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On intuitionistic fuzzy ψ-2-normed spaces

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Abstract. The aim of this paper is to study intuitionistic fuzzy ψ-2-normed space. We have discussed bounded linear operators, strong boundedness, weak boundedness, continuity in intuitionistic fuzzy ψ-2-normed space. Also, we have provided relation between boundedness and continuity. Further we have established some topological results in this new set up.

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1. Introduction

In 1965, L. A. Zadeh[22] introduced the fuzzy set theory. R.Lowen[11] studied the basic results in fuzzy set theory. Fuzzy set theory has wide application in almost all brances of Science and Engineering. A. George and P. V. Veeramani[7] studied fuzzy metric spaces. The concept of intuitionistic fuzzy set was introduced by Atanassov[3]. Many authors[1, 2, 5, 8, 10, 13, 14, 17] have studied topological properties in fuzzy metric spaces, intuitionistic fuzzy metric spaces and related topics. Saadati and Park[15] coined the notion of intuitionistic fuzzy normed space. Continuity, boundedness, completeness and compactness in intuitionistic fuzzy normed spaces are studied in [4, 9, 19, 20, 21] . M. Mursaleen[12] defined the new structure intuitionistic fuzzy 2-normed space and studied some basic results of normed linear spaces. Recently, T.K. Samanta and Sumit Mohinta[16] have introduced the concept of intuitionistic fuzzy ψ -normed space and discussed continuity and boundedness in this structure. Some basic results of intuitionistic fuzzy ψ-2-normed space are discussed in[6].

In this paper, we have discussed bounded linear operators, strong boundedness, weak boundedness, strong continuity, weak continuity, sequentially continuity in intuitionistic fuzzy ψ-2-normed space.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1 ([18])**.** A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Example 2.2. Two typical examples of continuous t-norms are

$$a * b = ab \text{ and } a * b = \min\{a, b\}.$$

Definition 2.3 ([18])**.** A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a) $\diamond$ is associative and commutative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Example 2.4. Two typical examples of continuous t-conorms are

$$a \diamond b = \min\{a + b, 1\} \text{ and } a \diamond b = \max\{a, b\}.$$

Definition 2.5 ([12])**.** Let V be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on V is a function $\|\cdot, \cdot\| : V \times V \to \mathbb{R}$ which satisfies for every $x, y, z \in V$,

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\| = \|y, x\|$;
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$;
(d) $\|x, y + z\| \leq \|x, y\| + \|y, z\|$.

The pair $V, \|\cdot, \cdot\|$ is then called a 2-normed space.

As an example of a 2-normed space take $V = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula $\|x, y\| = |x_1y_2 - x_2y_1|, x = (x_1, x_2), y = (y_1, y_2)$.

Definition 2.6 ([12])**.** The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times V \times (0, \infty)$ satisfying the following conditions. For every $x, y, z \in V$ and $s, t > 0$,

(a) $\mu(x, y, t) + \nu(x, y, t) \leq 1$;
(b) $\mu(x, y, t) > 0$;
(c) $\mu(x, y, t) = 1$ if and only if x and y are linearly dependent;
(d) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;

(e) $\mu(x,y,t) * \mu(x,z,s) \leq \mu(x,y+z,t+s)$;
(f) $\mu(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(g) $\lim_{t\to\infty} \mu(x,y,t) = 1$ and $\lim_{t\to 0} \mu(x,y,t) = 0$;
(h) $\mu(x,y,t) = \mu(y,x,t)$;
(i) $\nu(x,y,t) < 1$;
(j) $\nu(x,y,t) = 0$ if and only if x and y are linearly dependent;
(k) $\nu(\alpha x,y,t) = \nu(x,y,\frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(l) $\nu(x,y,t) \diamond \nu(x,z,s) \geq \nu(x,y+z,t+s)$;
(m) $\nu(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(n) $\lim_{t\to\infty} \nu(x,y,t) = 0$ and $\lim_{t\to 0} \nu(x,y,t) = 1$,
(o) $\nu(x,y,t) = \nu(y,x,t)$.

In this case $(\mu,\nu)_2$ is called an intuitionistic fuzzy 2-norm on V. We denote it by $(\mu,\nu)_2$.

