Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 3, (September 2015), pp. 397–412
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Soft Banach Algebra

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Received 26 December 2014; Revised 28 January 2015; Accepted 17 February 2015

Abstract. In this paper we introduce a new concept of soft convergence of a sequence of soft elements and deduce some of its basic properties in a soft normed linear space. We also introduce a soft topology generated by soft norm. Using these concepts we further introduce, for the first time, a definition of soft Banach algebra and study some of its properties.

2010 AMS Classification: 03E72, 46S40

Keywords: Soft sets, Soft elements, Soft normed linear spaces, Soft open sets, Soft closed sets, Soft topology, soft convergence, Soft Banach algebra.

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1. Introduction

In dealing with the real life problems involving uncertainties in different fields of study such as economics, engineering, mathematical modeling, environmental science, social science etc, several techniques like probability theory, fuzzy set theory, rough set theory have been used for a quite long time. But those theories have their own limitations. Particularly the inadequacy of parametrization tools had been a matter of concern in fuzzy set theory and other relevant theories. In 1999, Molodtsov [13] addressed this problem and initiated the theory of soft sets by introducing a concept of parametrized family of sets. Molodtsov applied this theory in many fields like smoothness of function, probability theory, measure theory and game theory, Riemann integration, parron integration etc.

Later, P.K. Maji et al [11, 12] defined and studied several basic notions of soft set theory. In 2011, Shabir and Naz [15] introduced soft topological spaces. Later Hazra et al [10], Cagman et al [2] also studied soft topology in different ways. Algebraic structure of soft setting was introduced by Aktas and Cagman [1] in studying soft group. Recently studies on soft vector spaces and soft normed linear spaces have been initiated by Das and Samanta [3, 4, 5, 6] and later on studied by Yazar et. al. [18].

Banach algebra is an important field of functional analysis, which has many applications in various branches of mathematics. Many examples of classical Banach algebras are known, among them are $B(X)$, the space of bounded linear operators on X and $C(X)$, the space of continuous functions on X. When X is a Hilbert space, the space of bounded linear operators play a key role in quantum mechanics and differential equations. We have introduced fuzzy Banach algebra in [16, 17]. Thus it is a natural query to extend the concept of Banach algebra in soft setting.

In this paper we introduce a definition soft Banach algebra and study some of its properties. In section 2, preliminary results are given. In section 3, we introduce a new concept of convergence of a sequence of soft elements. With this convergence we have shown that the condition of finiteness of parameter set is not required in many cases like completeness of finite dimensional soft normed linear spaces etc [3, 6]. In this section it is also shown that the norm axiom $N(5)$ is redundant, which was used frequently to prove most of the theorems on soft normed linear spaces [3, 6] and we are also able to define a soft topology generated by a soft norm in a soft normed linear space. In section 4, we introduce the concept of soft Banach algebra and some of its preliminary properties are studied. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1 ([14])**.** Let U be a universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Definition 2.2 ([9])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

- 1. $A \subseteq B$ and
- 2. for all $e \in A$, $F(e) \subseteq G(e)$.

We write $(F, A)\tilde{\subset}(G, B)$.

(F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\tilde{\supset}(G, B)$.

Definition 2.3 ([9])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([9])**.** The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.5 ([12])**.** A soft set (F, E) over U is said to be an absolute soft set denoted by $\check{U}$ if for all $e \in E$, $F(e) = U$.

Definition 2.6 ([12])**.** A soft set (F, E) over U is said to be a null soft set denoted by Φ if for all $e \in E$, $F(e) = \phi$.

Definition 2.7 ([12])**.** The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C) , where $C = A \bigcup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \bigcup G(e) & \text{if e} \in \text{A} \bigcap \text{B}. \end{cases}$$

We express it as $(F, A)\tilde{\bigcup}(G, B) = (H, C)$.

Definition 2.8 ([8])**.** The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \bigcap B$ and for all $e \in C$, $H(e) = F(e) \bigcap G(e)$. We write $(F, A)\tilde{\bigcap}(G, B) = (H, C)$.

Definition 2.9 ([15])**.** Let X be an initial universal set and E be the non-empty set of parameters. The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) - (G, E)$, is defined by $H(e) = F(e) - G(e)$ for all $e \in E$.

Proposition 2.10 ([15])**.** *Let (F, E) and (G, E) be two soft sets over X. Then*

- *(i)* $((F, E)\tilde{\bigcup}(G, E))^c = (F, E)^c\tilde{\bigcap}(G, E)^c$
- *(ii)* $((F, E)\tilde{\bigcap}(G, E))^c = (F, E)^c\tilde{\bigcup}(G, E)^c$.

Definition 2.11 ([7])**.** Let X be a non-empty set and E be a non-empty parameter set. Then a function $\epsilon : E \to X$ is said to be a soft element of X. A soft element ϵ of X is said to belong to a soft set A of X, which is denoted by $\epsilon\tilde{\in}A$, if $\epsilon(e) \in A(e)$, $\forall e \in E$. Thus for a soft set A of X with respect to the index set E, we have $A(e) = \{\epsilon(e); \epsilon\tilde{\in}A\}$, $e \in E$.

It is to be noted that every singleton soft set (a soft set (F, E) for which $F(e)$ is a singleton set, $\forall e \in E$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in E$.

Definition 2.12 ([7])**.** Let R be the set of real numbers and $\mathcal{B}(R)$, the collection of all non-empty bounded subsets of R and A be taken as the set of parameters. Then a mapping $F : A \to \mathcal{B}(R)$ is called a soft real set. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number. We use notations $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$.
For two soft real numbers $\tilde{r}, \tilde{s}$ it is defined

- $\tilde{r}\tilde{\leq}\tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$.
- $\tilde{r}\tilde{\geq}\tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$.
- $\tilde{r}\tilde{<}\tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$.
- $\tilde{r}\tilde{>}\tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

Let X be an initial universal set and A be the non-empty set of parameters. Let us consider the collection of those soft sets (F, A) over X for which $F(\lambda) \neq \phi$, for all $\lambda \in A$, which is denoted by $\mathcal{S}(\check{X})$. For any soft set $(F, A) \in \mathcal{S}(\check{X})$, the collection of all soft elements of (F, A) is denoted by $SE(F, A)$ and let Y be any collection of soft elements of (F, A), then $SS(Y)$ is the soft set generated by Y such that $(SS(Y))(\lambda) = \{\tilde{x}(\lambda); \tilde{x} \in Y\}$, $\forall \lambda \in A$.

Definition 2.13 ([3]). **(Sums and Scalar products of soft sets)** Let $F_1, F_2, ..., F_n$ be n soft sets in (V, A). Then $F = F_1 + F_2 + ... + F_n$ is a soft set over (V, A) and is defined as $F(\lambda) = \{x_1 + x_2 + ... + x_n; x_i \in F_i(\lambda), i = 1, 2, ..., n\}$, $\forall \lambda \in A$. Let $\alpha \in K(\mathbb{R}$ or $\mathbb{C})$ be a scalar and F be a soft set over (V, A), then αF is a soft set over (V, A) and is defined as follows: $\alpha F = G$, $G(\lambda) = \{\alpha x; x \in F(\lambda)\}$, $\lambda \in A$.

Definition 2.14 ([3]). Let V be a vector space over a field $K(\mathbb{R}$ or $\mathbb{C})$ and let A be a parameter set. Let G be a soft set over (V, A). Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector subspace of V, $\forall \lambda \in A$.

Proposition 2.15 ([3]). *$\alpha(F + G) = \alpha F + \alpha G$ for all soft sets F, G over (V, A) and $\alpha \in K$.*

Definition 2.16 ([3]). **(Soft Vector Sub spaces)** Let F be a soft vector space of V over K. Let $G : A \to \mathcal{P}(V)$ be a soft set over (V, A). Then G is said to be a soft vector subspace of F if
(i) for each $\lambda \in A$, $G(\lambda)$ is a vector subspace of V over K and
(ii) $F(\lambda) \supseteq G(\lambda)$, $\forall \lambda \in A$.

