Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 3, (September 2015), pp. 361–379
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

A generalization of rough sets via filter by using increasing and $\mathcal{I}$-increasing sets

A. Kandil, O. Tantawy, S. A. El-Sheikh, M. Hosny

Received 12 January 2015; Revised 9 February 2015; Accepted 9 March 2015

Abstract. The main aim of this paper is to present another notion of generalized rough set by using filters, increasing and $\mathcal{I}$-increasing sets. The important of the current results is reducing the boundary region by increasing the lower approximation and decreasing the upper approximation which is the main aim of rough set. Moreover, the properties of the new lower and upper approximations are obtained. Comparisons between the current approximations and the previous approximations are introduced.

2010 AMS Classification: 06A06, 06B30, 54A05, 54F05, 54H99.

Keywords: Increasing sets, $\mathcal{I}$-increasing sets, Rough sets, Lower and upper approximations, Ideal, Filter.

Corresponding Author: M. Hosny (`moona_hosny@yahoo.com`)

1. Introduction

In the early of 1982, Pawlak [18] had proposed rough set theory. It has achieved a large amount of applications in various real-life fields, like economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, medicine, pharmacology, banking, market research, engineering, speech recognition, material science, information analysis, data analysis, data mining, linguistics, networking and other fields can be found in [12, 14, 17].

Rough set is dealing with vagueness (ambiguous) of the set by using the concept of the lower and upper approximations [18]. The set with the same lower and upper approximations, called crisp (exact) set, otherwise known as rough (inexact) set. The boundary region is defined as the difference between the upper and lower approximations, and then the accuracy of the set or ambiguous depending on the boundary region is empty or not respectively. A nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely. The main aim of rough set is reducing the boundary region by increasing the lower approximation and decreasing the upper approximation.

The standard rough set theory was based on an equivalence relation on a finite universe X. Various generalized rough set models have been established and their properties or structures have been investigated intensively [2, 11, 19, 20]. An interesting and natural research topic in rough set theory is to study rough set theory via topology [1, 9, 10, 12, 13]. The original rough set theory does not consider attributes with preference-ordered domains, that is, criteria. In fact, in many real-world situations, we are often faced with the problems in which the ordering of properties of the considered attributes plays a crucial role. Recently, in [4, 7, 8] had studied rough set theory via ordered topology.

This paper concerns with investigate another notion of generalized rough set by using filters, ideals and ordered relation. We consider the filter $\mathfrak{F}^*_R$ which is generated by the after-fore sets $\xi^* = \{RxR : x \in X\}$ that has a nonempty finite intersection. In addition, we use a partially order relation to construct the increasing and decreasing sets and also use ideal to construct the $\mathcal{I}$-increasing and $\mathcal{I}$-decreasing sets. Hence, we define the lower and upper approximations. Some examples are given to illustrate the new lower and upper approximations. Moreover, the main properties of lower and upper approximations are obtained and compared to the previous approximations [4, 7]. It is therefore shown that the current approximations are more generally.

2. Preliminaries

In this section, the needed definitions and results are given.

Definition 2.1 ([3, 12])**.** If R is a binary relation on X and $A \subseteq X$, then

(1) the after set of $x \in X$ is denoted by xR, where $xR = \{y \in X : xRy\}$.
(2) the fore set of $x \in X$ is denoted by Rx, where $Rx = \{y \in X : yRx\}$.
(3) the fore-fore set of $x \in X$ is denoted by RxR, where $RxR = xR \cap Rx$.

Definition 2.2 ([16])**.** Let (X, R) be a poset. A set $A \subseteq X$ is said to be:

(1) decreasing if for every $a \in A$ and $x \in X$ such that xRa, then $x \in A$.
(2) increasing if for every $a \in A$ and $x \in X$ such that aRx, then $x \in A$.

Theorem 2.1 ([16])**.** *Let (X, R) be a poset and $A \subseteq X$. Then, the class of all increasing (decreasing) sets forms a topology on X which is denoted by $\tau_{inc}(\tau_{dec})$.*

Definition 2.3 ([15])**.** Let $\mathfrak{F}$ be a non-empty collection of subsets of X. Then, $\mathfrak{F}$ is called a filter if it satisfies the following conditions:

(1) $\phi \notin \mathfrak{F}$.
(2) $A_1, A_2 \in \mathfrak{F} \Rightarrow A_1 \cap A_2 \in \mathfrak{F}$.
(3) $A \in \mathfrak{F}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathfrak{F}$.

Definition 2.4 ([15])**.** Let $\mathfrak{B}$ be a non-empty collection of subsets of X. Then, $\mathfrak{B}$ is called a filter base if it satisfies the following conditions:

(1) $\phi \notin \mathfrak{B}$.
(2) $B_1, B_2 \in \mathfrak{B} \Rightarrow \exists B_3 \in \mathfrak{B} : B_3 \subseteq B_1 \cap B_2$.

A filter base $\mathfrak{B}$ can be turned into a filter by including all sets of $P(X)$ which contains a set of $\mathfrak{B}$, i.e., $\mathfrak{F}_{\mathfrak{B}} = \{A \in P(X) : A \supseteq B, B \in \mathfrak{B}\}$.

Definition 2.5 ([15])**.** Let ξ be a non-empty collection of subsets of X. Then, ξ is called a filter-subbases on X if it satisfies the finite intersection property, i.e., any finite subcollection of ξ has a non empty intersection.

Definition 2.6 ([6])**.** A non-empty collection $\mathcal{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions:

(1) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2.7 ([5])**.** Let (X, R) be a poset and $\mathcal{I}$ be an ideal on X. Then, a set $A \subseteq X$ is called:

(1) $\mathcal{I}$-decreasing set iff $Ra \cap A^{'} \in \mathcal{I}\ \forall a \in A$.
(2) $\mathcal{I}$-increasing set iff $aR \cap A^{'} \in \mathcal{I}\ \forall a \in A$.

Proposition 2.1 ([5])**.** *For every ideal $\mathcal{I}$ on X, any increasing set is $\mathcal{I}$-increasing set.*

Theorem 2.2 ([5])**.** *Let (X, R) be a poset, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then, $\tau_{\mathcal{I}-inc} = \{A \subseteq X : A \text{ is } \mathcal{I}\text{-inc set }\}$ is a topology on X, which is finer than the topology that is generated by the increasing sets. In other words, $\tau_{inc} \subseteq \tau_{\mathcal{I}-inc}$.*

Definition 2.8 ([18])**.** Let R be an equivalence relation on a finite universe X and $A \subseteq X$. Then, the lower and upper approximations respectively are defined:
$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}$.
$\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\}$.
Boundary, positive and negative regions are also defined:
$BN_R(A) = \overline{R}(A) - \underline{R}(A)$.
$POS_R(A) = \underline{R}(A)$.
$NEG_R(A) = X - \overline{R}(A)$.

