Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 3, (September 2015), pp. 381–395
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

Permanent of interval-valued and triangular number fuzzy matrices

Anjana Das, Madhumangal Pal, Monoranjan Bhowmik

Received 8 Januar 2015; Revised 16 February 2015; Accepted 8 March 2015

Abstract. In this paper, permanent of both interval-valued fuzzy matrices (IVFMs) and triangular number fuzzy matrices (TFMs) are defined with examples. Some properties due to the permanent nature of IVFM are presented here. Propositions related to the permanent of IVFMs and TFMs are proved. Also, a method to evaluate the permanent for large order IVFM is described.

2010 AMS Classification: 15B15, 94D05

Keywords: Fuzzy matrices, Interval-valued fuzzy matrices, Triangular number fuzzy matrices, Permanent.

Corresponding Author: Anjana Das (1988.anja@gmail.com)

1. Introduction

The permanent has a rich structure when restricted to certain classes of matrices, particularly, matrices of zeros and ones, (entrywise) nonnegative matrices and positive semidefinite matrices. Furthermore, there is a certain similarity of its properties over the class of nonnegative matrices and the class of positive semidefinite matrices. Romanowicz and Grabowski [19] used permanent of a square matrix. Permanent is used in graph-theoretic interpretations also. One is as the sum of weights of cycle covers of a directed graph, another is as the sum of weights of perfect matching in a bipartite graph.

The concept of fuzzy matrix (FM)[21] is one of the recent topics developed for dealing with the uncertainties present in engineering, agriculture, science, social science and also in most of our real life situations. Thomason [24] published the first work on fuzzy matrices and this work was based on the max-min operation. He studied the convergence of power of a fuzzy matrix with max-min operation. Kim [10] presented some properties on semigroup of fuzzy matrices. He also studied the inverse of fuzzy matrix. Kim and Roush [11] studied generalized fuzzy matrices and presented a systematic development of fuzzy matrix theory. Hashimoto [7, 8] investigated some properties of fuzzy matrices. Pal [15] defined first time the intuitionistic fuzzy determinant and Pal et al. [1, 2, 13, 16, 17, 18, 20] have introduced the concept of intuitionistic fuzzy matrices and studied several properties. Bhowmik and pal[4] presented some results on interval-valued intuitionistic fuzzy matrices [9]. The parameterization tool of interval-valued fuzzy matrix enhances the flexibility of its application. Most of our real life problems in medical sciences, engineering, management environment and social sciences often involved data which are not necessarily crisp, precise and deterministic in character due to various uncertainties associated with these problems. Such uncertainties are usually being handled with the help of the topics like probability, fuzzy sets, intuitionistic fuzzy sets, interval mathematics, rough sets, etc. The most useful representation of fuzziness is by membership function. Depending upon the nature and shape of the membership function the fuzzy number can be classified in different forms, such as triangular fuzzy number (TFN), trapezoidal fuzzy number, etc. Several researchers present various results on TFNs. Chen [5] gives the concept about generalized TFNs. Pal et al. [3, 23] presented some properties of TFNs and TFMs (matrices of TFNs). The concept of interval-valued fuzzy matrix (IVFM)[14] as a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [22], by extending the max-min operation on fuzzy algebra $F = [0, 1]$, for the elements $a,\ b \in F,\ a + b = max\{a,\ b\}$ and $a.b = min\{a,\ b\}$. Let F_{mn} be the set of all $m \times n$ order fuzzy matrices over the fuzzy algebra with support $[0,\ 1]$, that is matrices whose entries are intervals and all the intervals are subintervals of the interval [0, 1]. An IVFM is represented as $A = [a_{ij}] = \Big[[a_{ijL},\ a_{ijU}]\Big]$, where each a_{ij} is a subinterval of the interval $[0,\ 1]$, as the interval matrix $A = [A_L,\ A_U]$ whose ijth entry is the interval $[a_{ijL},\ a_{ijU}]$, where the lower limit matrix $A_L = [a_{ijL}]$ and the upper limit matrix $A_U = [a_{ijU}]$ are fuzzy matrices such that $a_{ijL} \leq a_{ijU}$ for all $i,\ j$, i.e. $A_L \leq A_U$ [12].

In this paper, in Section 2 some definitions of FM, IVFM are recalled with some algebraic operations and the definition of permanent with example. In Section 3, permanent of IVFM is defined with some of its properties. Also, a method is described in Section 4 to evaluate the permanent for large order IVFMs with examples. Finally, in Section 5 permanent of TFM is defined with some propositions.

2. Preliminaries

In this section, some basic definitions of FMs, IVFMs, TFNs, TFMs and permanent of crisp matrix are given. IVFM denotes the set of all interval-valued fuzzy matrices, that is, fuzzy matrices whose entries are all subintervals of the interval [0,1].

Definition 2.1. (Fuzzy Matrix) A fuzzy matrix A of order $m \times n$ is defined by $A = [< a_{ij}, a_{ij\mu} >]_{m\times n}$ where $a_{ij\mu}$ is the membership value of the element a_{ij} in A and $a_{ij\mu} \in [0, 1]$.
For simplicity, we write A as A $=[a_{ij\mu}]_{m\times n}$.

For a pair of fuzzy matrices $E = [e_{ij}]$ and $F = [f_{ij}]$ in F_{mn} such that $E \leq F$, let us define the interval matrix as $[E, F]$, whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}, that is $[e_{ij},\ f_{ij}]$. In particular, for $E = F$, IVFM $[E, F]$ reduces to the fuzzy matrix $E \in F_{mn}$.

Definition 2.2. (Interval-Valued Fuzzy Matrix) An IVFM A over F_{mn} is defined as $A = [a_{ij}] = \Big[[a_{ijL},\ a_{ijU}]\Big]_{m\times n}$. Let us define $A_L = [a_{ijL}]_{m\times n}$ and $A_U = [a_{ijU}]_{m\times n}$, clearly A_L and A_U belong to F_{mn} such that $A_L \leq A_U$. In short, A can be written as $A = [A_L,\ A_U]$, where A_L and A_U are called the lower and upper limits of A, respectively. For simplicity, we write an IVFM as A $=\Big[[a_{ijL}, a_{ijU}]\Big]_{m\times n}$ with maintaining the condition $0 \leq a_{ijL} \leq a_{ijU} \leq 1$.