Theorem 2.17 ([3]). *A soft subset G of a soft vector space F is a soft vector sub-space of F if and only if for all scalars $\alpha, \beta \in K$, $\alpha G + \beta G \subset G$.*

Definition 2.18 ([3]). Let G be a soft vector space of V over K. Then a soft element of V is said to be a soft vector of G. In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Definition 2.19 ([3]). A soft vector $\tilde{x}$ in a soft vector space G is said to be the null soft vector if $\tilde{x}(\lambda) = \theta$, $\forall \lambda \in A$, θ being the zero element of V . It will be denoted by Θ. A soft vector is said to be non-null if it is not a null soft vector.

Definition 2.20 ([3]). Let $\tilde{x}, \tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. Then the addition $\tilde{x} + \tilde{y}$ of $\tilde{x}, \tilde{y}$ and scalar multiplication $\tilde{k}.\tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by $(\tilde{x} + \tilde{y})(\lambda) = \tilde{x}(\lambda) + \tilde{y}(\lambda)$, $\tilde{k}.\tilde{x}\ (\lambda) = \tilde{k}(\lambda).\ \tilde{x}(\lambda), |\tilde{k}|(\lambda) = |\tilde{k}(\lambda)|, \forall \lambda \in A$. Obviously, $\tilde{x} + \tilde{y}, \tilde{k}.\tilde{x}$ are soft vectors of G.

Theorem 2.21 ([3]). *In a soft vector space G of V over K,*

- *(i).* $\bar{0}.\tilde{\alpha} = \Theta$, *for all* $\tilde{\alpha} \tilde{\in} G$;
- *(ii).* $\tilde{k}.\Theta = \Theta$, *for all soft scalar* $\tilde{k}$.
- *(iii).* $(\bar{-1})\,\tilde{\alpha} = -\tilde{\alpha}$, *for all* $\tilde{\alpha} \tilde{\in} G$.

Definition 2.22 ([3]). Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $||.|| : SE(\check{X}) \to R(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$ if $||.||$ satisfies the following conditions:

- $(N1)$. $||\tilde{x}|| \tilde{\geq} \bar{0}$, for all $\tilde{x} \tilde{\in} \check{X}$;
- $(N2)$. $||\tilde{x}|| = \bar{0}$ if and only if $\tilde{x} = \Theta$;
- $(N3)$. $||\tilde{\alpha}.\tilde{x}|| = |\tilde{\alpha}|||\tilde{x}||$ for all $\tilde{x} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$;
- $(N4)$. For all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $||\tilde{x} + \tilde{y}|| \tilde{\leq} ||\tilde{x}|| + ||\tilde{y}||$. The soft vector space $\check{X}$ with a soft norm $||.||$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, ||.||, A)$ or $(\check{X}, ||.||)$.

$(N1), (N2), (N3)$and $(N4)$ are said to be soft norm axioms.

Example 2.23. [[3]] Every parametrized family of crisp norms $||.||_\lambda : \lambda \in A$ on a crisp vector space X can be considered as a soft norm on the soft vector space $\check{X}$.

Definition 2.24 ([3])**.** Let $(\check{X}, ||.||, A)$ be a soft normed linear space and $\tilde{r}\tilde{>}\bar{0}$ be a soft real number. We define the followings;

$$B(\tilde{x}, \tilde{r}) = \left\{\tilde{y}\tilde{\in}\check{X} : ||\tilde{x} - \tilde{y}||\tilde{<}\tilde{r}\right\} \subset SE(\check{X})$$

$$\bar{B}(\tilde{x}, \tilde{r}) = \left\{\tilde{y}\tilde{\in}\check{X} : ||\tilde{x}\text{-}\tilde{y}||\tilde{\leq}\tilde{r}\right\} \subset SE(\check{X})$$

$$S(\tilde{x}, \tilde{r}) = \left\{\tilde{y}\tilde{\in}\check{X} : ||\tilde{x} - \tilde{y}|| = \tilde{r}\right\} \subset SE(\check{X})$$

$B(\tilde{x}, \tilde{r})$, $\bar{B}(\tilde{x}, \tilde{r})$ and $S(\tilde{x}, \tilde{r})$ are respectively called an open ball, a closed ball and a sphere with centre at $\tilde{x}$ and radius $\tilde{r}$. $SS(B(\tilde{x}, \tilde{r}))$, $SS(B(\tilde{x}, \tilde{r}))$ and $SS(S(\tilde{x}, \tilde{r}))$ are respectively called a soft open ball, a soft closed ball and a soft sphere with centre at $\tilde{x}$ and radius $\tilde{r}$.

Definition 2.25 ([3])**.** A sequence of soft elements $\{\tilde{x}_n\}$ in a soft normed linear space $(\check{X}, ||.||, A)$ is said to be convergent and converges to a soft element $\tilde{x}$ if $||\tilde{x}_n - \tilde{x}|| \to \bar{0}$ as $n \to \infty$. This means for every $\tilde{\epsilon}\tilde{>}\bar{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\tilde{\epsilon})$, such that $\bar{0}\tilde{\leq}||\tilde{x}_n - \tilde{x}||\tilde{<}\tilde{\epsilon}$, whenever $n > N$. i.e., $n > N \Rightarrow \tilde{x}_n \in B(\tilde{x}, \tilde{\epsilon})$. We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $lim_{n\to\infty}\tilde{x}_n = \tilde{x}$. $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

Definition 2.26 ([15])**.** Let τ be the collection of soft sets over X, then τ is said to be soft topology on X if
(1) ϕ, X belong to τ
(2) the union of any number of soft sets in τ belongs to τ
(3) the intersection of any two soft sets in τ belongs to τ.
The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.27 ([10])**.** Let τ be a family of soft sets over (U, E). Define $\tau(e) = \{F(e) : F \in \tau\}$ for $e \in E$. Then τ is said to be a topology of soft subsets over (U, E) if $\tau(e)$ is a crisp topology on $U \forall e \in E$. In this case, $((U, E), \tau)$ is said to be a topological space of soft subsets. If τ is a topology of soft subsets over (U, E), then the members of τ are called open soft sets and a soft set F over (U, E) is said to be closed soft set if $F^c \in \tau$.

3. Soft convergence and soft topology

In this section we discuss about a new type of convergence on a soft normed linear space and introduce soft topology generated by soft norm on a soft normed linear space and study some of its basic properties.

Lemma 3.1. *In a soft normed linear space $(\check{X}, ||.||)$, for any $\tilde{x}\tilde{\in}\check{X}$ and $\lambda \in A$, $||\tilde{x}||(\lambda) = 0$ if and only if $\tilde{x}(\lambda) = \theta$.*

Proof. Let us consider a soft scalar $\tilde{\alpha}$ such that $\tilde{\alpha}(\mu) = 1$ if $\mu = \lambda$, $\tilde{\alpha}(\mu) = 0$ if $\mu \neq \lambda$. Then $(\tilde{\alpha}\tilde{x})(\mu) = \theta$ for $\mu \neq \lambda$, $(\tilde{\alpha}\tilde{x})(\mu) = \tilde{x}(\lambda)$ for $\mu = \lambda$. From $N(3)$ we have $||\tilde{\alpha}\tilde{x}|| = |\tilde{\alpha}|\,||\tilde{x}||$. This shows that $||\tilde{x}||(\lambda) = 0$ iff $|\tilde{\alpha}|\,||\tilde{x}|| = \bar{0}$ iff $||\tilde{\alpha}\tilde{x}|| = \bar{0}$ iff $\tilde{\alpha}\tilde{x} = \bar{\theta}$ iff $\tilde{x}(\lambda) = \theta$. □

Lemma 3.2. *In a soft normed linear space* $(\check{X}, ||.||)$, *for each* $\xi \in X$ *and* $\lambda \in A$, $\{||\tilde{x}||(\lambda) : \tilde{x}(\lambda) = \xi\}$ *is a singleton set.*

Proof. In a soft normed linear space, by $N(4)$, we have, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ $||\tilde{x}|| \tilde{=} ||\tilde{x} - \tilde{y} + \tilde{y}|| \tilde{\leq} ||\tilde{x} - \tilde{y}|| + ||\tilde{y}|| \Rightarrow ||\tilde{x}|| - ||\tilde{y}|| \tilde{\leq} ||\tilde{x} - \tilde{y}||$. Similarly $||\tilde{y}|| - ||\tilde{x}|| \tilde{\leq} ||\tilde{x} - \tilde{y}||$. So $|\,||\tilde{x}|| - ||\tilde{y}||\,| \tilde{\leq} ||\tilde{x} - \tilde{y}||$. Now if $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \tilde{y}(\lambda)$ then $|\,||\tilde{x}||(\lambda) - ||\tilde{y}||(\lambda)\,| \tilde{\leq} ||\tilde{x} - \tilde{y}||(\lambda) = 0$(by Lemma 3.1) since $(\tilde{x} - \tilde{y})(\lambda) = \tilde{x}(\lambda) - \tilde{y}(\lambda) = \theta$. i.e. $||\tilde{x}||(\lambda) = ||\tilde{y}||(\lambda)$, which proves the lemma. □