Definition 2.9 ([4])**.** A triple (X, τ_R, ρ) is called an order topological approximation space "OTAS", where τ_R is the topology generated by any relation R and ρ is a partially order relation,

Definition 2.10 ([4]). Let (X, τ_R, ρ) be an OTAS , $A \subseteq X$. Then, the lower (respectively upper) approximation is given by:
$\underline{R}_{inc}(A) = \cup\{G \in \tau_R : G$ is an increasing, $G \subseteq A\}$.
$\underline{R}_{dec}(A) = \cup\{G \in \tau_R : G$ is a decreasing, $G \subseteq A\}$.
$\overline{R}^{inc}(A) = \cap\{F \in \tau_R^{'} : F$ is an increasing, $A \subseteq F\}$.
$\overline{R}^{dec}(A) = \cap\{F \in \tau_R^{'} : F$ is a decreasing, $A \subseteq F\}$.
$BN_{inc}(A) = \overline{R}^{inc}(A) \setminus \underline{R}_{inc}(A)$.
$BN_{dec}(A) = \overline{R}^{dec}(A) \setminus \underline{R}_{dec}(A)$.
$\alpha^{inc}(A) = \frac{|\underline{R}_{inc}(A)|}{|\overline{R}^{inc}(A)|}$.
$\alpha^{dec}(A) = \frac{|\underline{R}_{dec}(A)|}{|\overline{R}^{dec}(A)|}$, α^{inc} is an increasing accuracy and α^{dec} is a decreasing accuracy.

Definition 2.11 ([7]). A quadrable $(X, \tau_R, \rho, \mathcal{I})$ is said to be ideal order topological approximation space (IOTAS, for short), where τ_R is a topology generated by any relation R, ρ is a partially order relation and $\mathcal{I}$ is an ideal on X.

Definition 2.12 ([7]). Let $(X, \tau_R, \rho, \mathcal{I})$ be an IOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:
$\underline{R}_{\mathcal{I}-inc}(A) = \cup\{G \in \tau_R : G$ is an $\mathcal{I}$-increasing, $G \subseteq A\}$.
$\underline{R}_{\mathcal{I}-dec}(A) = \cup\{G \in \tau_R : G$ is an $\mathcal{I}$-decreasing, $G \subseteq A\}$.
$\overline{R}^{\mathcal{I}-inc}(A) = \cap\{F \in \tau_R^{'} : F$ is an $\mathcal{I}$-increasing, $A \subseteq F\}$.
$\overline{R}^{\mathcal{I}-dec}(A) = \cap\{F \in \tau_R^{'} : F$ is an $\mathcal{I}$-decreasing, $A \subseteq F\}$.
$BN_{\mathcal{I}-inc}(A) = \overline{R}^{\mathcal{I}-inc}(A) \setminus \underline{R}_{\mathcal{I}-inc}(A)$.
$BN_{\mathcal{I}-dec}(A) = \overline{R}^{\mathcal{I}-dec}(A) \setminus \underline{R}_{\mathcal{I}-dec}(A)$.
$\alpha^{\mathcal{I}-inc}(A) = \frac{|\underline{R}_{\mathcal{I}-inc}(A)|}{|\overline{R}^{\mathcal{I}-inc}(A)|}$.
$\alpha^{\mathcal{I}-dec}(A) = \frac{|\underline{R}_{\mathcal{I}-dec}(A)|}{|\overline{R}^{\mathcal{I}-dec}(A)|}$, $\alpha^{\mathcal{I}-inc}$ is an $\mathcal{I}$-increasing accuracy and $\alpha^{\mathcal{I}-dec}$ is an $\mathcal{I}$-decreasing accuracy.

Definition 2.13 ([7]). A triple $(X, \mathfrak{F}_R, \rho)$ is said to be generalized order topological approximation space (GOTAS, for short), where $\mathfrak{F}_R$ is a filter generated by any relation R and ρ is a partially ordered relation.

Definition 2.14 ([7]). Let $(X, \mathfrak{F}_R, \rho)$ be a GOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:

$$R_{*inc}(A) = \cup\{G \in \mathfrak{F}_R : G \textit{ is an increasing}, G \subseteq A\}.$$

$$R_{*dec}(A) = \cup\{G \in \mathfrak{F}_R : G \textit{ is a decreasing}, G \subseteq A\}.$$

$$R^{*inc}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}_R^{'} : H \textit{ is an increasing}, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}_R^{'} : \text{H is an increasing}, A \subseteq H. \end{cases}$$

$$R^{*dec}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}_R^{'} : H \text{ } is \text{ } a \text{ } decreasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}_R^{'} : \text{H is a decreasing}, A \subseteq H. \end{cases}$$

$$BN_{*inc}(A) = R^{*inc}(A) \setminus R_{*inc}(A).$$

$$BN_{*dec}(A) = R^{*dec}(A) \setminus R_{*dec}(A).$$

$$\alpha^{*inc}(A) = \frac{|R_{*inc}(A)|}{|R^{*inc}(A)|}.$$

$$\alpha^{*dec}(A) = \frac{|R_{*dec}(A)|}{|R^{*dec}(A)|}.$$

Definition 2.15 ([7]). A quadrable $(X, \mathfrak{F}_R, \rho, \mathcal{I})$ is said to be generalized ideal order topological approximation space (GIOTAS, for short), where $\mathfrak{F}_R$ is a filter generated by any relation R, ρ is a partially order relation and $\mathcal{I}$ an ideal on X.

Definition 2.16 ([7]). Let $(X, \mathfrak{F}_R, \rho, \mathcal{I})$ be a GIOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy of a set A with respect to a relation R by using the notion of $\mathcal{I}$-increasing and $\mathcal{I}$-decreasing sets are given by:

$$R_{*\mathcal{I}-inc}(A) = \cup\{G \in \mathfrak{F}_R : G \text{ } is \text{ } an \text{ } \mathcal{I} - increasing, G \subseteq A\}.$$

$$R_{*\mathcal{I}-dec}(A) = \cup\{G \in \mathfrak{F}_R : G \text{ } is \text{ } an \text{ } \mathcal{I} - decreasing, G \subseteq A\}.$$

$$R^{*\mathcal{I}-inc}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}_R^{'} : H \text{ } is \text{ } an \text{ } \mathcal{I} - increasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}_R^{'} : \text{H is an } \mathcal{I} - \text{increasing}, A \subseteq H. \end{cases}$$

$$R^{*\mathcal{I}-dec}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}_R^{'} : H \text{ is an } \mathcal{I} - decreasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}_R^{'} : \text{H is an } \mathcal{I} - \text{decreasing}, A \subseteq H. \end{cases}$$

$$BN_{*\mathcal{I}-inc}(A) = R^{*\mathcal{I}-inc}(A) \setminus R_{*\mathcal{I}-inc}(A).$$

$$BN_{*\mathcal{I}-dec}(A) = R^{*\mathcal{I}-dec}(A) \setminus R_{*\mathcal{I}-dec}(A).$$

$$\alpha^{*\mathcal{I}-inc}(A) = \frac{|R_{*\mathcal{I}-inc}(A)|}{|R^{*\mathcal{I}-inc}(A)|}.$$

$$\alpha^{*\mathcal{I}-dec}(A) = \frac{|R_{*\mathcal{I}-dec}(A)|}{|R^{*\mathcal{I}-dec}(A)|}.$$