Example 2.7 ([12]). Let $(V, \|\cdot,\cdot\|)$ be 2-normed space over F and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$, for all $a,b \in [0,1]$ and every $t > 0$, consider $\mu(x,y,t) = \dfrac{t}{t+\|x,y\|}$, $\nu(x,y,t) = \dfrac{\|x,y\|}{t+\|x,y\|}$. Then $(V,\mu,\nu,*,\diamond)$ is an intuitionistic fuzzy 2-normed space.

3. Intuitionistic fuzzy ψ-2-normed space

Definition 3.1. Let ψ be a function defined on the real field $\mathbb{R}$ into itself satisfying the following properties;

(a) $\psi(-t) = \psi(t)$ for all $t \in \mathbb{R}$
(b) $\psi(1) = 1$
(c) ψ is strictly increasing and continuous on $(0,\infty)$
(d) $t \leq \psi(t) \quad \forall t \in (0,\infty)$
(e) $\lim_{\alpha\to 0} \psi(\alpha) = 0$ and $\lim_{\alpha\to\infty} \psi(\alpha) = \infty$.

Example 3.2. (1) $\psi(\alpha) = |\alpha|$.
(2) $\psi(\alpha) = |\alpha|^n, n \in \mathbb{R}^+$.
(3) $\psi(\alpha) = \dfrac{2\alpha^{2n}}{|\alpha|+1}, n \in \mathbb{N}^+$.

Definition 3.3. The five-tuple $(V,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy ψ-2-normed space, if V is a vector space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ,ν are fuzzy sets on $V \times V \times (0,\infty)$ satisfying the following conditions. For every $x,y,z \in V$ and $s,t > 0$,

(a) $\mu(x,y,t) + \nu(x,y,t) \leq 1$;
(b) $\mu(x,y,t) > 0$;
(c) $\mu(x,y,t) = 1$ if and only if x and y are linearly dependent;
(d) $\mu(\alpha x,y,t) = \mu(x,y,\frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
(e) $\mu(x,y,t) * \mu(x,z,s) \leq \mu(x,y+z,t+s)$;
(f) $\mu(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(g) $\lim_{t\to\infty} \mu(x,y,t) = 1$ and $\lim_{t\to 0} \mu(x,y,t) = 0$;
(h) $\mu(x,y,t) = \mu(y,x,t)$;
(i) $\nu(x,y,t) < 1$;

(j) $\nu(x, y, t) = 0$ if and only if x and y are linearly dependent;
(k) $\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
(l) $\nu(x, y, t) \diamond \nu(x, z, s) \geq \nu(x, y + z, t + s)$;
(m) $\nu(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
(n) $\lim_{t\to\infty} \nu(x, y, t) = 0$ and $\lim_{t\to 0} \nu(x, y, t) = 1$;
(o) $\nu(x, y, t) = \nu(y, x, t)$.

In this case $(\mu, \nu)_2$ is called an intuitionistic fuzzy ψ-2-norm on V.

Definition 3.4. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A sequence $\{x_n\}$ is said to be convergent to $x \in V$ with respect to the intuitionistic fuzzy ψ-2-norm $(\mu, \nu)_2$, if for every $r > 0$ and $t > 0$, $r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$ for all $n \geq n_0$ and for all $z \in V$.

Definition 3.5. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A sequence $\{x_n\}$ in V is said to be Cauchy if for each $r > 0$ and each $t > 0$,$r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, z, t) > 1 - r$ and $\nu(x_n - x_m, z, t) < r$ for all $n, m \geq n_0$ and for all $z \in V$.

Definition 3.6. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space and let $r \in (0, 1), t > 0$ and $x \in X$.The set $B(x, r, t) = \{y \in V : \mu(y - x, z, t) > 1 - r, \ \ \nu(y - x, z, t) < r, \forall z \in V\}$ is called the open ball with center x and radius r with respect to t.

Definition 3.7. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A set $U \subset V$ is said to an open set if each of its points is the centre of some open ball contained in U. The open set in an intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ is denoted by $\mathbb{U}$.