Proposition 3.3. (Decomposition Theorem) *In a soft normed linear space* $(\check{X}, ||.||)$, *if we define for each* $\lambda \in A$, $||.||_\lambda : X \to \mathbb{R}^+$ *be a mapping such that for each* $\xi \in X$, $||\xi||_\lambda = ||\tilde{x}||(\lambda)$, *where* $\tilde{x} \tilde{\in} \check{X}$ *is such that* $\tilde{x}(\lambda) = \xi$. *Then for each* $\lambda \in A$, $(X, ||.||_\lambda)$ *is a normed linear space.*

Proof. Since for $\lambda \in A$, $\{||\tilde{x}||(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set, the mapping $||.||_\lambda : X \to \mathbb{R}^+$ is well defined. Hence from soft norm axioms, it follows that $(X, ||.||_\lambda)$ is a normed linear space $\lambda \in A$. □

Definition 3.4. In a soft normed linear space a sequence $\tilde{x}_n$ of soft elements is said to be soft convergent and soft converges to a soft element $\tilde{x}$, if for any soft real number $\tilde{\epsilon} \tilde{>} \bar{0}$ there exists a soft natural number $\tilde{N}$ such that $||\tilde{x}_n - \tilde{x}||(\lambda) < \tilde{\epsilon}(\lambda)$ $\forall$ $n \geq \tilde{N}(\lambda)$, $\forall$ $\lambda \in A$ and is denoted by $lim_{n\to\infty}\tilde{x}_n = \tilde{x}$ or $\tilde{x}_n \hookrightarrow \tilde{x}$, where $\tilde{x}$ is called the soft limit of the sequence $\tilde{x}_n$.

Proposition 3.5. *Soft limit of a sequence of soft elements in a soft normed linear space is unique.*

Proof. Let $\tilde{x}_n$ is sequence of soft elements in a soft normed linear space $(\check{X}, ||.||)$ such that $lim_{n\to\infty}\tilde{x}_n \tilde{=} \tilde{x}$ and $lim_{n\to\infty}\tilde{x}_n = \tilde{y}$. Then for any soft real number $\tilde{\epsilon} \tilde{>} \bar{0}$ there exist soft natural numbers $\tilde{N}_1$ and $\tilde{N}_2$ such that $||\tilde{x}_n - \tilde{x}||(\lambda) < \frac{\tilde{\epsilon}(\lambda)}{2}$ $\forall$ $n \geq \tilde{N}_1(\lambda)$, $\forall$ $\lambda \in A$ and $||\tilde{x}_n - \tilde{y}||(\lambda) < \frac{\tilde{\epsilon}(\lambda)}{2}$ $\forall$ $n \geq \tilde{N}_2(\lambda)$, $\forall$ $\lambda \in A$. i.e.$||\tilde{x}_n - \tilde{x}||(\lambda) < \frac{\tilde{\epsilon}(\lambda)}{2}$ and $||\tilde{x}_n - \tilde{y}|||(\lambda) < \frac{\tilde{\epsilon}(\lambda)}{2}$ $\forall$ $n \geq \tilde{N}(\lambda)$, ($\tilde{N}$= $max\left\{\tilde{N}_1, \tilde{N}_2\right\}$, where maximum of these soft natural numbers taken as component wise) $\forall$ $\lambda \in A$. Now for $\forall$ $n \geq \tilde{N}(\lambda)$, $||\tilde{x} - \tilde{y}||(\lambda) < ||\tilde{x}_n - \tilde{x}||(\lambda) + ||\tilde{x}_n - \tilde{y}||(\lambda) < \tilde{\epsilon}(\lambda) \forall$ $\lambda \in A$, which shows that $\tilde{x} = \tilde{y}$. □

Proposition 3.6. *A sequence* $\tilde{x}_n$ *of soft elements in a soft normed linear space* $(\check{X}, ||.||)$ *is soft convergent to* $\tilde{x}$ *iff* $\tilde{x}_n(\lambda)$ *is convergent to* $\tilde{x}(\lambda)$ *in* $(X, ||.||_\lambda)$ $\forall \lambda \in A$, *where* $||.||_\lambda$ *defined as in Proposition 3.3.*

Proof. Let $\tilde{x}_n$ be sequence soft converging to the soft element $\tilde{x}$ in $(\check{X}, ||.||)$. Take $\epsilon > 0$, then since $\tilde{x}_n \hookrightarrow \tilde{x}$, so there exists a soft natural number $\tilde{N}$ such that $||\tilde{x}_n - \tilde{x}||(\lambda) < \bar{\epsilon}(\lambda) = \epsilon$ $\forall$ $n \geq \tilde{N}(\lambda)$, $\forall$ $\lambda \in A$. But $||\tilde{x}_n - \tilde{x}||(\lambda) = ||\tilde{x}_n(\lambda) - \tilde{x}(\lambda)||_\lambda$, which shows that $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$ $\forall \lambda \in A$.

Conversely, $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)\ \forall \lambda \in A$. Take $\tilde{\epsilon}\tilde{>}\bar{0}$, since $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)\ \forall\lambda \in A$, so for each $\lambda \in A\ \exists N_\lambda$, $||\tilde{x}_n - \tilde{x}||(\lambda) = ||\tilde{x}_n(\lambda) - \tilde{x}(\lambda)||_\lambda < \tilde{\epsilon}(\lambda)\ \forall n > N_\lambda$. Now if we define $\tilde{N}(\lambda) = N_\lambda\ \forall\lambda \in A$ then $||\tilde{x}_n - \tilde{x}||(\lambda) < \tilde{\epsilon}(\lambda)\ \forall\ n \geq \tilde{N}(\lambda),\ \forall\ \lambda \in A$. This proves the proposition. □

Definition 3.7. A sequence $\tilde{x}_n$ in a soft normed linear space is said to be soft Cauchy if for any soft real number $\tilde{\epsilon}\tilde{>}\bar{0}$ there exists a soft natural number $\tilde{N}$ such that $||\tilde{x}_n - \tilde{x}_m||(\lambda) < \tilde{\epsilon}(\lambda)\ \forall\ n, m \geq \tilde{N}(\lambda)\ ,\ \forall\ \lambda \in A$.

Proposition 3.8. *A sequence $\tilde{x}_n$ in a soft normed linear space $\left(\check{X}, ||.||\right)$ is soft Cauchy iff $\tilde{x}_n(\lambda)$ is Cauchy in $(X, ||.||_\lambda)\ \forall\lambda \in A$, where $||.||_\lambda$ is defined as in Proposition 3.3.*

Proof. Proof is same as in Proposition 3.6 □

Proposition 3.9. *Every soft convergent sequence of soft elements is soft Cauchy.*

Proof. Let $\tilde{x}_n \hookrightarrow \tilde{x}$, then the relation $||\tilde{x}_n - \tilde{x}_m||(\lambda) \leq ||\tilde{x}_n - \tilde{x}||(\lambda) + ||\tilde{x}_m - \tilde{x}||(\lambda)\forall$ $\lambda \in A$ gives the result. □

Definition 3.10. A sequence $\tilde{x}_n$ of soft elements in a soft normed linear space $\left(\check{X}, ||.||\right)$ is said to be bounded if there exists a soft real number $\tilde{M}$ such that $||\tilde{x}_n||\tilde{\leq}\tilde{M}$, $\forall n \in \mathbb{N}$ (The set of all natural numbers).