Lemma 2.1 ([7]). *Let R be a binary relation on X. Then, $\tau_R \setminus \phi \subseteq \mathfrak{F}_R$, where τ_R is the topology generated by the subbase $\xi = \{xR : x \in X\}$ and $\mathfrak{F}_R$ is a filter generated by the same subbase.*

3. Generalized rough sets via filters by using increasing and decreasing sets

The goal of this section is to introduce a new notion of lower and upper approximations to decrease the boundary region and increase the accuracy of sets. This new notion is generated by using filters and increasing (decreasing) sets. We consider the filter which is generated by the after-fore sets that has a nonempty finite intersection. To construct the filter $\mathfrak{F}^*_R$, let $\xi^* = \{RxR : x \in X\}$ be a subbase of a filter $\mathfrak{F}^*_R$ also we use partially order relation to construct the increasing and decreasing sets and hence define the lower and upper approximation by using the increasing and decreasing sets. The current approximations are compared with El-Shafei et al.'s approximations [4] and Kandil et al.'s approximations [7].

Definition 3.1. A triple $(X, \mathfrak{F}^*_R, \rho)$, is said to be generalized order topological approximation space (GOTAS, for short), where $\mathfrak{F}^*_R$ is a filter generated by any relation R and ρ is a partially ordered relation.

Definition 3.2. Let $(X, \mathfrak{F}^*_R, \rho)$ be a GOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:

$$R_{**inc}(A) = \cup\{G \in \mathfrak{F}^*_R : G \; is \; an \; increasing, G \subseteq A\}.$$
$$R_{**dec}(A) = \cup\{G \in \mathfrak{F}^*_R : G \; is \; a \; decreasing, G \subseteq A\}.$$
$$R^{**inc}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}^{*'}_R : H \; is \; an \; increasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}^{*'}_R : \text{H is an increasing}, A \subseteq H. \end{cases}$$
$$R^{**dec}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}^{*'}_R : H \; is \; a \; decreasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}^{*'}_R : \text{H is a decreasing}, A \subseteq H. \end{cases}$$
$$BN_{**inc}(A) = R^{**inc}(A) \setminus R_{**inc}(A).$$
$$BN_{**dec}(A) = R^{**dec}(A) \setminus R_{**dec}(A).$$
$$\alpha^{**inc}(A) = \frac{|R_{**inc}(A)|}{|R^{**inc}(A)|}.$$
$$\alpha^{**dec}(A) = \frac{|R_{**dec}(A)|}{|R^{**dec}(A)|}.$$

The following Lemma 3.1 presents the relationship between the filters $\mathfrak{F}^*_R$, $\mathfrak{F}_R$ and topology τ_R which is necessary to prove Propositions 3.1 and 3.2.

Lemma 3.1. *Let $\xi = \{xR : x \in X\}$ be a subbase of the filter $\mathfrak{F}_R$ and $\xi^* = \{RxR : x \in X\}$ be a subbase of the filter $\mathfrak{F}^*_R$. Then, we have*

(1) *if $\mathfrak{B}$ is a filterbase for $\mathfrak{F}_R$ and $B_x \in \mathfrak{B}$, then $\exists B^*_x \in \mathfrak{B}^*$ which is a filterbase for $\mathfrak{F}^*_R$, such that $B^*_x \subseteq B_x$.*

(2) $\mathfrak{F}_R \subseteq \mathfrak{F}^*_R$.

(3) $\tau_R \setminus \phi \subseteq \mathfrak{F}^*_R$.

Proof. (1) Let $\mathfrak{B}$ be a filterbase for $\mathfrak{F}_R$ and $B_x \in \mathfrak{B}$. Then, we have two cases:
Case 1 if $B_x = xR$, then $\exists\ B^*_x = RxR \in \mathfrak{B}^*$ such that $RxR = B^*_x \subseteq B_x = xR$
Case 2 if $B_x = \cap_{x\in X}(xR)$, then $\exists\ B^*_x = \cap_{x\in X}(RxR) \in \mathfrak{B}^*$ such that $\cap_{x\in X}(RxR) = B^*_x \subseteq B_x = \cap_{x\in X}(xR)$

(2) Let $A \in \mathfrak{F}_R$. Then, $\exists\ B_x \in \mathfrak{B}$ such that $B_x \subseteq A$
$\Rightarrow \exists\ B^*_x \in \mathfrak{B}^*$ such that $B^*_x \subseteq B_x$
$\Rightarrow B^*_x \subseteq B_x \subseteq A$
$\Rightarrow A \in \mathfrak{F}^*_R$. Hence, $\mathfrak{F}_R \subseteq \mathfrak{F}^*_R$.

(3) From Lemma 2.1 $\tau_R \setminus \phi \subseteq \mathfrak{F}_R$. Hence, $\tau_R \setminus \phi \subseteq \mathfrak{F}_R \subseteq \mathfrak{F}^*_R$.

□

The following proposition presents the relationship between the current approximations and the approximations in [4] (Definition 2.10).

Proposition 3.1. *Let (X, τ_R, ρ) be an OTAS, $(X, \mathfrak{F}^*_R, \rho)$ be a GOTAS and $A \subseteq X$. Then,*

(1) $\underline{R}_{inc}(A) \subseteq R_{**inc}(A)$ $(\underline{R}_{dec}(A) \subseteq R_{**dec}(A))$.
(2) $R^{**inc}(A) \subseteq \overline{R}_{inc}(A)$ $(R^{**dec}(A) \subseteq \overline{R}_{dec}(A))$.
(3) $BN_{**inc}(A) \subseteq BN_{inc}(A)$ $(BN_{**dec}(A) \subseteq BN_{dec}(A))$.
(4) $\alpha^{**inc}(A) \geq \alpha^{inc}(A)$ $(\alpha^{**dec}(A) \geq \alpha^{dec}(A))$.

Proof. The proof is straightforward from Definitions 2.10, 3.2 and Lemma 3.1. □

The following proposition presents the relationship between the current approximations and Kandil et al.'s approximations [7] (Definition 2.14).

Proposition 3.2. *Let $(X, \mathfrak{F}_R, \rho)$ be a GOTAS, $(X, \mathfrak{F}^*_R, \rho)$ be a GOTAS and $A \subseteq X$. Then,*

(1) $R_{*inc}(A) \subseteq R_{**inc}(A)$ $(R_{*dec}(A) \subseteq R_{**dec}(A))$.
(2) $R^{**inc}(A) \subseteq R^{*inc}(A)$ $(R^{**dec}(A) \subseteq R^{*dec}(A))$.
(3) $BN_{**inc}(A) \subseteq BN_{*inc}(A)$ $(BN_{**dec}(A) \subseteq BN_{*dec}(A))$.
(4) $\alpha^{**inc}(A) \geq \alpha^{*inc}(A)$ $(\alpha^{**dec}(A) \geq \alpha^{*dec}(A))$.