Here we shall follow the basic operations on IVFM.

Let $A = [a_{ij}] = \Big[[a_{ijL},\ a_{ijU}]\Big]$ and $B = [b_{ij}] = \Big[[b_{ijL},\ b_{ijU}]\Big]$ be two IVFMs of order $m \times n$, their sum is denoted by $A + B$ and is defined as

$$A + B = [\max\{a_{ij},\ b_{ij}\}] = [\max\{a_{ijL},\ b_{ijL}\},\ \max\{a_{ijU},\ b_{ijU}\}].$$

Product of two IVFMs $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ denoted by AB and is defined as

$$\begin{aligned} AB = [c_{ij}] &= \Big[\sum_{k=1}^{n} a_{ik}b_{kj}\Big], \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, p \\ &= \Big[\sum_{k=1}^{n}(a_{ikL}.b_{kjL}),\ \sum_{k=1}^{n}(a_{ikU}.\ b_{kjU})\Big] \\ &= [\max_k\ \min(a_{ikL}, b_{kjL}),\ \max_k\ \min(a_{ikU}, b_{kjU})]. \end{aligned}$$

If $A = [A_L,\ A_U]$ and $B = [B_L,\ B_U]$, then $A + B = [A_L + B_L,\ A_U + B_U]$, $AB = [A_L B_L, A_U B_U]$.
$A \geq B$ if and only if $a_{ijL} \geq b_{ijL}$ and $a_{ijU} \geq b_{ijU}$. Also, it can be proved that $A \geq B$ if and only if $A + B = A$.

Definition 2.3. (Triangular Fuzzy Number) A triangular fuzzy number (TFN) can be represented as $M = \langle m, \alpha, \beta\rangle$, where m is the point whose membership grade is 1 called the mean value and α, β are the left hand and right hand spreads of M, respectively. The membership function of M is,

$$\mu_M(x) = \begin{cases} 0 & for \quad x \leq m - \alpha \\ 1 - \frac{m-x}{\alpha} & for \quad m - \alpha < x < m \\ 1 & for \quad x = m \\ 1 - \frac{x-m}{\beta} & for \quad m < x < m + \beta \\ 0 & for \quad x \geq m + \beta. \end{cases}$$

Here we introduce some arithmetic operations of TFNs due to Dubois and Prade [6].

Let $M = \langle m, \alpha, \beta\rangle$ and $N = \langle n, \gamma, \delta\rangle$ be two TFNs.

(1) **Addition:** For two TFNs M and N, addition is given by $M + N = \langle m + n, \alpha + \gamma, \beta + \delta\rangle$.

(2) **Scalar multiplication:** Let λ be a scalar, then multiplication of TFM by a scalar is given by
$\lambda M = \langle \lambda m, \lambda\alpha, \lambda\beta\rangle$, when $\lambda \geq 0$.
$\lambda M = \langle \lambda m, -\lambda\beta, -\lambda\alpha\rangle$, when $\lambda \leq 0$.
In particular, $-M = \langle -m, \beta, \alpha\rangle$.

(3) **Subtraction:** For two TFMs M and N, subtraction is defined as $M - N = \langle m, \alpha, \beta\rangle - \langle n, \gamma, \delta\rangle = \langle m - n, \alpha + \delta, \beta + \gamma\rangle$.

(4) **Multiplication:** Rules for multiplication are as follows:
(a) When $M \geq 0$ and $N \geq 0$ ($M \geq 0,\ if\ m \geq 0$)
$M.N = \langle m, \alpha, \beta\rangle.\langle n, \gamma, \delta\rangle = \langle mn, m\gamma + n\alpha, m\delta + n\beta\rangle$.
(b) When $M \leq 0$ and $N \geq 0$
$M.N = \langle m, \alpha, \beta\rangle.\langle n, \gamma, \delta\rangle = \langle mn, n\alpha - m\delta, n\beta - m\gamma\rangle$.
(c) When $M \leq 0$ and $N \leq 0$
$M.N = \langle m, \alpha, \beta\rangle.\langle n, \gamma, \delta\rangle = \langle mn, -n\beta - m\delta, -n\alpha - m\gamma\rangle$.

Definition 2.4. (Matrix of Triangular Fuzzy Numbers (TFM)) A TFM of order $m \times n$ is defined as $A = [a_{ij}]_{m\times n}$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij}\rangle$ is the ijth element of A, m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are the left and right spreads of a_{ij}, respectively.

The following operations are defined on TFMs.
(i) $A + B = [a_{ij} + b_{ij}]$
(ii) $A - B = [a_{ij} - b_{ij}]$
(iii) If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, then $A.B = \sum\limits_{k=1}^{n} a_{ik}.b_{kj}, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.
(iv) $A^T = [a_{ji}]_{n\times m}$
(v) $k.A = [ka_{ij}]$, where k is a scalar.

The definition of permanent[25] is similar to the definition of determinant except the sign of each term in summation. The number of terms over summation are same in both cases but the signs associated in each term are all positive in case of permanent. The permanent cannot compete with determinant, in terms of the depth of theory and breadth of applications, but it is safe to say that the permanent also exhibits both these characteristics in ample measure, a fact that has not received enough attention.

Definition 2.5. (Permanent) If $A = [a_{ij}]_{n\times n}$ is a crisp matrix of order $n \times n$, then the permanent of A is denoted by Per(A) and defined as

$$Per(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n.

Let us consider an example to illustrate the permanent of a crisp matrix.

Example 2.6. Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 7 \\ 3 & 9 & 6 \end{bmatrix}$ be a crisp matrix of order 3×3.

$$\begin{aligned} \text{Then } Per(A) &= a_{11} \cdot a_{22} \cdot a_{33} \cdot + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{22} \cdot a_{31} + a_{11} \cdot a_{23} \cdot a_{32} \\ &\quad + a_{12} \cdot a_{21} \cdot a_{33} + a_{13} \cdot a_{21} \cdot a_{32} \\ &= 2 \cdot 3 \cdot 6 + 5 \cdot 7 \cdot 3 + 4 \cdot 3 \cdot 3 + 2 \cdot 7 \cdot 9 + 5 \cdot 1 \cdot 6 + 4 \cdot 1 \cdot 9 = 369. \end{aligned}$$

3. Permanent of IVFMs

In this section, we introduce the permanent of IVFMs and some of its properties.