Proposition 3.11. *Every soft Cauchy sequence $\tilde{x}_n$ of soft elements in a soft normed linear space $\left(\check{X}, ||.||\right)$ is bounded.*

Proof. Let $\tilde{x}_n$ be a soft Cauchy sequence in $\left(\check{X}, ||.||\right)$. Then there exists a soft real number $\tilde{N}$ such that $||\tilde{x}_n - \tilde{x}_m||(\lambda) < \bar{1}(\lambda) = 1\ \forall\ n, m \geq \tilde{N}(\lambda),\ \forall\ \lambda \in A$. $||\tilde{x}_n||(\lambda) \leq ||\tilde{x}_n - \tilde{x}_{\tilde{N}(\lambda)}||(\lambda) + ||\tilde{x}_{\tilde{N}(\lambda)}||(\lambda) < 1 + ||\tilde{x}_{\tilde{N}(\lambda)}||(\lambda)\ \forall\ n \geq N(\lambda),\ \forall\ \lambda \in A$.
Now if we take
$\tilde{M}(\lambda) = max\left\{||\tilde{x}_1||(\lambda), ||\tilde{x}_2||(\lambda), ||\tilde{x}_3||(\lambda), ...||\tilde{x}_{\tilde{N}(\lambda)-1}||(\lambda),\ 1 + ||\tilde{x}_{\tilde{N}(\lambda)}||(\lambda)\right\}$, then clearly $||\tilde{x}_n||(\lambda) < \tilde{M}(\lambda)\ \forall n \in \mathbb{N}\ \forall\ \lambda \in A$. i.e. $||\tilde{x}_n||\tilde{\leq}\tilde{M}\ \forall n \in \mathbb{N}$. □

Corollary 3.12. *Every soft convergent sequence $\tilde{x}_n$ of soft elements in a soft normed linear space $\left(\check{X}, ||.||\right)$ is bounded.*

Definition 3.13. A soft normed linear space $\left(\check{X}, ||.||\right)$ is said to be soft complete if every soft Cauchy sequence in $\left(\check{X}, ||.||\right)$ is soft convergent in $\left(\check{X}, ||.||\right)$.

Proposition 3.14. *A soft normed linear space $\left(\check{X}, ||.||\right)$ is soft complete iff $(X, ||.||_\lambda)$ is complete $\forall\lambda \in A$, where $||.||_\lambda$ defined as in Proposition 3.3.*

Proof. Let $\left(\check{X}, ||.||\right)$ is soft complete and $\lambda \in A$. Consider $(X, ||.||_\lambda)$. Let $\{x_n\}$ be a Cauchy sequence in $(X, ||.||_\lambda)$. Now if we construct a sequence of soft elements $\{\tilde{x}_n\}$ such that

$$\tilde{x}_n(\mu) = \begin{cases} x_n & \text{if } \mu = \lambda \\ \theta & \text{if } \mu \neq \lambda \end{cases} \quad .$$

Then clearly $\{\tilde{x}_n\}$ is soft Cauchy. So by soft completeness of $\left(\check{X}, ||.||\right)$, $\{\tilde{x}_n\}$ is convergent $\Rightarrow \{x_n\}$ is convergent. i.e. $(X, ||.||_\lambda)$ is complete.

Converse of the proof directly follows from Proposition 3.3, Proposition 3.6 and Proposition **3.8**. □

Corollary 3.15. *The soft set $R(A)$ over $\mathbb{R}$(set of real numbers) is soft complete.*

Proposition 3.16. *In a soft normed linear space $\left(\check{X}, ||.||\right)$, $SS(B(\tilde{x},\tilde{r}))\,(\lambda) = S\left(\tilde{x}\left(\lambda\right), \tilde{r}\left(\lambda\right)\right) \forall\lambda \in A$, where $B(\tilde{x},\tilde{r}) = \left\{\tilde{y}\tilde{\in}\check{X} : ||\tilde{x}-\tilde{y}||\tilde{<}\tilde{r}\right\} \subset SE(\check{X})$ and $S(\tilde{x}\left(\lambda\right), \tilde{r}\left(\lambda\right)) = \{z \in X : ||\tilde{x}\left(\lambda\right) - z||_\lambda < \tilde{r}\left(\lambda\right)\}$.*

Proof. Let $\lambda \in A$ and $z \in SS(B(\tilde{x},\tilde{r}))\,(\lambda)$. Then there exists a soft element $\tilde{y}$ such that $||\tilde{x}-\tilde{y}||\tilde{<}\tilde{r}$ and $\tilde{y}\left(\lambda\right) = z$. So $||\tilde{x}-\tilde{y}||\left(\lambda\right) < \tilde{r}\left(\lambda\right) \Rightarrow ||\tilde{x}\left(\lambda\right) - \tilde{y}\left(\lambda\right)||_\lambda < \tilde{r}\left(\lambda\right)$. i.e. $||\tilde{x}\left(\lambda\right) - z||_\lambda < \tilde{r}\left(\lambda\right) \Rightarrow z \in S\left(\tilde{x}\left(\lambda\right), \tilde{r}\left(\lambda\right)\right)$.
Now let $z \in S\left(\tilde{x}\left(\lambda\right), \tilde{r}\left(\lambda\right)\right)$. Then if we take a soft element $\tilde{z}$ such that $\tilde{z}\left(\mu\right) = z$ when $\mu = \lambda$, $\tilde{z}\left(\mu\right) = \tilde{x}\left(\mu\right)$ when $\mu \neq \lambda$. Then clearly $\tilde{z}\tilde{\in}B(\tilde{x},\tilde{r})$. Hence $z \in SS(B(\tilde{x},\tilde{r}))\,(\lambda)$. □

Corollary 3.17. *If $S\left(x_\lambda, r_\lambda\right)$ are open balls in $(X, ||.||_\lambda)$ $\forall\lambda \in A$. Then the soft set U such that $U\left(\lambda\right) = S\left(x_\lambda, r_\lambda\right)$ $\forall\lambda \in A$ is the soft open ball in $\left(\check{X}, ||.||\right)$ with centre $\tilde{x}$ and radius $\tilde{r}$, where $\tilde{x}\left(\lambda\right) = x_\lambda$ and $\tilde{r}\left(\lambda\right) = r_\lambda$,$\forall\lambda \in A$.*

Proof. Consider the soft element $\tilde{x}$ and soft real number $\tilde{r}$ such that $\tilde{x}\left(\lambda\right) = x_\lambda$ and $\tilde{r}\left(\lambda\right) = r_\lambda$,$\forall\lambda \in A$. Now if we consider the open ball $B(\tilde{x},\tilde{r})$. Then clearly, by the previous proposition, $SS(B(\tilde{x},\tilde{r})) = U$. □

Proposition 3.18. *Let in a soft normed linear space $\left(\check{X}, ||.||\right)$, τ be the set of all soft sets in X such that $\tilde{U}\tilde{\in}\tau$ iff $\tilde{U}$ can be expressed as a union of finite intersections of soft open balls of $\left(\check{X}, ||.||\right)$. Then τ forms a soft topology [15] on X.*

Proof. The proof is straightforward. □

All the members of τ are said to be the soft open in $\left(\check{X}, ||.||\right)$. A soft set $\tilde{F}$ is said to be the soft closed in $\left(\check{X}, ||.||\right)$ if $\tilde{F}^c \in \tau$. The topology defined as in Proposition 3.18 is called the topology generated by the soft norm $||.||$ on $\check{X}$.

Proposition 3.19. *For any $\alpha \in A$ the collection $\tau_\alpha = \left\{\tilde{U}\left(\alpha\right) : \tilde{U}\tilde{\in}\tau\right\}$ is a topology in X. i.e. τ is a topology of soft sets on X.*

Proof. Proof directly follows from the Definition of soft topology τ and Proposition 3.16. □

Corollary 3.20. *Let U be soft set in a soft normed linear space $\left(\check{X}, ||.||\right)$. If U is soft open in $\left(\check{X}, ||.||\right)$ then $U\left(\lambda\right)$ is open in $(X, ||.||_\lambda)$ $\forall\lambda \in A$. Further, if $X \neq \{\theta\}$, then the converse is also true.*

Proof. Let U be soft open in $(X, ||.||)$. Then $U = \cup_{i\in\Delta}\cap_{j=1}^{n} G_{i,j}$, where n is a positive integer and $G_{i,j}$ is a soft open ball, $i \in \Delta$. Now $U\left(\lambda\right) = \cup_{i\in\Delta}\cap_{j=1}^{n} G_{i,j}\left(\lambda\right)$ and $G_{i,j}\left(\lambda\right)$ is a open ball in $(X, ||.||_\lambda)$, so $U\left(\lambda\right)$ is an open ball in $(X, ||.||_\lambda)$ $\forall\lambda \in A$. For the converse part consider the following Cases:

- **Case-1:** Let $U \in \mathcal{S}\left(X\right)$**.** i.e. $U\left(\lambda\right) \neq \phi$ $\forall\lambda \in A$. Now since for each $\lambda \in A$, $U\left(\lambda\right)$ is open in $(X, ||.||_\lambda)$. So $U\left(\lambda\right)$ can be expressed as a union of open balls in $(X, ||.||_\lambda)$, $\forall\lambda \in A$. Choose for each $\lambda \in A$ one such open ball in

$\left(\check{X}, ||.||\right)$ and thereby construct a soft set. Then this soft sets are soft open balls in $\left(\check{X}, ||.||\right)$ and their union is the soft set U. Hence U is soft open in $\left(\check{X}, ||.||\right)$.