Proof. The proof is straightforward from Definitions 2.14 [7], 3.2 and Lemma 3.1. □

Propositions 3.1 and 3.2 show that the current method in Definition 3.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of El-Shafei et al.'s method in Definition 2.10 [4] and Kandil et al.'s method in Definition 2.14 [7]. Moreover, it shows that the current accuracy in Definition 3.2 is greater than the previous one in [4, 7].

The following example is computed the lower, upper approximations, boundary region and accuracy for all subset of X by using El-Shafei et al.'s Definition 2.10 [4], Kandil et al.'s Definition 2.14 [7] and the present method in Definition 3.2.

Example 3.1. Let $X = \{a, b, c, d\}, \rho = \Delta \cup \{(a, c), (a, d), (b, c), (d, c)\}$, and
$R = \{(b, b), (c, c), (d, d), (a, b), (a, d), (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c)\}$.
Then, $\xi = \mathfrak{B} = \{\{b, d\}, \{b, c, d\}, X\}, \tau_R = \{X, \phi, \{b, d\}, \{b, c, d\}\}$,
$\mathfrak{F}_R = \{\{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}, \xi^* = \{\{d\}, \{b, c, d\}, X\}, \mathfrak{B}^* = \{\{d\}, \{b, c, d\}, X\}$
and $\mathfrak{F}^*_R = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Table 1. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.10 [4], A.Kandil et al.'s method in Definition 2.14[7] and the present Definition 3.2 in the case of increasing sets.

A	El-Shafei et al.'s method in Definition 2.10 [4]				Kandil et al.'s method in Definition 2.14 [7]				The current method in Definition 3.2			
	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$BN_{inc}(A)$	$\alpha^{inc}(A)$	$R_{*inc}(A)$	$R^{*inc}(A)$	$BN_{*inc}(A)$	$\alpha^{*}{}_{inc}(A)$	$R_{**inc}(A)$	$R^{**inc}(A)$	$BN_{**inc}(A)$	$\alpha^{**}{}_{inc}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{c\}$	ϕ	X	X	0	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{b,c\}$	$\{b,c\}$	0
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0	$\{c,d\}$	X	$\{a,b\}$	0.5
$\{a,b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	$\{a,c,d\}$	X	$\{b\}$	0.75
$\{b,c,d\}$	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

For example, take $A=\{a,c,d\}$, then the boundary and accuracy by the present Definition 3.2 are $\{b\}$ and 0.75 respectively. Whereas the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.10 [4] are X and 0 respectively and by using Kandil et al.'s method in Definitions 2.14 [7] are also X and 0 respectively.

TABLE 2. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.10 [4], Kandil et al.'s method in Definitions 2.14 [7] and the present Definition 3.2 in the case of decreasing sets.

A	El-Shafei et al.'s method in Definition 2.10 [4]				Kandil et al.'s method in Definition 2.14 [7]				The current method in Definition 3.2			
	$\underline{R}_{dec}(A)$	$\overline{R}^{dec}(A)$	$BN_{dec}(A)$	$\alpha^{dec}(A)$	$R_{*dec}(A)$	$R^{*dec}(A)$	$BN_{*dec}(A)$	$\alpha^{*dec}(A)$	$R_{**dec}(A)$	$R^{**dec}(A)$	$BN_{**dec}(A)$	$\alpha^{**dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0	$\{a,d\}$	$\{a,d\}$	ϕ	1
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,d\}$	ϕ	X	X	0	$\{a,b,d\}$	X	$\{c\}$	0.75	$\{a,b,d\}$	X	$\{c\}$	0.75
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

The following Proposition 3.3 studies the main properties of the current lower and upper approximations.

Proposition 3.3. *Let $(X, \mathfrak{F}^*_R, \rho)$ be a GOTAS and $A, B \subseteq X$. Then,*

(1) $R_{**inc}(A) \subseteq A \subseteq R^{**inc}(A)$ $(R_{**dec}(A) \subseteq A \subseteq R^{**dec}(A))$, *equality hold if* $A = \phi$ *or* X.
(2) $A \subseteq B \Rightarrow R^{**inc}(A) \subseteq R^{*inc}(B)$ $(R^{*inc}(A) \subseteq R^{**dec}(B))$.
(3) $A \subseteq B \Rightarrow R_{**inc}(A) \subseteq R_{**inc}(B)$ $(R_{**dec}(A) \subseteq R_{**dec}(B))$.
(4) $R^{**inc}(A \cap B) \subseteq R^{**inc}(A) \cap R^{**inc}(B)$ $(R^{**dec}(A \cap B) \subseteq R^{**dec}(A) \cap R^{**dec}(B))$.
(5) $R_{**inc}(A \cup B) \supseteq R_{**inc}(A) \cup R_{**inc}(B)$ $(R_{**dec}(A \cup B) \supseteq R_{**dec}(A) \cap R_{**dec}(B))$.
(6) $R^{**inc}(A \cup B) = R^{**inc}(A) \cup R^{**inc}(B)$ $(R^{**dec}(A \cup B) = R^{**dec}(A) \cup R^{*dec}(B))$.
(7) $R_{**inc}(A \cap B) = R_{**inc}(A) \cap R_{**inc}(B)$ $(R_{**dec}(A \cap B) = R_{**dec}(A) \cap R_{**dec}(B))$.
(8) $x \in R^{**inc}(A) \Leftrightarrow G \cap A \neq \phi, \forall\, G \in \mathfrak{F}^*_R, G$ *is a decreasing set containing* x. *(*$x \in R^{**dec}(A) \Leftrightarrow G \cap A \neq \phi, \forall\, G \in \mathfrak{F}^*_R, G$ *is an increasing set containing* x*).*
(9) $x \in R_{**inc}(A) \Leftrightarrow \exists G \in \mathfrak{F}^*_R, G$ *is an increasing set containing* x *such that* $G \subseteq A$.
(10) $R^{**inc}(R^{**inc}(A)) = R^{**inc}(A)$ $(R^{**dec}(R^{**dec}(A)) = R^{**dec}(A))$.
(11) $R_{**inc}(R_{**inc}(A)) = R_{**inc}(A)$ $(R_{**dec}(R_{**dec}(A)) = R_{**dec}(A))$.

Proof. **1.:** Straightforward.

2.: Let $x \notin R^{**inc}(B)$. Then, $\exists\, F \in \mathfrak{F}^{*'}_R, F$ is an increasing, $F \supseteq B \supseteq A, x \notin F \Rightarrow x \notin R^{**inc}(A)$.

3.: Similar to part 2.

4.: It is directly from part 2.

5.: It is directly from part 3.