Theorem 3.8. *In intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$. A sequence $\{x_n\}$ converges to x if and only if $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$.*

Theorem 3.9. *The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$.*

Theorem 3.10. *In IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$. Every convergent sequence is a cauchy sequence.*

Theorem 3.11. *In IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$. A sequence $\{x_n\}$ is a cauchy sequence if and only if $\mu(x_{n+p} - x, z, t) \to 1$ and $\nu(x_{n+p} - x, z, t) \to 0$ as $n \to \infty$.*

Definition 3.12. An intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ is said to be complete if every cauchy sequence in IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$ is convergent.

Theorem 3.13. *Let $(V, \mu, \nu, *, \diamond)$ be a IF ψ-2-NS. A sufficient condition for the IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$ to be complete is that every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ has a convergent subsequence.*

The proofs of theorems 3.8, 3.9, 3.10, 3.11, 3.13 are easily obtained using [6].

Definition 3.14. Let$(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A subset A of V is said to be IF_2-bounded if there exist $t > 0$ and $0 < r < 1$ such that $\mu(y-x, z, t) > 1-r$ and $\nu(y-x, z, t) < r$ for all $x, y \in A$ and for all $z \in V$.

Theorem 3.15. *In an intuitionistic fuzzy ψ-2-normed space every compact set is closed and IF_2-bounded.*

Lemma 3.16. *A subset A of $\mathbb{R}$ is IF_2-bounded in $(\mathbb{R}, \mu, \nu, *, \diamond)$ if and only if it is bounded in $\mathbb{R}$.*

Lemma 3.17. *A sequence $\{\beta_n\}$ is convergent in the intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ if and only if it is convergent in $(\mathbb{R}, |\cdot|)$.*

Corollary 3.18. *If the real sequence $\{\beta_n\}$ is IF_2-bounded, then it has at least one limit point.*

The proofs of theorem 3.15, lemmas 3.16, 3.17 and corollary 3.18 are easy so omitted.

Note that if $t_1 \leq t_2$ then $\psi(t_1) \leq \psi(t_2)$ which gives $\frac{t}{\psi(t_1)} \geq \frac{t}{\psi(t_2)}$. Now $\mu(xt_1, z, t) = \mu(x, z, \frac{t}{\psi(t_1)}) \geq \mu(x, z, \frac{t}{\psi(t_2)}) = \mu(xt_2, z, t)$. Hence,

$$\mu(xt_1, z, t) \geq \mu(xt_2, z, t) \text{ if } t_1 \leq t_2. \tag{3.1}$$

Theorem 3.19. *Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in vector space V and $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. Then there are numbers $c, d \neq 0$ and an intuitionistic fuzzy ψ-2-normed space $(\mathbb{R}, \mu, \nu, *, \diamond)$ such that for every choice of real scalars $\alpha_1, \cdots, \alpha_n$, we have,*

$$\mu(\alpha_1 x_1 + \cdots + \alpha_n x_n, z, t) \leq \mu_0(c[\psi(\alpha_1) + \cdots + \psi(\alpha_n)], z, t) \tag{3.2}$$

and

$$\nu(\alpha_1 x_1 + \cdots + \alpha_n x_n, t) \geq \nu_0(d[\psi(\alpha_1) + \cdots + \psi(\alpha_n)], z, t). \tag{3.3}$$

Proof. Put $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, all α_j's must be zero so all $\psi(\alpha_j)$ must be zero, hence (3.2) and (3.3) holds for any $c, d \neq 0$. If $s > 0$, by using definition 3.1, we have,

$$\psi(\alpha_1) \geq |\alpha_1|, \cdots, \psi(\alpha_n) \geq |\alpha_n|$$
$$\therefore [\psi(\alpha_1) + \cdots + \psi(\alpha_n)] \geq [|\alpha_1| + \cdots + |\alpha_n|].$$

By using (3.1), we get,

$$\mu_0(c[\psi(\alpha_1) + \cdots + \psi(\alpha_n)], z, t) \leq \mu_0(c[|\alpha_1| + \cdots + |\alpha_n|], z, t).$$

Hence, (3.2) takes the form,

$$\mu(\alpha_1 x_1 + \cdots + \alpha_n x_n, z, t) \leq \mu_0(c[|\alpha_1| + \cdots + |\alpha_n|], z, t).$$