- **Case-2:** If $U \notin \mathcal{S}(X)$. Let $A_1 = \{\lambda \in A; U(\lambda) = \phi\}$. Since $X \neq \{\theta\}$, $\exists x (\neq \theta) \in X$. Take two disjoint balls $B(x, r)$ and $B(\theta, r)$ in X. Now construct the soft sets U_1 and U_2 as in Case-1 by taking $U_1(\lambda) = B(x, r)$ if $\lambda \in A_1$, $U_1(\lambda) = U(\lambda)$ otherwise and $U_2(\lambda) = B(\theta, r)$ if $\lambda \in A_1$, $U_2(\lambda) = U(\lambda)$ otherwise. Then U_1 and U_2 are soft open by Case-1. Hence $U = U_1 \cap U_2$ is soft open in $\left(\check{X}, ||.||\right)$.

□

Definition 3.21. In a soft normed linear space $\left(\check{X}, ||.||\right)$, a soft element $\tilde{x}$ is said to be an interior point of a soft set U if there exists a open ball $B(\tilde{x}, \tilde{r})$ containing $\tilde{x}$ such that $SS(B(\tilde{x}, \tilde{r})) \tilde{\subseteq} U$.

Proposition 3.22. *In a soft normed linear space $\left(\check{X}, ||.||\right)$ a soft set $U \in \mathcal{S}(X)$ is soft open in $\left(\check{X}, ||.||\right)$ iff any soft element $\tilde{x} \tilde{\in} U$ is an interior point of U.*

Proof. Let $U (\in \mathcal{S}(X))$ be soft open in $\left(\check{X}, ||.||\right)$ and $\tilde{x} \tilde{\in} U$. Then $U(\lambda)$ is open in $(X, ||.||_\lambda)$ $\forall \lambda \in A$(by first part of Corollary 3.20). Now $(X, ||.||_\lambda)$ is a normed linear space so $U(\lambda)$ can be expressed as a union of open balls in $(X, ||.||_\lambda)$ $\forall \lambda \in A$, where at least one of the balls contains the point $\tilde{x}(\lambda)$ $\forall \lambda \in A$, since $\tilde{x} \tilde{\in} U$. If we take the soft set whose λ components are these open balls containing $\tilde{x}(\lambda)$ in $(X, ||.||_\lambda)$ $\forall \lambda \in A$, by Proposition 3.16, this soft set will be soft open ball in $\left(\check{X}, ||.||\right)$ containing $\tilde{x}$ and contained in U, which proves that $\tilde{x}$ is an interior point of U.
Conversely, let any soft element $\tilde{x} \tilde{\in} U$ be an interior point of U. Then for each $\tilde{x} \tilde{\in} U$ there exists a open ball $B(\tilde{x}, \tilde{r})$ such that $SS(B(\tilde{x}, \tilde{r})) \tilde{\subseteq} U$. Now if we take all soft elements of U then $U = SS\left(\cup_{\tilde{x} \tilde{\in} U} \{\tilde{x}\}\right) \tilde{\subseteq} \cup_{\tilde{x} \tilde{\in} U} \{SS(B(\tilde{x}, \tilde{r}))\} \tilde{\subseteq} U$. i.e. U is a soft open set. □

Proposition 3.23. *Let $\left(\check{X}, ||.||\right)$ be a soft normed linear space and $\tilde{x}_n$ be any sequence in a soft closed set F. If $\tilde{x}_n \hookrightarrow \tilde{x}$ then $\tilde{x} \tilde{\in} F$.*

Proof. Let $\tilde{x} \tilde{\notin} F$, then $\tilde{x}(\lambda) \notin F(\lambda)$ for some $\lambda \in A \Rightarrow \tilde{x}(\lambda) \in X - F(\lambda)$, where $X - F(\lambda)$ is open in $(X, ||.||_\lambda)$. Now since $\tilde{x}_n \hookrightarrow \tilde{x} \Rightarrow \tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$, so the sequence $\tilde{x}_n(\lambda)$ is eventually in $X - F(\lambda)$, which contradicts that the sequence $\tilde{x}_n$ is in F. Hence $\tilde{x} \tilde{\in} F$. □

4. Soft Banach algebra and its properties

Definition 4.1. Let V be an algebra over a field $\mathbb{C}$ of complex numbers and let A be the parameter set and (G, A) be a soft set over V. Now (G, A) is said to be a soft algebra of V over C if $G(\lambda)$ is a sub algebra of V $\forall \lambda \in A$.
It is very easy to see that in a soft algebra the soft elements satisfy the properties:

$$(\tilde{x}\tilde{y})\tilde{z} = \tilde{x}(\tilde{y}\tilde{z})$$

$$\tilde{x}(\tilde{y} + \tilde{z}) = \tilde{x}\tilde{y} + \tilde{x}\tilde{z}, (\tilde{x} + \tilde{y})\tilde{z} = \tilde{x}\tilde{z} + \tilde{y}\tilde{z}$$

$$\tilde{\alpha}(\tilde{x}\tilde{y}) = (\tilde{\alpha}\tilde{x})\tilde{y} = \tilde{x}(\tilde{\alpha}\tilde{y})$$

where for all $\tilde{x}, \tilde{y}, \tilde{z}\tilde{\in}\check{G}$ and for any soft scalar $\tilde{\alpha}$, $\tilde{x}\tilde{y}(\lambda) = \tilde{x}(\lambda)\tilde{y}(\lambda)$ and $\tilde{\alpha}\tilde{x}(\lambda) = \tilde{\alpha}(\lambda)\tilde{x}(\lambda)$. If $\left(\check{G}, A\right)$ is also a soft Banach space with respect to a soft norm that satisfies the inequality $||\tilde{x}\tilde{y}||\tilde{\leq}||\tilde{x}||||\tilde{y}||$ and if $\left(\check{G}, A\right)$ contains an identity $\bar{e}$ such that $\tilde{x}\bar{e}=\bar{e}\tilde{x}=\tilde{x}$ with $||\bar{e}|| = \bar{1}$, then (G, A) is called a soft Banach algebra. In addition, if in a soft Banach algebra (G, A), $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$, $\forall\tilde{x}, \tilde{y}\tilde{\in}\check{G}$ then (G, A) is called a commutative soft Banach algebra.

Proposition 4.2. *(G, A) is a soft Banach algebra iff $G(\lambda)$ is a Banach algebra $\forall\lambda \in A$.*

Proof. Proof follows from the definition of soft algebra and Proposition 3.14. □

Proposition 4.3. *In a soft Banach algebra if $\tilde{x}_n \hookrightarrow \tilde{x}$ and $\tilde{y_n} \hookrightarrow \tilde{y}$ then $\tilde{x}_n\tilde{y}_n \hookrightarrow \tilde{x}\tilde{y}$. i.e. multiplication in a soft Banach algebra is continuous.*

Proof. Since $\tilde{x}_n \hookrightarrow \tilde{x}$ and $\tilde{y_n} \hookrightarrow \tilde{y}$ in (G, A). So $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$ and $\tilde{y}_n(\lambda) \to \tilde{y}(\lambda)$ $\forall\lambda \in A$ in $(G(\lambda), ||.||_\lambda)$. Now since $G(\lambda)$ is Banach algebra $\forall\lambda \in A$ (by Proposition 4.2) and in Banach algebra multiplication is continuous so, $\tilde{x}_n(\lambda)\tilde{y}_n(\lambda) \to \tilde{x}(\lambda)\tilde{y}(\lambda)\forall\lambda \in A$, which proves that $\tilde{x}_n\tilde{y}_n \hookrightarrow \tilde{x}\tilde{y}$ (by Proposition 3.6). □