6.: $R^{**inc}(A \cup B) \supseteq R^{**inc}(A) \cup R^{**inc}(B)$ (by part 4) and to prove $R^{**inc}(A \cup B) \subseteq R^{**inc}(A) \cup R^{**inc}(B)$, let $x \notin R^{**inc}(A) \cup R^{**inc}(B)$. Then, $x \notin R^{**inc}(A)$ and $x \notin R^{**inc}(B) \Rightarrow \exists F_1, F_2 \in \mathfrak{F}^{*'}_R, F_1, F_2$ are increasing, such that $x \notin F_1, F_1 \supseteq A, x \notin F_2, F_2 \supseteq B \Rightarrow x \notin F_1 \cup F_2$, (which is an increasing by Theorem 2.1), $F_1 \cup F_2 \supseteq A \cup B \Rightarrow x \notin R^{**inc}(A \cup B)$. Then, $R^{**inc}(A \cup B) \subseteq R^{**inc}(A) \cup R^{**inc}(B)$. Hence, $R^{**inc}(A \cup B) = R^{**inc}(A) \cup R^{**inc}(B)$.

7.: Similar to No. 6.

8.: "$\Rightarrow$" Let $x \in R^{**inc}(A)$. Then, there exists $G \in \mathfrak{F}^*_R, G$ is a decreasing set containing x such that $G \cap A = \phi$.
$\Rightarrow A \subseteq G', G' \in \mathfrak{F}^{*'}_R$ which is an increasing set.
$\Rightarrow R^{**inc}(A) \subseteq G'$.

$\Rightarrow R^{**inc}(A) \cap G = \phi$, which is a contradiction.

"$\Leftarrow$" Let $G \cap A \neq \phi, \forall\ G \in \mathfrak{F}^*_R, G$ is a decreasing set containing x, $x \notin R^{**inc}(A)$.

$\Rightarrow x \in R^{**inc'}(A) \in \mathfrak{F}^*_R, R^{**inc'}(A)$ is a decreasing set.

$\Rightarrow R^{**inc'}(A) \cap A \neq \phi$, which is a contradiction as $(A \subseteq R^{**inc}(A))$.

9.: It is directly from Definition 3.2.

10.: It is directly from parts 1 and 2.

11.: It is directly from parts 1 and 3.

□

Example 3.1 shows that the inclusion in Proposition 3.3 parts 1, 4 and 5 can not be replaced by equality relation (for part 1, if $A = \{d\}, R^{**inc}(A) = X, R_{**inc}(A) = \phi$, also if $A = \{d\}, R^{**dec}(A) = X, R_{**dec}(A) = \phi$. Then, $R^{**inc}(A) \nsubseteq A \nsubseteq R_{**inc}(A)$ and also $R^{**dec}(A) \nsubseteq A \nsubseteq R_{**dec}(A)$). In a similar way, we can add examples to part 4 and 5). Moreover, the converse of parts 2 and 3 is not necessarily true (i.e., $R_{**inc}(A) \subseteq R_{**inc}(B) \nRightarrow A \subseteq B$, take $A = \{a, b, c\}, B = \{b, c, d\}$, then $R_{**inc}(A) = \phi, R_{**inc}(B) = \{b, c, d\}$. Therefore, $R_{**inc}(A) \subseteq R_{**inc}(B)$, but $A \nsubseteq B$. In a similar way, we can add examples to show that $R_{**dec}(A) \subseteq R_{**dec}(B)$ but $A \nsubseteq B$).

4. Generalized rough sets via filter by using $\mathcal{I}$-increasing and $\mathcal{I}$-decreasing sets

In this section, we use the $\mathcal{I}$-increasing and $\mathcal{I}$-decreasing sets instead of increasing and decreasing sets which are used in Section 3 to introduce a new notion of lower and upper approximations. Moreover, the main properties of the current lower and upper approximations are studied. Furthermore, comparisons between the current approximations in this section, Sections 3 and the previous approximations in [4] and [7] are introduced.

Definition 4.1. A quadrable $(X, \mathfrak{F}^*_R, \rho, \mathcal{I})$ is said to be generalized ideal order topological approximation space (GIOTAS, for short), where $\mathfrak{F}^*_R$ is a filter generated by any relation R, ρ is a partially ordered relation and $\mathcal{I}$ is an ideal.

Definition 4.2. Let $(X, \mathfrak{F}^*_R, \rho, \mathcal{I})$ be a GIOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy of a set A with respect to a relation R by using the notion of $\mathcal{I}$-increasing and $\mathcal{I}$-decreasing sets are given by:

$$R_{**\mathcal{I}-inc}(A) = \cup\{G \in \mathfrak{F}^*_R : G\ is\ an\ \mathcal{I}-increasing, G \subseteq A\}.$$

$$R_{**\mathcal{I}-dec}(A) = \cup\{G \in \mathfrak{F}^*_R : G\ is\ an\ \mathcal{I}-decreasing, G \subseteq A\}.$$

$$R^{**\mathcal{I}-inc}(A) = \begin{cases} & \cap\{H \in \mathfrak{F}^{*'}_R : H \text{ is an } \mathcal{I}-increasing, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}^{*'}_R : H\text{is an } \mathcal{I}-\text{increasing}, A \subseteq H. \end{cases}$$

$$R^{**\mathcal{I}-dec}(A)=\begin{cases} & \cap\{H\in\mathfrak{F}_R^{*'}: H \textit{ is an } \mathcal{I}-\textit{decreasing}, A\subseteq H\}.\\ X & \text{if not exists } H\in\mathfrak{F}_R^{*'}: \text{H is an } \mathcal{I}-\text{decreasing}, A\subseteq H.\end{cases}$$

$$BN_{*\mathcal{I}-inc}(A)=R^{**\mathcal{I}-inc}(A)\setminus R_{**\mathcal{I}-inc}(A).$$

$$BN_{**\mathcal{I}-dec}(A)=R^{**\mathcal{I}-dec}(A)\setminus R_{**\mathcal{I}-dec}(A).$$

$$\alpha^{**\mathcal{I}-inc}(A)=\frac{|R_{**\mathcal{I}-inc}(A)|}{|R^{**\mathcal{I}-inc}(A)|}.$$

$$\alpha^{**\mathcal{I}-dec}(A)=\frac{|R_{**\mathcal{I}-dec}(A)|}{|R^{**\mathcal{I}-dec}(A)|}.$$

The following proposition presents the relationship between El-Shafei et al.'s method in Definition 2.10 [4] and the current approximations in Definition 4.2.

Proposition 4.1. *Let (X,τ_R,ρ) be an OTAS, $(X,\mathfrak{F}_R^*,\rho,\mathcal{I})$ be a GIOTAS and $A\subseteq X$. Then,*

(1) $\underline{R}_{inc}(A)\subseteq R_{**\mathcal{I}-inc}(A)$ $(\underline{R}_{dec}(A)\subseteq R_{**\mathcal{I}-dec}(A))$.
(2) $R^{**\mathcal{I}-inc}(A)\subseteq \overline{R}^{inc}(A)$ $(R^{**\mathcal{I}-dec}(A)\subseteq \overline{R}^{dec}(A))$.
(3) $BN_{**\mathcal{I}-inc}(A)\subseteq BN_{inc}(A)$ $(BN_{*\mathcal{I}-dec}(A)\subseteq BN_{dec}(A))$.
(4) $\alpha^{**\mathcal{I}-inc}(A)\geq\alpha^{inc}(A)$ $(\alpha^{**\mathcal{I}-dec}(A)\geq\alpha^{dec}(A))$.