Definition 3.1. Let $A = [a_{ij}]_{m \times n}$ be an interval-valued fuzzy matrix, where $a_{ij} = [a_{ijL}, a_{ijU}]$ be an interval and $0 \leq a_{ijL} \leq a_{ijU} \leq 1$.
Then the permanent of A is denoted by Per(A) and defined by

$$\begin{aligned} Per(A) &= \sum_{\sigma \in S} \prod_{i=1}^{m} a_{i\sigma(i)} \ for\ m \leq n \\ &\quad \text{[where S is the set of all one-to-one mappings} \\ &\quad \text{ from } \{1, 2, \ldots m\} \ to \ \{1, 2, \ldots n\}] \\ &= \sum_{\sigma \in S} \prod_{j=1}^{n} a_{\sigma(j)j} \ for\ m > n \\ &\quad \text{[where S is the set of all one-to-one mappings} \\ &\quad \text{ from } \{1, 2, \ldots n\} \ to \ \{1, 2, \ldots m\}] \end{aligned}$$

Two expressions are written for the permanent of a matrix, because for $m > n$, there are one-to-one mappings from $\{1, 2, \ldots n\}$ to $\{1, 2, \ldots m\}$. In this case, no one-to-one mapping is possible from $\{1, 2, \ldots m\}$ to $\{1, 2, \ldots n\}$. But for $m \leq n$ the one-to-one mappings are possible from $\{1, 2, \ldots m\}$ to $\{1, 2, \ldots n\}$.

Following two examples are considered to illustrate the definition.

Example 3.2. Let $A = \begin{bmatrix} [0.2,0.5] & [0.1,0.7] & [0.5,0.6] \\ [0.3,0.4] & [0.5,0.8] & [0.4,0.6] \end{bmatrix}$.

$$\begin{aligned} \text{Then } Per(A) &= max\Big\{min([0.2, 0.5], [0.5, 0.8]), min([0.2, 0.5], [0.4, 0.6]), \\ &\quad min([0.1, 0.7], [0.3, 0.4]), min([0.1, 0.7], [0.4, 0.6]), \\ &\quad min([0.5, 0.6], [0.3, 0.4]), min([0.5, 0.6], [0.5, 0.8])\Big\} \\ &= max\{[0.2, 0.5], [0.2, 0.5], [0.1, 0.4], [0.1, 0.6], [0.3, 0.4], [0.5, 0.6]\} \\ &= [0.5, 0.6]. \end{aligned}$$

Example 3.3. If we take $B = \begin{bmatrix} [0.1{,}0.5] & [0.4{,}0.5] \\ [0.3{,}0.4] & [0.2{,}0.3] \\ [0.2{,}0.6] & [0.6{,}0.8] \end{bmatrix}$,

$$\begin{aligned} \text{Then } Per(B) &= max\Big\{min([0.1,0.5],[0.2,0.3]), min([0.1,0.5],[0.6,0.8]), \\ &\quad min([0.3,0.4],[0.4,0.5]), min([0.3,0.4],[0.6,0.8]), \\ &\quad min([0.2,0.6],[0.4,0.5]), min([0.2,0.6],[0.2,0.3])\Big\} \\ &= max\Big\{[0.1,0.3],[0.1,0.5],[0.3,0.4],[0.3,0.4],[0.2,0.5],[0.2,0.3],\Big\} \\ &= [0.3,0.5]. \end{aligned}$$

3.1. **Some properties of permanent of IVFMs.** Some trivial properties of permanent of IVFMs are presented below.

1. For any triangular or diagonal IVFM A, $Per(A) = min\{$of its diagonal entries$\}$.
2. For any row IVFM or column IVFM A, $Per(A) = max\{$of the entries$\}$.
3. For any two IVFMs A and B such that, number of columns of A = number of rows of B, then $Per(AB) \geq min\big\{Per(A), Per(B)\big\}$.
4. If A and B are any two IVFMs such that both AB and BA are defined, then $Per(AB) \neq Per(BA)$, in general.

The proofs of the above properties are straightforward, they are illustrated by the following examples.

Example 3.4. Let $A = \begin{bmatrix} [0.2{,}0.5] & [0.1{,}0.3] & [0.3{,}0.4] \\ [0.3{,}0.6] & [0.2{,}0.3] & [0.4{,}0.5] \end{bmatrix}$ and
$B = \begin{bmatrix} [0.1{,}0.2] & [0.7{,}0.8] \\ [0.1{,}0.6] & [0.6{,}0.9] \\ [0.3{,}0.8] & [0.2{,}0.5] \end{bmatrix}$.
Then $AB = \begin{bmatrix} [0.3{,}0.4] & [0.2{,}0.5] \\ [0.3{,}0.5] & [0.3{,}0.6] \end{bmatrix}$.
$Per(A) = max\{[0.2,0.5],[0.1,0.3],[0.3,0.4]\} = [0.3,0.5]$,
$Per(B) = max\{[0.1,0.2],[0.1,0.6],[0.3,0.8]\} = [0.3,0.8]$
and $Per(AB) = max\{[0.3,0.4],[0.2,0.5]\} = [0.3,0.5]$
Thus $Per(AB) = min\big\{Per(A), Per(B)\big\}$.

In this example the equality part of the property 3 is satisfied.