Dividing by s, we get,

$$\begin{aligned}\mu\Big(\frac{\alpha_1 x_1}{s} + \cdots + \frac{\alpha_n x_n}{s}, z, \frac{t}{s}\Big) &\leq \mu_0\Big(c\frac{|\alpha_1| + \cdots + |\alpha_n|}{|\alpha_1| + \cdots + |\alpha_n|}, z, \frac{t}{s}\Big) \\ &\leq \mu_0\Big(c, z, \frac{t}{s}\Big).\end{aligned}$$

Putting $\frac{\alpha_j}{s} = \beta_j$,

$$\text{(3.4)}\quad \mu(\beta_1 x_1 + \cdots + \beta_n x_n, z, t') \leq \mu_0(c, z, t'), \qquad \text{where} \quad t' = \frac{t}{s}, \qquad \sum_{j=1}^{n} |\beta_j| = 1.$$

Similarly,

$$\text{(3.5)}\quad \nu(\beta_1 x_1 + \cdots + \beta_n x_n, z, t') \geq \nu_0(d, z, t'), \qquad \text{where} \quad t' = \frac{t}{s}, \qquad \sum_{j=1}^{n} |\beta_j| = 1.$$

Hence, it suffices to prove the existence of $c, d \neq 0$ and an intuitionistic fuzzy-ψ-2-norm $(\mu_0, \nu_0)_2$ such that (3.4) and (3.5) holds.

Suppose that this is not true. Then there exists a sequence $\{y_m\}$ of vectors

$$y_m = \beta_{1,m} x_1 + \cdots + \beta_{n,m} x_n, \qquad (\sum_{j=1}^{n} |\beta_{j,m}| = 1),$$

such that $\mu(y_m, z, t) \to 1$ and $\nu(y_m, z, t) \to 0$ as $m \to \infty$ for every $t > 0$. Since $\sum_{j=1}^{n} |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \leq 1$ and then by Lemma 3.16, the sequence of $\{\beta_{j,m}\}$ is IF-bounded. In according to Corollary 3.18, $\{\beta_{1,m}\}$ has a convergent subsequence, let β_1 denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding sequence of real scalars $\{\beta_{2,m}\}$ converges to some β_2.

Continuing this process, after n steps we obtain a subsequence $\{y_{n,m}\}$ of $\{y_m\}$ such that

$$y_{n,m} = \sum_{j=1}^{n} \gamma_{j,m} x_j, \quad \Big(\sum_{j=1}^{n} |\gamma_{j,m}| = 1\Big) \quad \text{and} \quad \gamma_{j,m} \to \beta_j \quad \text{as} \quad m \to \infty.$$

Since,

$$\begin{aligned}
\lim_m \mu\Big(y_{n,m} - \sum_{j=1}^{n} \beta_j x_j, z, t\Big) &= \lim_m \mu\Big(\sum_{j=1}^{n} (\gamma_{j,m} - \beta_j) x_j, z, t\Big) \\
&\geq \lim_m \Big(\mu((\gamma_{1,m} - \beta_1) x_1, z, \frac{t}{n}) * \cdots * \mu((\gamma_{n,m} - \beta_n) x_n, z, \frac{t}{n})\Big) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\lim_m \nu\Big(y_{n,m} - \sum_{j=1}^{n} \beta_j x_j, z, t\Big) &= \lim_m \nu(\sum_{j=1}^{n} (\gamma_{j,m} - \beta_j) x_j, z, t) \\
&\leq \lim_m \Big(\nu((\gamma_{1,m} - \beta_1) x_1, z, \frac{t}{n}) \diamond \cdots \diamond \mu((\gamma_{n,m} - \beta_n) x_n, z, \frac{t}{n})\Big) \\
&= 0.
\end{aligned}$$

We have, $\lim_{m \to \infty} y_{n,m} = \sum_{j=1}^{n} \beta_j x_j, \quad (\sum_{j=1}^{n} |\beta_j| = 1),$ so that not all β_j can be zero.