Proof. The proof is straightforward from Definitions 2.10 [4], 4.2, Propositions 2.1 and Lemma 3.1. □

The following proposition presents the relationship between A.Kandil et al.'s approximations in Definition 2.12 [7] and the current approximations in Definition 4.2.

Proposition 4.2. *Let $(X,\tau_R,\rho,\mathcal{I})$ be an IOTAS, $(X,\mathfrak{F}_R^*,\rho,\mathcal{I})$ be a GIOTAS and $A\subseteq X$. Then,*

(1) $\underline{R}_{\mathcal{I}-inc}(A)\subseteq R_{**\mathcal{I}-inc}(A)$ $(\underline{R}_{\mathcal{I}-dec}(A)\subseteq R_{**\mathcal{I}-dec}(A))$.
(2) $R^{**\mathcal{I}-inc}(A)\subseteq \overline{R}^{\mathcal{I}-inc}(A)$ $(R^{**\mathcal{I}-dec}(A)\subseteq \overline{R}^{\mathcal{I}-dec}(A))$.
(3) $BN_{**\mathcal{I}-inc}(A)\subseteq BN_{\mathcal{I}-inc}(A)$ $(BN_{**\mathcal{I}-dec}(A)\subseteq BN_{\mathcal{I}-dec}(A))$.
(4) $\alpha^{**\mathcal{I}-inc}(A)\geq\alpha^{\mathcal{I}-inc}(A)$ $(\alpha^{**\mathcal{I}-dec}(A)\geq\alpha^{\mathcal{I}-dec}(A))$.

Proof. The proof is straightforward from Definitions 2.12 [7], 4.2 and Lemma 3.1. □

The following proposition presents the relationship between Kandil et al.'s approximations in Definition 2.14 [7] and the current approximations in Definition 4.2.

Proposition 4.3. *Let $(X,\mathfrak{F}_R,\rho)$ be a GOTAS, $(X,\mathfrak{F}_R^*,\rho,\mathcal{I})$ be a GIOTAS and $A\subseteq X$. Then,*

(1) $R_{*inc}(A)\subseteq R_{**\mathcal{I}-inc}(A)$ $(R_{*dec}(A)\subseteq R_{**\mathcal{I}-dec}(A))$.
(2) $R^{**\mathcal{I}-inc}(A)\subseteq R^{*inc}(A)$ $(R^{**\mathcal{I}-dec}(A)\subseteq R^{*dec}(A))$.

(3) $BN_{**\mathcal{I}-inc}(A) \subseteq BN_{*inc}(A)$ $(BN_{**\mathcal{I}-dec}(A) \subseteq BN_{*dec}(A))$.
(4) $\alpha^{**\mathcal{I}-inc}(A) \geq \alpha^{*inc}(A)$ $(\alpha^{**\mathcal{I}-dec}(A) \geq \alpha^{*dec}(A))$.

Proof. The proof is straightforward from Definitions 2.14 [7], 4.2, Propositions 2.1 and Lemma 3.1. □

The following proposition presents the relationship between Kandil et al.'s approximations in Definition 2.16 [7] and the current approximations in Definition 4.2.

Proposition 4.4. *Let $(X, \mathfrak{F}_R, \rho, \mathcal{I})$ be a GIOTAS, $(X, \mathfrak{F}^*_R, \rho, \mathcal{I})$ be a GIOTAS and $A \subseteq X$. Then,*

(1) $R_{*\mathcal{I}-inc}(A) \subseteq R_{**\mathcal{I}-inc}(A)$ $(R_{*\mathcal{I}-dec}(A) \subseteq R_{**\mathcal{I}-dec}(A))$.
(2) $R^{**\mathcal{I}-inc}(A) \subseteq R^{*inc}(A)$ $(R^{**\mathcal{I}-dec}(A) \subseteq R^{*\mathcal{I}-dec}(A))$.
(3) $BN_{**\mathcal{I}-inc}(A) \subseteq BN_{*\mathcal{I}-inc}(A)$ $(BN_{**\mathcal{I}-dec}(A) \subseteq BN_{*\mathcal{I}-dec}(A))$.
(4) $\alpha^{**\mathcal{I}-inc}(A) \geq \alpha^{*\mathcal{I}-inc}(A)$ $(\alpha^{**\mathcal{I}-dec}(A) \geq \alpha^{*\mathcal{I}-dec}(A))$.

Proof. The proof is straightforward from Definitions 2.16 [7], 4.2 and Lemma 3.1. □

The following proposition presents the relationship between the present approximations in Definitions 3.2 and 4.2.

Proposition 4.5. *Let $(X, \mathfrak{F}^*_R, \rho)$ be a GOTAS, also $(X, \mathfrak{F}^*_R, \rho, \mathcal{I})$ be a GIOTAS and $A \subseteq X$. Then,*

(1) $R_{**inc}(A) \subseteq R_{**\mathcal{I}-inc}(A)$ $(R_{**dec}(A) \subseteq R_{**\mathcal{I}-dec}(A))$.
(2) $R^{**\mathcal{I}-inc}(A) \subseteq R^{**}{}_{inc}(A)$ $(R^{**\mathcal{I}-dec}(A) \subseteq R^{**}{}_{dec}(A))$.
(3) $BN_{**\mathcal{I}-inc}(A) \subseteq BN_{**inc}(A)$ $(BN_{**\mathcal{I}-dec}(A) \subseteq BN^{**}{}_{dec}(A))$.
(4) $\alpha_{**\mathcal{I}-inc}(A) \geq \alpha_{**inc}(A)$ $(\alpha_{**\mathcal{I}-dec}(A) \geq \alpha_{**dec}(A))$.

Proof. The proof is straightforward from Definitions 3.2, 4.2 and Proposition 2.1 and Lemma 3.1. □

Let $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$, in Example 3.1. Then, we calculate the lower, upper approximations, boundary region and accuracy for all subset of X by using El-Shafei et al.'s Definition 2.10 [4], A.Kandil et al.'s Definitions 2.12, 2.14, 2.16 [7] and the current approximations in Definitions 3.2, 4.2.

TABLE 3. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.10 [4], A.Kandil et al.'s approximations in Definitions 2.12, 2.14, 2.16 [7] the current approximations in Definitions 3.2,4.2 in the case of increasing ($\mathcal{I}$-increasing) set.

