{V} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \subseteq \mathcal{W}$.*
(iii) *For every pair of fuzzy rough sets $\mathcal{E}$ and $\mathcal{U}$ with $\mathcal{E} \cap \mathcal{U} = \varnothing$, one of which is fuzzy rough $\mathscr{T}\mathscr{E}$ closed and the other fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{E} \subseteq \mathcal{V}$, $\mathcal{U} \subseteq \mathcal{W}$ and $FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{W}) = \varnothing$.*
(iv) *For any two fuzzy rough $\mathscr{T}\mathscr{E}$ closed sets $\mathcal{E}$ and $\mathcal{U}$, one of which is fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed and $E \not{q} F$, there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{E} \subseteq \mathcal{V}$, $\mathcal{U} \subseteq \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W} = \varnothing$.*

Proof. (i) ⇒ (ii) : Let $\mathcal{E}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed set and $\mathcal{W}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ open set containing $\mathcal{E}$. Then, $\mathcal{W}' \subseteq \mathcal{E}'$, where $\mathcal{E}'$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ regular open set containing the fuzzy rough $\mathscr{T}\mathscr{E}$ closed set $\mathcal{W}'$. Thus there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{G}$ such that $\mathcal{W}' \subseteq \mathcal{G} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{G}) \subseteq \mathcal{E}'$. Therefore, $\mathcal{E} \subseteq (FR\mathscr{T}\mathscr{E}cl(\mathcal{G}))' \subseteq \mathcal{G}' \subseteq \mathcal{W}$. Let $(FR\mathscr{T}\mathscr{E}cl(\mathcal{G}))' = \mathcal{V}$. Then $\mathcal{V}$ is fuzzy rough $\mathscr{T}\mathscr{E}$ open and $\mathcal{E} \subseteq \mathcal{V} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \subseteq \mathcal{W}$.

(ii) ⇒ (iii) : Let $\mathcal{E}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed set and let $\mathcal{U}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ closed set such that $\mathcal{E} \cap \mathcal{U} = \varnothing$. Then, $\mathcal{E} \subseteq \mathcal{U}'$, where $\mathcal{U}'$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ open set. Hence there exists a fuzzy rough $\mathscr{T}\mathscr{E}$ open set $\mathcal{V}$ such that $\mathcal{E} \subseteq \mathcal{V} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \subseteq \mathcal{U}'$. By (ii), we get a fuzzy $\mathscr{T}\mathscr{E}$ open set $\mathcal{G}$ such that $\mathcal{E} \subseteq \mathcal{G} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{G}) \subseteq \mathcal{V}$. Put $\mathcal{W} = (FR\mathscr{T}\mathscr{E}cl(\mathcal{V}))'$. Then, $\mathcal{E} \subseteq \mathcal{G}$ and $\mathcal{U} \subseteq \mathcal{W}$.

Therefore,

$$\begin{aligned} FR\mathscr{T}\mathscr{E}cl(\mathcal{W}) &= FR\mathscr{T}\mathscr{E}cl((FR\mathscr{T}\mathscr{E}cl(\mathcal{V}))') \\ &= (FR\mathscr{T}\mathscr{E}int(FR\mathscr{T}\mathscr{E}cl(\mathcal{V})))' \\ &= \mathcal{V}' \\ &\subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{G}), \end{aligned}$$

and hence, $FR\mathscr{T}\mathscr{E}cl(\mathcal{G}) \cap FR\mathscr{T}\mathscr{E}cl(\mathcal{W}) = \varnothing$.

(iii) $\Rightarrow$ (iv) : The proof is obvious.

(iv) $\Rightarrow$ (i) : Let $\mathcal{E}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ closed set and $\mathcal{U}$ be any fuzzy rough $\mathscr{T}\mathscr{E}$ regular open set containing $\mathcal{E}$ in $(X, \mathbb{T})$. Then $\mathcal{E} \cap \mathcal{U}' = \varnothing$, where $\mathcal{U}'$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ regular closed set. Therefore there exist fuzzy rough $\mathscr{T}\mathscr{E}$ open sets $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{E} \subseteq \mathcal{V}$, $\mathcal{U}' \subseteq \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W} = \varnothing$. Thus, $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{W}' \subseteq \mathcal{U}$. Now, $\mathcal{W}'$ is a fuzzy rough $\mathscr{T}\mathscr{E}$ closed set. Hence,

$$\mathcal{E} \subseteq \mathcal{V} \subseteq FR\mathscr{T}\mathscr{E}cl(\mathcal{V}) \subseteq \mathcal{W}' \subseteq \mathcal{U}. \qquad \square$$

Acknowledgements. The authors are greatful to the referees for their useful comments and suggestions. The first author would like to thank the UGC for financial support. The authors thank the referees for his suggestion in improving this paper.

References

[1] C. L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[2] T. K. Mondal and S. K. Samanta, Intuitionistic fuzzy rough sets and Rough intuitionistic fuzzy sets, J. Fuzzy Math. 9 (3) (2001) 561–582.
[3] S. Nanda and S. Majundar, Fuzzy Rough Sets, Fuzzy Sets and Systems 45 (1992) 157–160.
[4] S. Padmapriya, M. K. Uma and E. Roja, Fuzzy Rough Topological Entropy of Fuzzy Rough Continuous Functions, American J. of Mathematics and Sciences 3 (1) (2014) 141-144.
[5] Z. Pawlak, Rough Sets, Inter. J. Inform. Comput. Sci. 11 (5) (1982) 341–356.
[6] P. Smets, The Degree of Belief in a Fuzzy Event, Inform. Sci. 25 (1981) 1–19.
[7] M. Sugeno, An Introductory Survey of Control, Inform. Sci. 36 (1985) 59–83.
[8] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

S. Padmapriya (priyasathi17@gmail.com)
Department of Mathemathics, Sri Sarada College for Women, Salem - 16, Tamil Nadu, India

M. K. Uma (ar.udhay@yahoo.co.in)
Department of Mathemathics, Sri Sarada College for Women, Salem - 16, Tamil Nadu, India

E. Roja (ar.udhay@yahoo.co.in)
Department of Mathemathics, Sri Sarada College for Women, Salem - 16, Tamil Nadu, India