Proposition 4.4. *Every parametrized family of crisp Banach algebras on a crisp vector space V can be considered as a soft Banach algebra on the soft vector space $\check{V}$.*

Proof. Let $||.||_\lambda : \lambda \in A$ be a family of crisp norms on the vector space V such that $(V, ||.||_\lambda)$ are Banach algebra $\forall\lambda \in A$. Now let us define $||.|| : \check{V} \to R(A)^*$ by $||\tilde{x}||(\lambda) = ||\tilde{x}(\lambda)||_\lambda$, $\forall\lambda \in A, \forall\tilde{x}\tilde{\in}\check{V}$. Then by Example 2.23 $(\check{V}, ||.||)$ is a soft normed linear space. Now to show that $(\check{V}, ||.||)$ is a soft Banach algebra we have to show that $||\tilde{x}\tilde{y}||\tilde{\leq}||\tilde{x}||||\tilde{y}||$ $\forall$ $\tilde{x},\tilde{y}\tilde{\in}\check{V}$ and $(\check{V}, ||.||)$ is complete.
Now $||\tilde{x}\tilde{y}||(\lambda) = ||\tilde{x}(\lambda)\tilde{y}(\lambda)||_\lambda \leq ||\tilde{x}(\lambda)||_\lambda||\tilde{y}(\lambda)||_\lambda \leq ||\tilde{x}||(\lambda)||\tilde{y}||(\lambda)$ $\forall\lambda \in A$, which shows that $||\tilde{x}\tilde{y}||\tilde{\leq}||\tilde{x}||||\tilde{y}||$.
Now let $\tilde{x}_n$ be a Cauchy sequence in $\check{V}$. Then for any $\tilde{\epsilon}\tilde{>}\bar{0}$ there exists a soft natural number $\tilde{N}$ such that $||\tilde{x}_{n+p} - \tilde{x}_n||(\lambda)\tilde{<}\frac{\tilde{\epsilon}}{2}(\lambda)$ $\forall n \geq \tilde{N}(\lambda)\forall\lambda \in A \Rightarrow ||\tilde{x}_{n+p}(\lambda) - \tilde{x}_n(\lambda)||_\lambda\tilde{<}\frac{\tilde{\epsilon}}{2}(\lambda)$ $\forall n \geq \tilde{N}(\lambda)$, $\forall\lambda \in A$. i.e. $\tilde{x}_n(\lambda)$ is a Cauchy sequence in $(V, ||.||_\lambda)$ $\forall\lambda \in A$. Since $(V, ||.||_\lambda)$ are Banach algebra $\forall\lambda \in A$, so there exist x_λ such that $\tilde{x}_n(\lambda)$ converge to x_λ, $\forall\lambda \in A$. Hence there must exist some $N_\lambda(> N(\lambda))$ such that $||\tilde{x}_n(\lambda) - x_\lambda||_\lambda\tilde{<}\frac{\tilde{\epsilon}}{2}(\lambda)$ $\forall n \geq N_\lambda$, $\forall\lambda \in A$. Now $||\tilde{x}_n - \tilde{x}||(\lambda) = ||\tilde{x}_n(\lambda) - x_\lambda||_\lambda\tilde{<}||\tilde{x}_n(\lambda) - \tilde{x}_{N_\lambda}(\lambda)||_\lambda + ||\tilde{x}_{N_\lambda} - x_\lambda(\lambda)||_\lambda\tilde{<}\tilde{\epsilon}(\lambda)$ $\forall n > N(\lambda)$, $\forall\lambda \in A$, where $\tilde{x}(\lambda) = x_\lambda$. This shows that $(V, ||.||)$ is a soft Banach space. Hence $(V, ||.||)$ is a soft Banach algebra. □

Definition 4.5. A soft element $\tilde{x}\tilde{\in}\check{G}$ is said to be invertible if it has an inverse in $\check{G}$ i.e. if there exists a soft element $\tilde{y}\tilde{\in}\check{G}$ such that $\tilde{x}\tilde{y} = \tilde{y}\tilde{x} = \bar{e}$ and then $\tilde{y}$ is called the inverse of $\tilde{x}$, denoted by $\tilde{x}^{-1}$. Otherwise $\tilde{x}$ is said to be non-invertible soft element of $\check{G}$.

Remark 4.6. Clearly $\bar{e}$ is invertible. If $\tilde{x}$ is invertible, then we can verify that the inverse is unique. because if $\tilde{y}\tilde{x} = \bar{e} = \tilde{x}.\tilde{z}$ Then $\tilde{y} = \tilde{y}\bar{e} = \tilde{y}(\tilde{x}\tilde{z}) = (\tilde{y}\tilde{x})\tilde{z} = \bar{e}\tilde{z} = \tilde{z}$. Further, if $\tilde{x}$ and $\tilde{y}$ are both invertible then $\tilde{x}\tilde{y}$ is invertible and $(\tilde{x}\tilde{y})^{-1} = \tilde{y}^{-1}\tilde{x}^{-1}$. For $(\tilde{x}\tilde{y})(\tilde{y}^{-1}\tilde{x}^{-1}) = \tilde{x}(\tilde{y}\tilde{y}^{-1})\tilde{x}^{-1} = \tilde{x}\bar{e}\tilde{x}^{-1} = \bar{e}$ and similarly $(\tilde{y}^{-1}\tilde{x}^{-1})(\tilde{x}\tilde{y})= \bar{e}$.

Definition 4.7 ([1])**.** Let $(G, *)$ be a group and (F, A) be a soft set over G. Then (F, A) is said to be a soft group over G if and only if $F(\lambda)$ is a subgroup of $(G, *)$ for all $\lambda \in A$.

Proposition 4.8. *Let $(G, *)$ be a group and (F, A) be a soft set over G. If for any $\tilde{x}$, $\tilde{y}$ $\tilde{\in}(F, A)$*

(1) $\tilde{x}\tilde{*}\tilde{y}\tilde{\in}(F, A)$
(2) $\tilde{x}^{-1}\tilde{\in}(F, A)$,

where $\tilde{x}\tilde{*}\tilde{y}\,(\lambda) = \tilde{x}\,(\lambda) * \tilde{y}\,(\lambda)$ and $\tilde{x}^{-1}\,(\lambda) = (\tilde{x}\,(\lambda))^{-1}$.
Then (F, A) is a soft group over G.

Proof. Proof is obvious. □

Note 4.9. This shows that in a soft algebra, the soft set generated by the all invertible elements is a soft group with respect to the composition defined as in Proposition **??**.

Definition 4.10. A series $\sum\limits_{n=1}^{\infty}\tilde{x}_n$ of soft elements is said to be soft convergent if the partial sum of the series $\tilde{s}_k = \sum\limits_{n=1}^{k}\tilde{x}_n$ is soft convergent.

Proposition 4.11. *Let (G, A) be a soft Banach algebra. If $\tilde{x}\tilde{\in}\check{G}$ satisfies $||\tilde{x}||\tilde{<}\bar{1}$, then $(\bar{e} - \tilde{x})$ is is invertible and $(\bar{e} - \tilde{x})^{-1} = \bar{e} + \sum\limits_{n=1}^{\infty}\tilde{x}^n$.*

Proof. Since (G, A) is soft algebra, so we have $||\tilde{x}^j||\tilde{\leq}||\tilde{x}||^j$ for any positive integer j, so that the infinite series $\sum\limits_{n=1}^{\infty}||\tilde{x}||^n$ is soft convergent because $||\tilde{x}||\tilde{<}\bar{1}$. So the sequence of partial sum $\tilde{s}_k = \sum\limits_{n=1}^{k}\tilde{x}_n$ is a soft Cauchy sequence since $||\sum\limits_{n=k}^{k+p}\tilde{x}^n|| \;\tilde{<}\; \sum\limits_{n=k}^{k+p}||\tilde{x}||^n$. Since (G, A) is soft complete so $\sum\limits_{n=1}^{\infty}\tilde{x}^n$ is soft convergent. Now let $\tilde{s} = \bar{e} + \sum\limits_{n=1}^{\infty}\tilde{x}^n$. Now it is only we have to show that $\tilde{s} = (\bar{e} - \tilde{x})^{-1}$.