A	El-Shafei et al.'s method in Definition 2.10 [4]				Kandil et al.'s method in Definition 2.12 [7]				Kandil et al.'s method in Definition 2.14 [7]				A.Kandil et al.'s method in Definition 2.16 [7]				The current method in Definition 3.2				The current method in Definition 4.2			
	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$BN_{inc}(A)$	$\alpha^{inc}(A)$	$\underline{R}_{\mathcal{I}-inc}(A)$	$\overline{R}^{\mathcal{I}-inc}(A)$	$BN_{\mathcal{I}-inc}(A)$	$\alpha^{\mathcal{I}-inc}(A)$	$R_{*inc}(A)$	$R^{*inc}(A)$	$BN_{*inc}(A)$	$\alpha^{*}{}_{inc}(A)$	$R_{*\mathcal{I}-inc}(A)$	$R^{*\mathcal{I}-inc}(A)$	$BN_{*\mathcal{I}-inc}(A)$	$\alpha^{*}{}_{\mathcal{I}-inc}(A)$	$R_{**inc}(A)$	$R^{**inc}(A)$	$BN_{**inc}(A)$	$\alpha^{**}{}_{inc}(A)$	$R_{**\mathcal{I}-inc}(A)$	$R^{**\mathcal{I}-inc}(A)$	$BN_{**\mathcal{I}-inc}(A)$	$\alpha^{**}{}_{\mathcal{I}-inc}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	X	X	0	ϕ	$\{a\}$	$\{a\}$	0	ϕ	X	X	0	ϕ	$\{a\}$	$\{a\}$	0	ϕ	X	X	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	$\{d\}$	X	$\{a,b,c\}$	0.25
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	ϕ	X	X	0	ϕ	$\{a,c\}$	$\{a,c\}$	0	ϕ	X	X	0	ϕ	$\{a,c\}$	$\{a,c\}$	0	ϕ	X	X	0	ϕ	$\{a,c\}$	$\{a,c\}$	0
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{b,c\}$	$\{b,c\}$	0	ϕ	$\{b,c\}$	$\{b,c\}$	0
$\{b,d\}$	ϕ	X	X	0	$\{b,d\}$	X	$\{a,c\}$	0.5	ϕ	X	X	0	$\{b,d\}$	X	$\{a,c\}$	0.5	ϕ	X	X	0	$\{b,d\}$	X	$\{a,c\}$	0.5
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5
$\{a,b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{a,b,d\}$	ϕ	X	X	0	$\{b,d\}$	X	$\{a,c\}$	0.5	ϕ	X	X	0	$\{a,b,d\}$	X	$\{c\}$	0.75	ϕ	X	X	0	$\{a,b,d\}$	X	$\{c\}$	0.75
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	$\{a,c,d\}$	X	$\{b\}$	0.75	$\{a,c,d\}$	X	$\{b\}$	0.75
$\{b,c,d\}$	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

Table 3 shows that the current method in Definition 4.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of El-Shafei et al.'s method in Definition 2.10 [4], Kandil et al.'s approximations in Definitions 2.12, 2.14, 2.16 [7] and the current approximations in Definition 3.2.

TABLE 4. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.10 [4], Kandil et al.'s approximations in Definitions 2.12, 2.14, 2.16 [7] and the current approximations in Definitions 3.2, 4.2 in the case of decreasing ($\mathcal{I}$-decreasing) set.

A	El-Shafei et al.'s method in Definition 2.10 [4]				Kandil et al.'s method in Definition 2.12 [7]			
	$\underline{R}_{dec}(A)$	$\overline{R}^{dec}(A)$	$BN_{dec}(A)$	$\alpha^{dec}(A)$	$\underline{R}_{\mathcal{I}-dec}(A)$	$\overline{R}^{\mathcal{I}-dec}(A)$	$BN_{\mathcal{I}-dec}(A)$	$\alpha^{\mathcal{I}-dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1

A	Kandil et al.'s method in Definition 2.14 [7]				Kandil et al.'s method in Definition 2.16 [7]			
	$R_{*dec}(A)$	$R^{*dec}(A)$	$BN_{*dec}(A)$	$\alpha^{*}{}_{dec}(A)$	$R_{*\mathcal{I}-dec}(A)$	$R^{*\mathcal{I}-dec}(A)$	$BN_{*\mathcal{I}-dec}(A)$	$\alpha^{*}{}_{\mathcal{I}-dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,d\}$	$\{a,b,d\}$	X	$\{c\}$	0.75	$\{a,b,d\}$	X	$\{c\}$	0.75
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1

A	The current method in Definition 3.2				The current method in Definition 4.2			
	$R_{**inc}(A)$	$R^{**dec}(A)$	$BN_{**dec}(A)$	$\alpha^{**}{}_{dec}(A)$	$R_{**\mathcal{I}-dec}(A)$	$R^{**\mathcal{I}-dec}(A)$	$BN_{**\mathcal{I}-dec}(A)$	$\alpha^{**}{}_{\mathcal{I}-dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	$\{b\}$	$\{b\}$	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{a,d\}$	$\{a,d\}$	X	$\{b,c\}$	1	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c\}$	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{a,b,d\}$	$\{a,b,d\}$	X	$\{c\}$	0.75	$\{a,b,d\}$	X	$\{c\}$	0.75
$\{a,c,d\}$	$\{a,d\}$	X	$\{b,c\}$	0.5	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1

Table 4 shows that the current method in Definition 4.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of El-Shafei et al.'s method in Definition 2.10 [4], Kandil et al.'s approximations in Definitions 2.12, 2.14, 2.16 [7] and the current approximations in Definition 3.2.

Proposition 4.6. *Let* $(X, \mathfrak{F}_R, \rho, \mathcal{I})$ *be a GIOTAS and* $A, B \subseteq X$. *Then,*

(1) $R_{**\mathcal{I}-inc}(A) \subseteq A \subseteq R^{**\mathcal{I}-inc}(A)$ $(R_{**\mathcal{I}-dec}(A) \subseteq A \subseteq R^{**\mathcal{I}-dec}(A))$, *equality hold if* $A = \phi$ *or* X.

(2) $A \subseteq B \Rightarrow R^{**\mathcal{I}-inc}(A) \subseteq R^{**\mathcal{I}-inc}(B)$ $(R^{**\mathcal{I}-inc}(A) \subseteq R^{**\mathcal{I}-dec}(B))$.

(3) $A \subseteq B \Rightarrow R_{**\mathcal{I}-inc}(A) \subseteq R_{**\mathcal{I}-inc}(B)$ $(R_{**\mathcal{I}-dec}(A) \subseteq R_{**\mathcal{I}-dec}(B))$.

(4) $R^{**\mathcal{I}-inc}(A \cap B) \subseteq R^{**\mathcal{I}-inc}(A) \cup R^{**\mathcal{I}-inc}(B)$ $(R^{**\mathcal{I}-dec}(A \cap B) \subseteq R^{**\mathcal{I}-dec}(A) \cup R^{**\mathcal{I}-dec}(B))$.

(5) $R_{**\mathcal{I}-inc}(A \cup B) \supseteq R_{**\mathcal{I}-inc}(A) \cap R_{**\mathcal{I}-inc}(B)$ $(R_{**\mathcal{I}-dec}(A \cup B) \supseteq R_{**\mathcal{I}-dec}(A) \cap R_{**\mathcal{I}-dec}(B))$.

(6) $R^{**\mathcal{I}-inc}(A \cup B) = R^{**\mathcal{I}-inc}(A) \cup R^{**\mathcal{I}-inc}(B)$ $(R^{**\mathcal{I}-dec}(A \cup B) = R^{**\mathcal{I}-dec}(A) \cup R^{**\mathcal{I}-dec}(B))$.