Put $y = \sum_{j=1}^{n} \beta_j x_j$ then $y \neq 0$ because $\{x_1, \cdots, x_n\}$ is a linearly independent set. Since $\mu(y_m, z, t) \to 1$ and $\nu(y_m, z, t) \to 0$ by assumption, then we have $\mu(y_{n,m}, z, t) \to 1$ and $\nu(y_{n,m}, z, t) \to 0$. Hence

$$\mu(y, z, t) = \mu((y - y_{n,m}) + y_{n,m}, z, t) \geq \Big(\mu(y - y_{n,m}, z, \frac{t}{2}) * \mu(y_{n,m}, z, \frac{t}{2})\Big) \to 1$$

and

$$\nu(y, z, t) = \nu((y - y_{n,m}) + y_{n,m}, z, t) \leq \Big(\nu(y - y_{n,m}, z, \frac{t}{2}) \diamond \nu(y_{n,m}, z, \frac{t}{2})\Big) \to 0.$$

Thus $y = 0$ which is a contradiction. □

Definition 3.20. Let $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *', \diamond')$ be intuitionistic fuzzy-ψ-2-normed spaces then intuitionistic fuzzy-ψ-2-norms $(\mu, \nu)_2$ and $(\mu', \nu')_2$ are said to be equivalent whenever $x_n \to^{(\mu,\nu)} x$ in $(V, \mu, \nu, *, \diamond)$ if and only if $x_n \to^{(\mu'\nu')} x$ in $(V, \mu', \nu', *', \diamond')$.

Theorem 3.21. *On a finite dimensional vector space V, every two intuitionistic fuzzy-ψ-2-norms $(\mu, \nu)_2$ and $(\mu^{'}, \nu^{'})_2$ are equivalent.*

Proof. Let $dimV = n$ and $\{v_1, ..., v_n\}$ be a basis for V. Then every $x \in V$ has a unique representation $x = \sum_{j=1}^{n} \alpha_j v_j$. Let $x_m \to^{(\mu,\nu)} x$ in $(V, \mu, \nu, *, \diamond)$ but for each $m \in N$, x_m has a unique representation, i.e. $x_m = \alpha_{1,m} v_1 + \cdots + \alpha_{n,m} v_n$. By theorem 3.19, there are $c, d \neq 0$ and an intuitionistic fuzzy-ψ-2- norm $(\mu_0, \nu_0)_2$ such that (3.2) and (3.3) hold. So

$$\begin{aligned} \mu(x_m - x, z, t) &= \mu\Big((\alpha_{1,m} - \alpha_1)v_1 + (\alpha_{2,m} - \alpha_2)v_2 + \cdots + (\alpha_{n,m} - \alpha_n)v_n\Big) \\ &\leq \mu_0\Big(c\Big[\psi(\alpha_{1,m} - \alpha_1) + \psi(\alpha_{2,m} - \alpha_2) + \cdots + \psi(\alpha_{n,m} - \alpha_n)\Big]\Big) \\ &\leq \mu_0(c\ \psi(\alpha_{j,m} - \alpha_j), z, t) \end{aligned}$$

and

$$\nu(x_m - x, z, t) \geq \nu_0(d\ \psi(\alpha_{j,m} - \alpha_j), z, t).$$

Now, if $m \to \infty$ then $\mu(x_m - x, z, t) \to 1$ and $\nu(x_m - x, z, t) \to 0$ for every $t > 0$ and hence $\psi(\alpha_{j,m} - \alpha) \to 0$ in $\mathbb{R}$.

On the other hand,

$$\begin{aligned} \mu'(x_m - x, z, t) &\geq \mu'\Big((\alpha_{1,m} - \alpha_1)v_1, \frac{t}{n}\Big) *' \cdots *' \mu'\Big((\alpha_{n,m} - \alpha_n)v_n, z, \frac{t}{n}\Big) \\ &= \mu'\Big(v_1, z, \frac{t}{n\psi(\alpha_{1,m} - \alpha_1)}\Big) *' \cdots *' \mu'\Big(v_n, z, \frac{t}{n\psi(\alpha_{n,m} - \alpha_n)}\Big) \end{aligned}$$

and

$$\begin{aligned} \nu'(x_m - x, z, t) &\leq \nu'\Big((\alpha_{1,m} - \alpha_1)v_1, z, \frac{t}{n}\Big) \diamond' \cdots \diamond' \mu'\Big((\alpha_{n,m} - \alpha_n)v_n, z, \frac{t}{n}\Big) \\ &= \nu'\Big(v_1, z, \frac{t}{n\psi(\alpha_{1,m} - \alpha_1)}\Big) \diamond' \cdots \diamond' \nu'\Big(v_n, z, \frac{t}{n\psi(\alpha_{n,m} - \alpha_n)}\Big). \end{aligned}$$