Example 3.5. Let $A = \begin{bmatrix} [0.2,0.3] & [0.1,0.5] & [0.5,0.9] \end{bmatrix}$ and $B = \begin{bmatrix} [0.2{,}0.6] \\ [0.3{,}0.4] \\ [0.2{,}0.5] \end{bmatrix}$.
Then $AB = [[0.2,0.5]]$, and $BA = \begin{bmatrix} [0.2{,}0.3] & [0.1{,}0.5] & [0.2{,}0.6] \\ [0.2{,}0.3] & [0.1{,}0.4] & [0.3{,}0.4] \\ [0.2{,}0.3] & [0.1{,}0.5] & [0.2{,}0.5] \end{bmatrix}$
Now, $Per(BA) = max\Big\{[0.1,0.3],[0.1,0.3],[0.1,0.3]\Big\} = [0.1,0.3]$.
For this case, $Per(AB) \neq Per(BA)$. which illustrate property 4.

In the following some important results are presented for permanent of IVFMs.

Proposition 3.6. *For any two IVFMs A and B of same order, such that $A \leq B \Rightarrow Per(A) \leq Per(B)$.*

Proof. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two IVFMs where $a_{ij} = [a_{ijL}, a_{ijU}]$ and $b_{ij} = [b_{ijL}, b_{ijU}]$.
Then, $A \leq B \Rightarrow a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$.
When $m \leq n$,

$$\begin{aligned} Per(A) &= \sum_{\sigma\in S}\prod_{i=1}^{m} a_{i\sigma(i)} = \sum_{\sigma\in S}\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}] \\ &\leq \sum_{\sigma\in S}\prod_{i=1}^{m}[b_{i\sigma(i)L}, b_{i\sigma(i)U}] = Per(B). \end{aligned}$$

When $m > n$,

$$\begin{aligned} Per(A) &= \sum_{\sigma\in S}\prod_{j=1}^{n} a_{\sigma(j)j} = \sum_{\sigma\in S}\prod_{j=1}^{n}[a_{\sigma(j)jL}, a_{\sigma(j)jU}] \\ &\leq \sum_{\sigma\in S}\prod_{j=1}^{n}[b_{\sigma(j)jL}, b_{\sigma(j)jU}] = Per(B). \end{aligned}$$

Hence $A \leq B \Rightarrow Per(A) \leq Per(B)$ □

Proposition 3.7. *For any IVFM A, $Per(A) = Per(A^T)$.*

Proof. Let $A = [a_{ij}]_{m\times n}$ and $a_{ij} = [a_{ijL}, a_{ijU}]$,
When $m \leq n$,

$$Per(A) = \sum_{\sigma\in S}\prod_{i=1}^{m} a_{i\sigma(i)} = \sum_{\sigma\in S}\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}].$$

Let $A^T = B = [b_{ij}]_{m\times n},\ n \geq m$.
Then, $b_{ij} = a_{ji}$ i.e. $b_{ijL} = a_{jiL}$ and $b_{ijU} = a_{jiU}$.

$$\begin{aligned} Per(A^T) = Per(B) &= \sum_{\sigma\in S}\Big(\prod_{j=1}^{m} b_{\sigma(j)j}\Big) = \sum_{\sigma\in S}\prod_{j=1}^{m} a_{j\sigma(j)} \\ &= \sum_{\sigma\in S}\prod_{i=1}^{m} a_{i\sigma(i)} = Per(A). \end{aligned}$$

For $m > n$, the proof is similar as before. □

Proposition 3.8. *Interchanging of rows or columns does not effect to the permanent value of the matrix.*

Proof. Let $A = [a_{ij}]$ be an IVFM of order $m \times n$ and $B = [b_{ij}]_{m\times n}$ is obtained from A by interchanging the rth and sth row ($r < s$) of A. Then, it is clear that,

$b_{ij} = a_{ij},\ i \neq r,\ i \neq s$ and $b_{rj} = a_{sj},\ b_{sj} = a_{rj}$.

$$\begin{aligned}
\text{Now, } Per(B) &= \sum_{\sigma \in S} \Big(\prod_{i=1}^{m} b_{i\sigma(i)} \Big) \\
&= \sum_{\sigma \in S} b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{r\sigma(r)} \cdots b_{s\sigma(s)} \cdots b_{m\sigma(m)} \\
&= \sum_{\sigma \in S} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{s\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{m\sigma(m)} \\
&= \sum_{\sigma \in S} [a_{1\sigma(1)L},\ a_{1\sigma(1)U}][a_{2\sigma(2)L},\ a_{2\sigma(2)U}] \cdots [a_{s\sigma(r)L},\ a_{s\sigma(r)U}] \cdots \\
&\quad [a_{r\sigma(s)L},\ a_{r\sigma(s)U}] \cdots [a_{m\sigma(m)L},\ a_{m\sigma(m)U}].
\end{aligned}$$

Let $\lambda = \begin{pmatrix} 1 & 2 \cdots r \cdots s \cdots m \\ 1 & 2 \cdots s \cdots r \cdots m \end{pmatrix}$ and $\sigma\lambda = \phi$, then $\phi(i) = \sigma(i)$ for $i \neq r,\ i \neq s$ and $\phi(r) = \sigma(s)$ and $\phi(s) = \sigma(r)$.

$$\begin{aligned}
\text{Then, } Per(B) &= \sum_{\sigma \in S} [a_{1\sigma(1)L},\ a_{1\sigma(1)U}][a_{2\sigma(2)L},\ a_{2\sigma(2)U}] \cdots [a_{s\sigma(r)L},\ a_{s\sigma(r)U}] \cdots \\
&\quad [a_{r\sigma(s)L},\ a_{r\sigma(s)U}] \cdots [a_{m\sigma(m)L},\ a_{m\sigma(m)U}] \\
&= \sum_{\phi \in S} [a_{1\phi(1)L},\ a_{1\phi(1)U}][a_{2\phi(2)L},\ a_{2\phi(2)U}] \cdots [a_{s\phi(s)L},\ a_{s\phi(s)U}] \cdots \\
&\quad [a_{r\phi(r)L},\ a_{r\phi(r)U}] \cdots [a_{m\phi(m)L},\ a_{m\phi(m)U}] \\
&= \sum_{\phi \in S} [a_{1\phi(1)L},\ a_{1\phi(1)U}][a_{2\phi(2)L},\ a_{2\phi(2)U}] \cdots [a_{r\phi(r)L},\ a_{r\phi(r)U}] \cdots \\
&\quad [a_{s\phi(s)L},\ a_{s\phi(s)U}] \cdots [a_{m\phi(m)L},\ a_{m\phi(m)U}] \\
&= Per(A).
\end{aligned}$$

Hence, interchanging of rows or columns does not alter the permanent value. □

A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere. An example of permutation matrix of order 3×3 is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

It also can be written as $A = \begin{bmatrix} [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] \\ [0,0] & [1,1] & [0,0] \end{bmatrix}$.