We have

$$(\bar{e} - \tilde{x})(\bar{e} + \tilde{x} + \tilde{x}^2 + ...\tilde{x}^n) = (\bar{e} + \tilde{x} + \tilde{x}^2 + ...\tilde{x}^n)(\tilde{e} - \tilde{x}) = \bar{e} - \tilde{x}^{n+1} \tag{4.1}$$

Now again since $||\tilde{x}||\tilde{<}\bar{1}$ so $\tilde{x}^{n+1} \hookrightarrow \bar{\theta}$ as $n\to \infty$. Therefore letting $n\to \infty$ in and remembering that multiplication in G is continuous we get,

$$(\bar{e} - \tilde{x})\tilde{s} = \tilde{s}(\bar{e} - \tilde{x}) = \tilde{e}$$

. So that $\tilde{s} = (\bar{e} - \tilde{x})^{-1}$. This proves the proposition. □

Corollary 4.12. *Let G be a soft Banach algebra. If $\tilde{x} \in G$ and $||\bar{e} - \tilde{x}|| < \bar{1}$, then $\tilde{x}^{-1}$ exists and $\tilde{x}^{-1} = \bar{e} + \sum_{j=1}^{\infty}(\bar{e} - \tilde{x})^j$.*

Corollary 4.13. *Let G be a soft Banach algebra. Let $\tilde{x} \in G$ and $\tilde{\mu}$ be a soft scalar such that $|\tilde{\mu}| \tilde{>} ||\tilde{x}||$. Then $(\tilde{\mu}\bar{e} - \tilde{x})^{-1}$ exists and $(\tilde{\mu}\bar{e} - \tilde{x})^{-1} = \sum_{n=1}^{\infty} \tilde{\mu}^{-n}\tilde{x}^{n-1}(\tilde{x}^0 = \tilde{e})$.*

Proof. $\tilde{y} \in G$ be such that $\tilde{y}^{-1}$exists in G and $\tilde{\alpha}$ be a soft scalar such that $\tilde{\alpha}(\lambda) \neq 0$, $\forall \lambda \in A$. Then it is clear that

$$(\tilde{\alpha}\tilde{y})^{-1} = \tilde{\alpha}^{-1}\tilde{y}^{-1}.$$

Having noted this we can write

$$\tilde{\mu}\bar{e} - \tilde{x} = \tilde{\mu}(\bar{e} - \tilde{\mu}^{-1}\tilde{x})$$

and now we show that $(\bar{e} - \tilde{\mu}^{-1}\tilde{x})^{-1}$ exists. We have $||\bar{e} - (\bar{e} - \tilde{\mu}^{-1}\tilde{x})|| = ||\tilde{\mu}^{-1}\tilde{x}|| = |\tilde{\mu}|^{-1}||\tilde{x}|| \,\tilde{<}\bar{1}$ by hypothesis. So, By Corollary 4.12 $(\bar{e} - \tilde{\mu}^{-1}\tilde{x})^{-1}$ exists and hence $(\tilde{\mu}\bar{e} - \tilde{x})^{-1}$ exists. For the infinite series representation, using the Proposition 4.11 we have

$$(\tilde{\mu}\bar{e} - \tilde{x})^{-1} = \tilde{\mu}^{-1}(\bar{e} - \mu^{-1}\tilde{x})^{-1}$$

$$= \tilde{\mu}^{-1}(\bar{e} + \sum_{n=1}^{\infty}[\bar{e} - (\bar{e} - \tilde{\mu}^{-1}\tilde{x})]^n$$

$$= \tilde{\mu}^{-1}(\bar{e} + \sum_{n=1}^{\infty}(\tilde{\mu}^{-1}\tilde{x})^n]$$

$$= \sum_{n=1}^{\infty} \tilde{\mu}^{-n}\tilde{x}^{n-1}.$$

This proves the corollary. □

Proposition 4.14. *Let G be a soft Banach algebra. The soft set S generated by the set of all invertible soft elements of G is a soft open subset in G.*

Proof. $\tilde{x}_0 \tilde{\in} S$. We have to show that $\tilde{x}_0$ is a soft interior point of G. Consider the open sphere $S(\tilde{x}_0, \frac{1}{||\tilde{x}_0^{-1}||})$ with centre at $\tilde{x}_0$ and radius $\frac{1}{||\tilde{x}_0^{-1}||}$. Every soft element $\tilde{x}$ of this sphere satisfies the inequality

$$||\tilde{x}_0 - \tilde{x}|| \tilde{<} \frac{1}{||\tilde{x}_0^{-1}||} \tag{4.2}$$

Let $\tilde{y} = \tilde{x}_0^{-1}\tilde{x}$ and $\tilde{z} = \bar{e} - \tilde{y}$ then we have $||\tilde{z}|| = ||\tilde{y} - \bar{e}|| = ||\tilde{x}_0^{-1}\tilde{x} - \tilde{x}_0^{-1}\tilde{x}_0|| \tilde{\leq} ||\tilde{x}_0^{-1}||||\tilde{x} - \tilde{x}_0|| \tilde{<} \bar{1}$. So by Proposition 4.11, $\bar{e} - \tilde{z}$ is invertible i.e. $\tilde{y}$ is invertible. Hence $\tilde{y} \tilde{\in} S$. Now $\tilde{x}_0 \tilde{\in} S$, $\tilde{y} \tilde{\in} S$ and so by Remark 4.6, $\tilde{x}_0\tilde{y} \in S$. But

$$\tilde{x}_0\tilde{y} = \tilde{x}_0\tilde{x}_0^{-1}\tilde{x} = \tilde{x}$$

So any $\tilde{x}$ satisfying the inequality (4.2) belongs to S. This shows that S is a soft open subset of G. □

Corollary 4.15. *The soft set $P(= S^c)$ of G is soft closed subset of G.*

Definition 4.16. A mapping T from a soft normed linear space G onto G is said to be continuous if for any sequence $\tilde{x}_n$, $\tilde{x}_n \hookrightarrow \tilde{x}$ implies $T(\tilde{x}_n) \hookrightarrow T(\tilde{x})$.

Proposition 4.17. *In a soft Banach algebra G, the mapping $\tilde{x} \hookrightarrow \tilde{x}^{-1}$ of S onto S is continuous.*

Proof. Let $\tilde{x}_0 \in S$ and let $\{\tilde{x}_n\}$ be a sequence of soft elements in S such that $\tilde{x}_n \hookrightarrow \tilde{x}_0$ as $n \to \infty$. To prove $\tilde{x} \hookrightarrow \tilde{x}^{-1}$ is continuous, it is enough to show that $\tilde{x}_n^{-1} \hookrightarrow \tilde{x}_0^{-1}$.

Now

$$||\tilde{x}_n^{-1} - \tilde{x}_0^{-1}|| = ||\tilde{x}_n^{-1}(\tilde{x}_0 - \tilde{x}_n)\tilde{x}_0^{-1}||$$

$$\tilde{\leq} ||\tilde{x}_n^{-1}||||\tilde{x}_0 - \tilde{x}_n||||\tilde{x}_0^{-1}||. \tag{4.3}$$

Since $\tilde{x}_n \hookrightarrow \tilde{x}_0$, for any given $\tilde{\epsilon} \tilde{>} \bar{0}$, there exists $\tilde{N}$ such that for all $n \geq \tilde{N}(\lambda)$,

$$||\tilde{x}_n - \tilde{x}_0||(\lambda) \tilde{<} \frac{\bar{1}}{2||\tilde{x}_0^{-1}||}(\lambda) \text{ where we have taken } \tilde{\epsilon} = \frac{\bar{1}}{2||\tilde{x}_0^{-1}||} \tag{4.4}$$

Now

$$||\bar{e} - \tilde{x}_0^{-1}\tilde{x}_n|| = ||\tilde{x}_0^{-1}(\tilde{x}_0 - \tilde{x}_n)|| \tilde{\leq} ||\tilde{x}_0^{-1}||||(\tilde{x} - \tilde{x}_n|| \tag{4.5}$$

Using (4.4) and (4.5) we get

$$||\bar{e} - \tilde{x}_0^{-1}\tilde{x}_n||(\lambda) \tilde{<} \frac{\bar{1}}{\bar{2}}(\lambda) = \frac{1}{2} \forall n \geq \tilde{N}(\lambda) \tag{4.6}$$