Since $\psi(\alpha_{j,m} - \alpha_j) \to 0$, $\frac{t}{n\psi(\alpha_{j,m} - \alpha_j)} \to \infty$ and thus we have $\mu'(v_j, z, \frac{t}{n\psi(\alpha_{j,m}\alpha_j)}) \to 1$ and $\nu'(v_j, z, \frac{t}{n\psi(\alpha_{j,m}\alpha_j)}) \to 0$ Then $x_m \to^{(\mu'\nu')} x$ in $(V, \mu', \nu', *, \diamond)$. With the same argument, $x_m \to^{(\mu',\nu')_2} x$ in $(V,\mu',\nu',*',\diamond')$ implies $x_m \to^{(\mu,\nu)_2} x$ in $(V,\mu,\nu,*,\diamond)$. □

4. Continuous linear operators

Definition 4.1. Let $(V,\mu,\nu,*,\diamond),(V',\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T:(V,\mu,\nu,*,\diamond)\to(V',\mu',\nu',*',\diamond')$ is said to be intuitionistic fuzzy-ψ-2-continuous (IF-ψ-2-C) at $x_0\in V$ if for given $\epsilon>0,\alpha\in(0,1)$ there exists some $\delta=\delta(\alpha,\epsilon)>0,\beta=\beta(\alpha,\epsilon)\in(0,1)$ such that for $x\in V$ and for all nonzero $z\in V$,

$$\begin{aligned}&\mu(x-x_0,z,\delta)>\psi(\beta) \text{ and } \nu(x-x_0,z,\delta)<(1-\psi(\beta))\\ \Rightarrow\ &\mu'(Tx-Tx_0,z,\epsilon)>\psi(\alpha) \text{ and } \nu'(Tx-Tx_0,z,\epsilon)<(1-\psi(\alpha)).\end{aligned}$$

If T is IF-ψ-2-C at each point of V then it is said to be IF-ψ-2-C on V.

Definition 4.2. Let $(V,\mu,\nu,*,\diamond),(V',\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T:(V,\mu,\nu,*,\diamond)\to(V',\mu',\nu',*',\diamond')$ is said to be intuitionistic fuzzy-ψ-2-strongly continuous (IF-ψ-2-SC) at $x_0\in V$ if for given $\epsilon>0$ there exists some $\delta=\delta(\epsilon)>0$ such that for $x\in V$ and for all nonzero $z\in V$,

$$\mu'(Tx-Tx_0,z,\epsilon)\geq\mu(x-x_0,z,\delta) \quad\text{and}\quad \nu'(Tx-Tx_0,z,\epsilon)\leq\nu(x-x_0,z,\delta).$$

If T is IF-ψ-2-SC at each point of V then T is said to be IF-ψ-2-SC on V.

Definition 4.3. Let $(V,\mu,\nu,*,\diamond),(V',\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T:(V,\mu,\nu,*,\diamond)\to(V',\mu',\nu',*',\diamond')$ is said to be intuitionistic fuzzy-ψ-2-weakly continuous (IF-ψ-2-WC) at $x_0\in V$ if for given $\epsilon>0,\alpha\in(0,1)$ there exists some $\delta=\delta(\alpha,\epsilon)>0$ such that for $x\in V$ and for all nonzero $z\in V$,

$$\begin{aligned}&\mu(x-x_0,z,\delta)\geq\psi(\alpha) \quad\text{and}\quad \nu(x-x_0,z,\delta)\leq(1-\psi(\alpha))\\ &\Rightarrow\mu'(Tx-Tx_0,z,\epsilon)\geq\psi(\alpha) \quad\text{and}\quad \nu'(Tx-Tx_0,z,\epsilon)\leq(1-\psi(\alpha))\\ &i.e.\quad \mu'(T(x-x_0),z,\epsilon)\geq\psi(\alpha) \quad\text{and}\quad \nu'(T(x-x_0),z,\epsilon)\leq(1-\psi(\alpha)).\end{aligned}$$

If T is IF-ψ-2-WC at each point of V then T is said to be IF-ψ-2-WC on V.