Next proposition is related to permutation matrices.

Proposition 3.9. *For any IVFM A, $Per(A) = Per(PAQ)$, where P and Q are permutation matrices.*

Proof. Let $A = [a_{ij}]_{m \times n}$, $P = [p_{ij}]_{m \times m}$, and $Q = [q_{ij}]_{n \times n}$ where $a_{ij} = [a_{ijL}, a_{ijU}]$, $p_{ij} = [p_{ijL}, p_{ijU}]$ and $q_{ij} = [q_{ijL}, q_{ijU}]$, where P and Q are permutation matrices.

The matrix AQ obtained from A by interchanging columns according as the matrix Q [i.e. by which change we get Q from I_n, we apply it to A].
Let the matrix AQ be A_c.

Now $PAQ = PA_c$, where the IVFM PA_c obtained from A_c by interchanging rows according as the matrix P.
Let $PA_c = A_{cr}$, where A_{cr} is the matrix obtained from A by interchanging some rows and columns.

Now $Per(PAQ) = Per(A_{cr}) = Per(A)$ [Since interchanging of rows and columns does not effect to the permanent value]. □

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two IVFMs where $a_{ij} = [a_{ijL}, a_{ijU}]$, $b_{ij} = [b_{ijL}, b_{ijU}]$.

Then we define
$$a_{ij} \bigwedge b_{ij} = [min\{a_{ijL}, b_{ijL}\}, min\{a_{ijU}, b_{ijU}\}],$$
$$a_{ij} \bigvee b_{ij} = [max\{a_{ijL}, b_{ijL}\}, max\{a_{ijU}, b_{ijU}\}].$$

Also, $A \bigwedge B = [a_{ij} \bigwedge b_{ij}]_{m\times n}$, $A \bigvee B = [a_{ij} \bigvee b_{ij}]_{m\times n}$.

Proposition 3.10. *For any two IVFMs A and B of same order, $Per(A \bigwedge B) \leq Per(A) \bigwedge Per(B)$.*

Proof. Let $A \bigwedge B = C$, where $A = [a_{ij}]_{m\times n},\ B = [b_{ij}]_{m\times n}, C = [c_{ij}]_{m\times n}$ are three IVFMs. Then $c_{ijL} = min\{a_{ijL},\ b_{ijL}\}$ and $c_{ijU} = min\{a_{ijU},\ b_{ijU}\}$.
Now for $m \leq n$,

$$\begin{aligned}
Per(A \bigwedge B) &= \sum_{\sigma\in S}\prod_{i=1}^{m} c_{i\sigma(i)} = \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[c_{i\sigma(i)L}, c_{i\sigma(i)U}]\Big)\\
&= \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[min\{a_{i\sigma(i)L},\ b_{i\sigma(i)L}\},\ min\{a_{i\sigma(i)U},\ b_{i\sigma(i)U}\}]\Big)\\
&\leq \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[a_{i\sigma(i)L},\ a_{i\sigma(i)U}]\Big) = Per(A).
\end{aligned}$$

So, $Per(A \bigwedge B) \leq Per(A)$.
Now for, $m > n$, the proof is similar.
Similarly, we can prove $Per(A \bigwedge B) \leq Per(B)$.
Therefore, $Per(A \bigwedge B) \leq Per(A) \bigwedge Per(B)$. □

Proposition 3.11. *For any two IVFMs A and B, $Per(A \bigvee B) \geq Per(B) \bigvee Per(B)$.*

Proof. Let $A \bigvee B = D$, where $A = [a_{ij}]_{m\times n},\ B = [b_{ij}]_{m\times n}, D = [d_{ij}]_{m\times n}$ are three IVFMs. Then, $d_{ijL} = max\{a_{ijL},\ b_{ijL}\}$ and $d_{ijU} = max\{a_{ijU},\ b_{ijU}\}$.

When $m \leq n$,

$$\begin{aligned} Per(A \bigvee B) &= \sum_{\sigma\in S}\prod_{i=1}^{m} d_{i\sigma(i)} = \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[d_{i\sigma(i)L}, d_{i\sigma(i)U}]\Big) \\ &= \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[max\{a_{i\sigma(i)L},\ b_{i\sigma(i)L}\},\ max\{a_{i\sigma(i)U},\ b_{i\sigma(i)U}\}]\Big) \\ &\geq \sum_{\sigma\in S}\Big(\prod_{i=1}^{m}[a_{i\sigma(i)L},\ a_{i\sigma(i)U}]\Big) = Per(A). \end{aligned}$$

So, $Per(A \bigvee B) \geq Per(A)$.
Now for, $m > n$, the proof is similar.
Similarly, we can prove $Per(A \bigvee B) \geq Per(B)$.
Therefore, $Per(A \bigvee B) \geq Per(A) \bigvee Per(B)$.
Hence the proof. □

If A and B are two IVFMs satisfies the relation $AXA = A$, then X is called g-inverse of A and A is called regular.