So by Corollary 4.12, $\tilde{x}_0^{-1}\tilde{x}_n$ is invertible and its inverse is given by $\tilde{x}_n^{-1}\tilde{x}_0 = (\tilde{x}_0^{-1}\tilde{x}_n)^{-1} = \bar{e} + \sum_{n=1}^{\infty}(\bar{e} - \tilde{x}_0^{-1}\tilde{x}_n)^n$. Thus $||\tilde{x}_n^{-1}\tilde{x}_0|| \tilde{\leq} \bar{1} + \sum_{n=1}^{\infty} ||e - \tilde{x}_0^{-1}\tilde{x}_n||^n \tilde{\leq} \frac{\bar{1}}{1 - ||e - \tilde{x}_0^{-1}\tilde{x}_n||}$ $\tilde{\leq} \bar{2}$ by (4.6). This gives $||\tilde{x}_n^{-1}\tilde{x}_0|| \tilde{<} \bar{2}$ so that we have

$$||\tilde{x}_n^{-1}|| = ||\tilde{x}_n^{-1}\tilde{x}_0\tilde{x}_0^{-1}|| \tilde{\leq} ||\tilde{x}_n^{-1}\tilde{x}_0||||\tilde{x}_0^{-1}|| \tilde{\leq} \bar{2}||\tilde{x}_0^{-1}|| \tag{4.7}$$

From (4.3))and (4.7) we get

$$||\tilde{x}_n^{-1} - \tilde{x}_0^{-1}||\,(\lambda) \tilde{\leq} 2||\tilde{x}_0^{-1}||\,(\lambda)\,||\tilde{x}_0 - \tilde{x}_n||\,(\lambda)\,||\tilde{x}_0^{-1}||\,(\lambda) \to 0 \text{ as } n \to \infty.$$

This proves that $\tilde{x}_n^{-1} \hookrightarrow \tilde{x}_0^{-1}$ as $n \to \infty$. So the mapping $\tilde{x} \hookrightarrow \tilde{x}^{-1}$ of S onto S is continuous. □

Corollary 4.18. *In a soft Banach algebra G, the mapping $\tilde{x}^{-1} \hookrightarrow \tilde{x}$ of S onto S is continuous.*

Definition 4.19. Let G be a soft Banach algebra. A soft element $\tilde{z} \in G$ is called a soft topological divisor of zero if there exists a sequence $\{\tilde{z}_n\}$, $\tilde{z}_n \in G$, $||\tilde{z}_n|| = \bar{1}$ for $n = 1, 2, 3.....$ and such that either $\tilde{z}\tilde{z}_n \hookrightarrow \Theta$ or $\tilde{z}_n\tilde{z} \hookrightarrow \Theta$.

Proposition 4.20. *The soft set Z is a soft subset of P, where Z denotes the set of all soft topological divisors of zero.*

Proof. Let $\tilde{z} \in Z$. The there exists a sequence $\{\tilde{z}_n\}$ such that $||\tilde{z}_n|| = \bar{1}$ for $n = 1, 2, 3.....$ and either $\tilde{z}\tilde{z}_n \hookrightarrow \Theta$ or $\tilde{z}_n\tilde{z} \to \Theta$ as $n \to \infty$. Suppose that $\tilde{z}\tilde{z}_n \hookrightarrow \Theta$. If possible, let $\tilde{z} \notin P$. Then $\tilde{z}(\lambda)^{-1}$ exists for some λ. Now as multiplication is continuous operation, we should have

$$\tilde{z}_n(\lambda) = \tilde{z}(\lambda)^{-1}(\tilde{z}\tilde{z}_n)(\lambda) \to \tilde{z}(\lambda)^{-1}\Theta(\lambda) = \theta \,\text{as}\, n \to \infty.$$

This contradicts the fact that $||\tilde{z}_n|| = \bar{1}$ for $n = 1, 2, 3.....$. Hence Z is a soft subset of P. □

Definition 4.21. Let $\left(\check{X}, ||.||\right)$ be a soft normed linear space and $Y \in S(X)$. A soft element $\tilde{\alpha} \in \check{X}$ is called a soft boundary elements of Y if there exist two sequence $\tilde{x}_n$ and $\tilde{y}_n$ of soft elements in Y and Y^c respectively such that $\tilde{x}_n \hookrightarrow \tilde{\alpha}$ and $\tilde{y}_n \hookrightarrow \tilde{\alpha}$.

Proposition 4.22. *The boundary of P is a soft subset of Z.*

Proof. Let $\tilde{z}$ be a boundary point of P. So there exist two sequences of soft elements $\tilde{r}_n$ in S and $\tilde{s}_n$ in P such that

$$\tilde{r}_n \hookrightarrow \tilde{z} \,\text{and}\, \tilde{s}_n \hookrightarrow \tilde{z}. \tag{4.8}$$

Since P is soft closed so $\tilde{z}\tilde{\in}P$. Now let us write $\tilde{r}_n^{-1}\tilde{z} - \bar{e} = \tilde{r}_n^{-1}(\tilde{z} - \tilde{r}_n)$. The sequence $\left\{\tilde{r}_n^{-1}(\lambda)\right\}$ given above is unbounded $\forall\lambda \in A$. If not, then there exists some $\lambda \in A$ and $n(\lambda)$ such that $||\tilde{r}_n^{-1}\tilde{z} - \bar{e}||(\lambda) < 1$ $\forall n \geq n(\lambda)$, $\forall\lambda \in A$. So that by Corollary 4.12, $\tilde{r}_n^{-1}\tilde{z}(\lambda)$ is regular and hence $\tilde{z}(\lambda) = \tilde{r}_n(\lambda)\left(\tilde{r}_n^{-1}\tilde{z}\right)(\lambda)$ is regular, contradicting $\tilde{z}\tilde{\in}P$. Hence $\left\{\tilde{r}_n^{-1}(\lambda)\right\}$ is unbounded $\forall\lambda \in A$ so that

$$||\tilde{r}_n^{-1}|| \hookrightarrow \bar{\infty} \,\text{as}\, n \to \infty. \tag{4.9}$$

Now let us define $\tilde{z}_n = \frac{\tilde{r}_n^{-1}}{||\tilde{r}_n^{-1}||}$. From the definition of $\tilde{z}_n$, we have

$$||\tilde{z}_n|| = \bar{1}. \tag{4.10}$$

Further

$$\tilde{z}\tilde{z}_n = \frac{\tilde{z}\tilde{r}_n^{-1}}{||\tilde{r}_n^{-1}||} = \frac{\bar{e} + \tilde{z}\tilde{r}_n^{-1} - \bar{e}}{||\tilde{r}_n^{-1}||} = \frac{\bar{e} + (\tilde{z} - \tilde{r}_n)\tilde{r}_n^{-1}}{||\tilde{r}_n^{-1}||}. \tag{4.11}$$

But

$$\frac{\bar{e} + (\tilde{z} - \tilde{r}_n)\tilde{r}_n^{-1}}{||\tilde{r}_n^{-1}||} = \frac{\bar{e}}{||\tilde{r}_n^{-1}||} + (\tilde{z} - \tilde{r}_n)\tilde{z}_n. \tag{4.12}$$

From (4.11) and (4.12), we get

$$\tilde{z}\tilde{z}_n = \frac{\bar{e}}{||\tilde{r}_n^{-1}||} + (\tilde{z} - \tilde{r}_n)\,\tilde{z}_n. \tag{4.13}$$

Using (4.8), (4.9) and (4.10) in (4.13) we see that $\tilde{z}\tilde{z}_n \hookrightarrow \Theta$ as $n \to \infty$. Hence $\tilde{z}$ is a topological divisor of zero. □

5. CONCLUSSION

In a soft normed linear space we have been able to define a new concept of convergence of a sequence of soft elements, which we call soft convergence. We have defined a soft topology [15] generated by a soft norm and which is also known to be a topology of sets [10]. We are also able to introduce a definition of soft Banach algebra and study some of its properties. It is just the begining to study in this direction. Since spectral theory in Banach algebra has various applications in quantum theory, string theory etc., there is an ample scope of further research in studying spectral theory of Banach algebra in soft setting.

Acknowledgements. The authors express their sincere thanks to the referees for their constructive comments in writing the paper in the present form. The authors are also thankful to the Editors-in-Chief and the Managing Editors for their valuable suggestions for improving the presentation of the paper.

This work is partially supported by the Department of Mathematics, Visva Bharati, under the project UGC SAP (DRS) Phase {II}.

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