(7) $R_{**\mathcal{I}-inc}(A \cap B) = R_{**\mathcal{I}-inc}(A) \cap R_{**\mathcal{I}-inc}(B)$ $(R_{**\mathcal{I}-dec}(A \cap B) = R_{**\mathcal{I}-dec}(A) \cap R_{**\mathcal{I}-dec}(B))$.

(8) $R^{**\mathcal{I}-inc}(R^{**\mathcal{I}-inc}(A)) \supseteq R^{**\mathcal{I}-inc}(A)$ $(R^{**\mathcal{I}-dec}(R^{**\mathcal{I}-dec}(A)) \supseteq R^{**\mathcal{I}-dec}(A))$.

(9) $R_{**\mathcal{I}-inc}(R_{**\mathcal{I}-inc}(A)) \subseteq R_{**\mathcal{I}-inc}(A)$ $(R_{**\mathcal{I}-dec}(R_{**\mathcal{I}-dec}(A)) \subseteq R_{**\mathcal{I}-dec}(A))$.

Proof. The proof is similar to Proposition 3.3. □

Let $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$, in Example 3.1. Then, it shows that the inclusion in Proposition 4.6 parts 1, 4 and 5 can not be replaced by equality relation (for part 1, if $A = \{d\}, R^{**\mathcal{I}-dec}(A) = X, R_{**\mathcal{I}-dec}(A) = \phi$. Then, $R^{**\mathcal{I}-dec}(A) \nsubseteq A \nsubseteq R_{**\mathcal{I}-dec}(A)$), also we can add examples to show that $R^{**\mathcal{I}-inc}(A) \nsubseteq A \nsubseteq R_{**\mathcal{I}-inc}(A)$. In a similar way, we can add examples to part 4 and 5). Moreover, the converse of parts 2 and 3 is not necessarily true (i.e., $R_{**\mathcal{I}-inc}(A) \subseteq R_{**\mathcal{I}-inc}(B) \not\Rightarrow A \subseteq B$, take $A = \{a, b, c\}, B = \{b, c, d\}$, then $R_{**\mathcal{I}-inc}(A) = \phi, R_{**\mathcal{I}-inc}(B) = \{b, c, d\}$. Therefore, $R_{**\mathcal{I}-inc}(A) \subseteq R_{**inc}(B)$, but $A \nsubseteq B$. In a similar way, we can add examples to show that $R_{**\mathcal{I}-dec}(A) \subseteq R_{**\mathcal{I}-dec}(B)$, but $A \nsubseteq B$).

5. Conclusion

In this paper, different methods are proposed to achieve the main aim of rough set which is reducing the boundary region and increasing the accuracy of sets. Comparison between the current approximation and previous approximation [4, 7] is presented. The current approximations are better than the previous approximation [4, 7], because it reduces the boundary region and increases the accuracy with comparison to the previous one [4, 7].

References

[1] A. A. Abo Khadra, B. M. Taher and M. K. El-Bably, Generalization of Pawlak approximation space, International Journal of Mathematical Archive 4 (11) (2013) 78–89.

[2] E.A. Abo–Tabl, A comparison of two kinds of definitions of rough approximations based on a similarity relation, Inform. Sci. 181 (2011) 2587–2596.

[3] M. Caldas, S. Jafari, S. A. Ponmani and M. L. Thivager, Some low separation axioms in bitopological spaces, Bol. Soc. Paran. Mat. (3s.) 24 (1-2) 2006 69–75.

[4] M. E. EL-Shafei, A. M. Kozae and M. Abo-Elhamayel, Rough Set Approximations via Topological Ordered Spaces, Annals of Fuzzy Sets, Fuzzy Logic and Fuzzy Systems 2 (2) (2013) 49–60.

[5] S. A. El-Sheikh and M. Hosny, $\mathcal{I}$-increasing (decreasing) sets and $\mathcal{I}P^*$-separation axioms in bitopological ordered spaces, Pensee Journal 76 (3) (2014) 429–443.

[6] D. Jankovic and T.R. Hamlet, New topologies from old via ideals, The American Mathematical Monthly 97 (1990) 295–310.

[7] A. Kandil, O. Tantawy, S. A. El-Sheikh and M. Hosny, A generalization of rough sets in topological ordered spaces, Journal of Mathematical and Computational Science 4 (2) (2014) 278–297.

[8] A. Kandil, O. Tantawy, S. A. El-Sheikh and M. Hosny, Topological ordered approach for approximation space. To appear in Journal of Uncertainty in Mathematics Science.

[9] M. Kondo and W. A. Dudek, Topological structures of rough sets induced by equivalence relations, Journal of Advanced Computational Intelligence and Intelligent Informatics 10 (5) (2006) 621–624.

[10] A. M. Kozae, A. Abo Khadra and T. Medhat, Topological approach for approximation space (TAS), Proceeding of the 5th International Conference on Informatics and Systems, Faculty of Computers and Information, Cairo University, Cairo, Egypt (2007) 289—302.

[11] A. M. Kozae, S. A. El-Sheikh and M. Hosny, On generalized rough sets and closure spaces, International Journal of Applied Mathematics 23 (6) (2010) 997–1023.

[12] A. M. Kozae, S. A. El-Sheikh, E.H. Aly and M. Hosny, Rough sets and its applications in a computer network, Ann. Fuzzy Math. Inform. 6 (3) (2013) 6051–624.

[13] E. F. Lashin, A. M. Kozae, A. A. Abo Khadra and T. Medhat, Rough set theory for topological spaces, International Journal of Approximate Reasoning 40 (2005) 35–43.

[14] H. J. Lee, J.B. Park and Y.H. Joo, Robust load–frequency control for uncertain nonlinear power systems: A fuzzy logic approach, Inform. Sci. 176 (2006) 3520–3537.

[15] G. M. Murdeshwar, General Topology, New Age International (P) Ltd., Publishers 1990.

[16] L. Nachbin, Topology and Order, Van NostrandMathematical studies, Princeton, New Jersey 1965.

[17] S. Pal and P. Mitra, Case generation using rough sets with fuzzy representation, IEEE Transactions on Knowledge and Data Engineering, 2004.

[18] Z. Pawlak, Rough sets, International Journal of Information and Computer Sciences 11 (5) (1982) 341–356.

[19] Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inform. Sci. 1119 (1-4) (1998) 239–259.

[20] Y. Y. Yao, Rough sets, neighborhood systems, and granular computing, Proceedings of IEEE Canadian Conference on Electrical and Computer Engineering, Edmonton, Alberta, Canada 3 (1999) 1553–1558.

A. KANDIL (`dr.ali kandil@yahoo.com`)

Department of Mathematics, Faculty of Science, Helan University, Cairo, Egypyt

O. TANTAWY (`drosamat@yahoo.com`)

Department of Mathematics, Faculty of Science, Zagazig University, Cairo, Egypyt

S. A. EL-SHEIKH (sobhyelsheikh@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11757, Cairo, Egypyt

M. HOSNY (moona_hosny@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11757, Cairo, Egypyt