Proposition 3.12. *If $A = [a_{ij}]_{m\times n}$ be a regular matrix and B is a g-inverse of A, then $Per(AB) = Per(AB)^2$.*

Proof. Since B is a g-inverse of A
Then $ABA = A$
$\Rightarrow ABAB = AB$
$\Rightarrow (AB)^2 = AB$
Then $Per(AB)^2 = Per(AB)$.
Hence the proof. □

Proposition 3.13. *If A is constant IVFM (i.e. all rows or all columns are equal) then*

$$\begin{aligned} Per(A) &= min\{its\ entries\} \\ &= min\{along\ row\ entries\}\ if\ rows\ are\ equal \\ &= min\{along\ column\ entries\}\ if\ columns\ are\ equal. \end{aligned}$$

Proof. Let $A = [a_{ij}]_{m\times n}$ be a constant IVFM whose rows are equal.
Here $a_{ij} = [a_{ijL},\ a_{ijU}]$ where $a_{1jL} = a_{ijL}$ and $a_{1jU} = a_{ijU}$ for all i.
When $m > n$,

$$\begin{aligned} Per(A) &= \sum_{\sigma\in S}\prod_{j=1}^{m}[a_{\sigma(j)jL}, a_{\sigma(j)jU}] \\ &= \sum_{\sigma\in S}\prod_{j=1}^{m}[a_{1jL}, a_{1jU}] \\ &= \prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}] \\ &= min\{along\ row\ entries\}. \end{aligned}$$

When $m \leq n$, the proof is similar.
The proof is similar to other cases. $\square$

4. Evaluation of permanent of large order IVFM

It is very difficult to handle a large order matrix, it may be crisp or any kind of fuzzy matrix. Hence, evaluation of permanent of such matrices is also a complicated task. In this section, a suitable method is described to evaluate the permanent of an IVFM.

Let $\Omega_{pt} = \{(\omega_1, \omega_2, \ldots, \omega_p) : 1 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_p \leq t\}$; ω_i is integer, $i = 1, 2, \ldots, p$.
For $\alpha \in \Omega_{rm},\ \beta \in \Omega_{sn}$, let $A[\alpha|\beta]$ denotes the $r \times s$ submatrix obtained from A by taking only rows of α and columns of β, where ijth entry of $A[\alpha|\beta]$ is $a_{\alpha_i \beta_j}$.
$A(\alpha|\beta)$ denotes the $(m - r) \times (n - s)$ submatrix of A complementary to $A[\alpha|\beta]$, i.e. $A(\alpha|\beta)$ is the submatrix which we obtained from A by deleting rows of α and columns of β.
Am example is given to illustrate $A[\alpha|\beta]$ and $A(\alpha|\beta)$.

Example 4.1. Let $A = \begin{bmatrix} [0.2,0.3] & [0.1,0.4] & [0.3,0.6] & [0.4,0.5] & [0.3,0.9] \\ [0.5,0.7] & [0.2,0.4] & [0.1,0.3] & [0.4,0.6] & [0.7,0.8] \\ [0.2,0.4] & [0.3,0.5] & [0.4,0.7] & [0.3,0.8] & [0.5,0.6] \\ [0.3,0.4] & [0.2,0.3] & [0.5,0.6] & [0.7,0.9] & [0.5,0.7] \\ [0.1,0.5] & [0.2,0.4] & [0.3,0.5] & [0.4,0.6] & [0.5,0.8] \\ [0.2,0.3] & [0.1,0.5] & [0.4,0.7] & [0.5,0.9] & [0.6,0.8] \end{bmatrix}$
and $\alpha = (2, 4, 5),\ \beta = (1, 3, 4)$.
Then $A[\alpha|\beta] = \begin{bmatrix} [0.5,0.7] & [0.1,0.3] & [0.4,0.6] \\ [0.3,0.4] & [0.5,0.6] & [0.7,0.9] \\ [0.1,0.5] & [0.3,0.5] & [0.4,0.6] \end{bmatrix}$
and $A(\alpha|\beta) = \begin{bmatrix} [0.1,0.4] & [0.3,0.9] \\ [0.3,0.5] & [0.7,0.8] \\ [0.1,0.5] & [0.6,0.8] \end{bmatrix}$

The permanent is calculated by using the following theorem.

Theorem 4.2. *For an $m \times n$ IVFM A, $m,\ n \geq 2$ and $\alpha \in \Omega_{rm}$, then*

$$Per(A) = \sum_{\beta \in \Omega_{sn}} \Big(\prod \{Per(A[\alpha|\beta]),\ Per(A(\alpha|\beta))\} \Big).$$

In particular,
$Per(A) = \sum\limits_{t=1}^{n} \Big(\prod\limits_{i=1}^{m} \{a_{it},\ Per(A(i|t))\} \Big)$.

Corollary 4.3. *Let A be $m \times n$ matrix and $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ be a block presentation, where $B = [b_{ij}]_{m_1 \times n_1},\ C = [c_{ij}]_{m_1 \times n_2},\ D = [d_{ij}]_{m_2 \times n_2}$, such that $m_1 + m_2 = m,\ n_1 + n_2 = n$.*
Then $Per(A) = Per(B)Per(D)$.

5. Permanent of TFMs

In this section, we introduce the permanent of TFM (i.e. matrices of triangular fuzzy numbers) and some of propositions.

Definition 5.1. Let $A = [a_{ij}]_{m\times n}$ be a TFM where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij}\rangle$ be a triangular fuzzy number.

Then the permanent of A is denoted by Per(A) and is defined by

$$\begin{aligned} Per(A) &= \sum_{\sigma\in S}\prod_{i=1}^{m}\langle m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)}\rangle \; for\ m \le n \\ &\quad [\text{ where S is the set of all one-to-one mapping} \\ &\quad \text{ from } \{1, 2, \ldots m\}\ to\ \{1, 2, \ldots n\}] \\ &= \sum_{\sigma\in S}\prod_{j=1}^{n}\langle m_{\sigma(j)j}, \alpha_{\sigma(j)j}, \beta_{\sigma(j)j}\rangle \; for\ m\,n \\ &\quad [\text{ where S is the set of all one-to-one mapping} \\ &\quad \text{ from } \{1, 2, \ldots n\}\ to\ \{1, 2, \ldots m\}] \end{aligned}$$

5.1. **Permanent of some special types of TFMs.** Here we deduce the permanent of some special types of TFMs.

(1) **Pure Null TFM:** If A is a pure null TFM (i.e. all its entries are zero), then $Per(A) = \langle 0, 0, 0\rangle$.
(2) **Fuzzy Null TFM:** If A is a fuzzy null TFM (i.e. all $a_{ij} = \langle 0, \epsilon_1, \epsilon_2\rangle$ for all i, j where $\epsilon_1\epsilon_2 \ne 0$), then $Per(A) = \langle 0, 0, 0\rangle$.
(3) **Pure Unit TFM:** If A is a pure unit TFM (i.e. a square TFM where $a_{ii} = \langle 1, 0, 0\rangle$ and $a_{ij} = \langle 0, 0, 0\rangle$ for all $i \ne j$), then $Per(A) = \langle 1, 0, 0\rangle$.
(4) **Fuzzy Unit TFM:** If A is a fuzzy unit TFM (i.e. a square TFM where $a_{ii} = \langle 1, \epsilon_1, \epsilon_2\rangle$ and $a_{ij} = \langle 0, \epsilon_3, \epsilon_4\rangle$ for all $i \ne j$ where $\epsilon_1\epsilon_2 \ne 0, \epsilon_3\epsilon_4 \ne 0$), then $Per(A) = \prod a_{ii}$.
(5) **Pure Triangular TFM:** If A is a pure triangular TFM (i.e. a square TFM where either $a_{ij} = \langle 0, 0, 0\rangle$ for all $i > j$ or $a_{ij} = \langle 0, 0, 0\rangle$ for all $i < j$), then $Per(A) = \prod a_{ii}$.
(6) **Fuzzy Triangular TFM:** If A is a fuzzy triangular TFM (i.e. a square TFM where either $a_{ij} = \langle 0, \epsilon_1, \epsilon_2\rangle$ for all $i > j$ or $a_{ij} = \langle 0, \epsilon_1, \epsilon_2\rangle$ for all $i < j$ and $\epsilon_1\epsilon_2 \ne 0$), then $Per(A) = \prod a_{ii}$.

Here some propositions related to the permanent of TFM are given.

Proposition 5.2. *Let A be a TFM of order $m \times n$. If all the elements of a row (column) of A are $\langle 0, 0, 0\rangle$, then $Per(A) = \langle 0, 0, 0\rangle$.*

Proof. Let $A = [a_{ij}]_{m\times n}$ be a TFM of order $m \times n$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij}\rangle$.
Let $m \le n$, then
$Per(A) = \sum_{\sigma\in S}\langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)}\rangle\langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)}\rangle$
$\cdots\langle m_{m\sigma(m)}, \alpha_{m\sigma(m)}, \beta_{m\sigma(m)}\rangle$.
Here $Per(A)$ is the sum of ${}^{n}p_m$ expressions, where in each expression there is a term $\langle 0, 0, 0\rangle$.

So each expression becomes $\langle 0,0,0\rangle$ and the sum is $\langle 0,0,0\rangle$.
Therefore, $Per(A) = \langle 0,0,0\rangle$
For $m > n$, the proof is similar. □

Proposition 5.3. *Let A be a TFM of order $m \times n$. If a row be multiplied by a scalar k then the permanent value will be $kPer(A)$.*

Proof. Let $A = [a_{ij}]_{m\times n}$ be a TFM of order $m \times n$ and $B = [b_{ij}]_{m\times n}$ be another TFM obtained by multiplying k to a row of A.

Let $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij}\rangle, b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij}\rangle$ and k is multiplied to the rth row.
Let $m \leq n$, then

$$\begin{aligned}
Per(B) &= \sum_{\sigma\in S}\langle n_{1\sigma(1)}, \gamma_{1\sigma(1)}, \delta_{1\sigma(1)}\rangle\langle n_{2\sigma(2)}, \gamma_{2\sigma(2)}, \delta_{2\sigma(2)}\rangle\cdots \\
&\quad \cdots\langle n_{r\sigma(r)}, \gamma_{r\sigma(r)}, \delta_{r\sigma(r)}\rangle\cdots\langle n_{m\sigma(m)}, \gamma_{m\sigma(m)}, \delta_{m\sigma(m)}\rangle \\
&= \sum_{\sigma\in S}\langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)}\rangle\langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)}\rangle\cdots \\
&\quad \cdots k\langle m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \beta_{r\sigma(r)}\rangle\cdots\langle m_{m\sigma(m)}, \alpha_{m\sigma(m)}, \beta_{m\sigma(m)}\rangle \\
&= k\sum_{\sigma\in S}\langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)}\rangle\langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)}\rangle\cdots \\
&\quad \cdots\langle m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \beta_{r\sigma(r)}\rangle\cdots\langle m_{m\sigma(m)}, \alpha_{m\sigma(m)}, \beta_{m\sigma(m)}\rangle \\
&= kPer(A).
\end{aligned}$$

Proof is similar for $m > n$. □

Proposition 5.4. *If any two rows (or columns) of a TFM A are interchanged, then permanent value remains unchanged.*

Proof. Let $A = [a_{ij}]_{m\times n}$ be a TFM of order $m \times n$ and $B = [b_{ij}]_{m\times n}$ is the TFM obtained from A by interchanging the rth and sth row ($r < s$) of A.

Then $b_{ij} = a_{ij}, i \neq r, j \neq s$ and $b_{rj} = a_{sj}$ and $b_{sj} = a_{rj}$.
Let $m \leq n$. Now,

$$\begin{aligned}
Per(B) &= \sum_{\sigma\in S} b_{1\sigma(1)}b_{2\sigma(2)}\cdots b_{r\sigma(r)}\cdots b_{s\sigma(s)}\cdots b_{m\sigma(m)} \\
&= \sum_{\sigma\in S} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{s\sigma(s)}\cdots a_{r\sigma(r)}\cdots a_{m\sigma(m)} \\
&= \sum_{\sigma\in S}\langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)}\rangle\langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)}\rangle\cdots\langle m_{s\sigma(s)}, \alpha_{s\sigma(s)}, \\
&\quad \beta_{s\sigma(s)}\rangle\cdots\langle m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \beta_{r\sigma(r)}\rangle\cdots\langle m_{m\sigma(m)}, \alpha_{m\sigma(m)}, \beta_{m\sigma(m)}\rangle \\
&= \sum_{\sigma\in S}\langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)}\rangle\langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)}\rangle\cdots\langle m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \\
&\quad \beta_{r\sigma(r)}\rangle\cdots\langle m_{s\sigma(s)}, \alpha_{s\sigma(s)}, \beta_{s\sigma(s)}\rangle\cdots\langle m_{m\sigma(m)}, \alpha_{m\sigma(m)}, \beta_{m\sigma(m)}\rangle \\
&= Per(A).
\end{aligned}$$

For $m > n$, the proof is similar as before. □

Proposition 5.5. *If A^T is the transpose of A, then $Per(A^T) = Per(A)$.*

Proof. Let $A = [a_{ij}]_{m\times n}$ be a TFM and let $A^T = B = [b_{ij}]_{m\times n}$
then $b_{ij} = a_{ji}$.
When $m > n$

$$\begin{aligned}
Per(A^T) &= Per(B) = \sum_{\sigma\in S} b_{1\sigma(1)}b_{2\sigma(2)}\cdots b_{n\sigma(n)} \\
&= \sum_{\sigma\in S} a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n} \\
&= \sum_{\sigma\in S} \langle m_{\sigma(1)1}, \alpha_{\sigma(1)1}, \beta_{\sigma(1)1}\rangle \langle m_{\sigma(2)2}, \alpha_{\sigma(2)2}, \beta_{\sigma(2)2}\rangle \\
&\quad \cdots \langle m_{\sigma(n)n}, \alpha_{\sigma(n)n}, \beta_{\sigma(n)n}\rangle \\
&= \sum_{\sigma\in S}\prod_{i=1}^{n} \langle m_{\sigma(j)j}, \alpha_{\sigma(j)j}, \beta_{\sigma(j)j}\rangle \\
&= Per(A).
\end{aligned}$$

For $m \le n$, the proof is similar. □

6. Conclusions

Permanents of both IVFM and TFM are defined where the matrices are not necessary to be square. Related properties are given with proper examples. Some propositions are stated and proved also. In future we should try to develop of the permanent of block fuzzy matrix as well as block IVFM.

Acknowledgements. Financial support offered by Council of Scientific and Industrial Research, New Delhi, India (Sanction no. 09/599(0051)/2012-EMR-I) is thankfully acknowledged. Authors would like to acknowledge the anonymous referees and chief editor of the journal for their kind support to improve the presentation of the article significantly.

References

[1] A. K. Adak, M. Bhowmik and M. Pal, Some properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice, Fuzzy Inf. and Eng. 4 (4) (2012) 371–387.
[2] A. K. Adak, M. Bhowmik and M. Pal, Intuitionistic fuzzy block matrix and its some properties, Annals of Pure and Applied Mathematics 1 (1) (2012) 13–31.
[3] M. Bhowmik, M. Pal and A. Pal, Circulant triangular fuzzy number matrices, Journal of Physical Sciences 12 (2008) 141–154.
[4] M.Bhowmik and M.Pal, Some results on generalized interval-valued intuitionistic fuzzy sets, International Journal of Fuzzy Systems 14 (2) (2012) 193–203.
[5] S. H. Chen, Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy Sets and Systems 17 (1985) 113–129.
[6] D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, London 1980.
[7] H. Hasimoto, Canonical form of a transitive fuzzy matrix, Fuzzy Sets and Systems 11 (1983) 157–162.
[8] H. Hashimoto, Convergence of powers of a fuzzy transitive matrix, Fuzzy Sets and Systems 9 (1983) 153–160.
[9] S. K. Khan and M. Pal, Interval-valued intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets 11(1) (2005) 16–27.

[10] J.B. Kim, A certain matrix semigroup, Math. Japonica 22 (1978) 519–522.
[11] K. H. Kim and F. W. Roush, Generalised fuzzy matrices, Fuzzy Sets and Systems 4 (1980) 293–315.
[12] A. R. Meenakshi and M. Kaliraja, An application of interval-valued fuzzy matrices in medical diagnosis, Int. Journal of Math. Analysis 5 (36) (2011) 1791–1802.
[13] S. Mondal and M. Pal, Intuitionistic fuzzy incline matrix and determinant, Ann. Fuzzy Math. Inform. 8 (1) (2014) 19–32.
[14] A. Pal and M. Pal, Some results on interval-valued fuzzy matrices, The 2010 International Conference on E-Business Intelligence, Org. by Tsinghua University, Kunming, China, Atlantis Press (2010) 554–559. doi:10.2991/icebi.2010.39.
[15] M. Pal, Intuitionistic fuzzy determinant, V.U.J. Physical Sciences 7 (2001) 87–93.
[16] M. Pal, S. K. Khan and A. K. Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets 8 (2) (2002) 51–62.
[17] R. Pradhan and M. Pal, Intuitionistic fuzzy linear transformations, Annals of Pure and Applied Mathematics 1 (1) (2012) 57–68.
[18] R. Pradhan and M. Pal, Generalized inverse of block intuitionistic fuzzy matrices, International Journal of Applications of Fuzzy Sets and Artificial Intelligence 3 (2013) 23–38.
[19] E. Romanowicz and A. Grabowski, On the permanent of a matrix, Formalized Mathematics 14 (1) (2006) 13–20.
[20] A. K. Shyamal and M. Pal, Distances between intuitionistic fuzzy matrices, V.U.J. Physical Sciences 8 (2002) 81–91.
[21] A. K. Shyamal and M. Pal, Two new operations on fuzzy matrices, Journal of Applied Mathematics and Computing 15(1-2) (2004) 91–107.
[22] A. K. Shyamal and M. Pal, Interval valued fuzzy matrices, J. Fuzzy Math. 14 (3) (2006) 582–592.
[23] A. K. Shyamal and M. Pal, Triangular fuzzy matrices, Iranian Journal of Fuzzy System 4(1) (2007) 75–87.
[24] M. G. Thomason, Convergence of powers of a fuzzy matrix, J. Math. Anal. Appl. 57 (1977) 476–480.
[25] Liu Wang-jin and Yi Liang-Zhong, The permanent of L-fuzzy matrices and some applications,(1990).

Anjana Das (1988.anja@gmail.com)
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721102, India

Madhumangal Pal (mmpalvu@gmail.com)
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721102, India

Monoranjan Bhowmik (mbvttc@gmail.com)
Department of Mathematics, V. T. T. College, Midnapore, Paschim Medinipur – 721101, India