Definition 4.4. Let $(V,\mu,\nu,*,\diamond),(V',\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T:(V,\mu,\nu,*,\diamond)\to(V',\mu',\nu',*',\diamond')$ is said to be intuitionistic fuzzy-ψ-2-sequentially continuous (Seq-IF-ψ-2-C) at $x_0\in V$ if for any sequence $\{x_k\},\ \ k\geq1$ with $x_k\to x_0$ implies $Tx_k\to Tx_0$.

$$\begin{aligned}&i.e.\quad \lim_{k\to\infty}\mu(x_k-x_0,z,t)=1 \quad\text{and}\quad \lim_{k\to\infty}\nu(x_k-x_0,z,t)=0 \quad \forall t>0\\ &\Rightarrow\lim_{k\to\infty}\mu'(Tx_k-Tx_0,z,t)=1 \quad\text{and}\quad \lim_{k\to\infty}\nu'(Tx_k-Tx_0,z,t)=0 \quad \forall t>0.\end{aligned}$$

If T is sequentially IF-ψ-2-C at each point of V then T is said to be sequentially IF-ψ-2-C on V.

Theorem 4.5. *Let $(V,\mu,\nu,*,\diamond),(V',\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. Let $T:(V,\mu,\nu,*,\diamond)\to(V',\mu',\nu',*',\diamond')$ be a linear operator.*

(a) *If T is IF-ψ-2-SC on V iff it is IF-ψ-2-SC at point x_0.*
(b) *If T is IF-ψ-2-WC on V iff it is IF-ψ-2-WC at point x_0.*
(c) *If T is Seq-IF-ψ-2-C on V iff it is Seq-IF-ψ-2-C at point x_0.*

(*d*) *If T is IF-ψ-2-C on V iff it is IF-ψ-2-C at point x_0.*

Proof. (*a*) Let T be IF-ψ-2-SC at $x_0 \in V$ then for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta). \tag{4.1}$$

Take any $y \in V$ and replace x by $x + x_0 - y$, by (4.1), we get

$$\begin{aligned}
&\mu'(Tx + Tx_0 - Ty - Tx_0, z, \epsilon) \geq \mu(x + x_0 - y - x_0, z, \delta) \text{ and}\\
&\nu'(Tx + Tx_0 - Ty - Tx_0, z, \epsilon) \leq \nu(x + x_0 - y - x_0, z, \delta)\\
\Rightarrow\ &\mu'(Tx - Ty, z, \epsilon) \geq \mu(x - y, z, \delta) \text{ and}\\
&\nu'(Tx - Ty, z, \epsilon) \leq \nu(x - y, z, \delta)\\
\Rightarrow\ &\text{T is IF-}\psi\text{-2-SC at y.}
\end{aligned}$$

Since, y is arbitrary, hence T is IF-ψ-2 SC on V.

Conversely, if T is IF-ψ-2 SC on V then by (4.2), T is IF-ψ-2 SC at any $x_o \in V$. The Proof of (*b*) follows easily by using (4.3) and (4.5)(*a*).
The Proof of (*c*) follows easily by using (4.4) and (4.5)(*a*).
The Proof of (*d*) follows easily by using (4.1) and (4.5)(*a*). □

Theorem 4.6. *If a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-SC then it sequentially IF-ψ-2-C but converse need not true.*

Proof. Let $\{x_k\}$, $k \geq 1$ be a sequence such that $x_k \to x_0$.

$$i.e.\quad \lim_{k\to\infty} \mu(x_k - x_0, z, t) = 1 \quad \text{and} \quad \lim_{k\to\infty} \nu(x_k - x_0, z, t) = 0, \quad \forall t > 0.$$

Let $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-SC at $x_0 \in V$ if for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$\begin{aligned}
&\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta)\\
\Rightarrow\ &\mu'(Tx_k - Tx_0, z, \epsilon) \geq \mu(x_k - x_0, z, \delta) \text{ and } \nu'(Tx_k - Tx_0, z, \epsilon) \leq \nu(x_k - x_0, z, \delta)\\
\Rightarrow\ &\lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, \epsilon) \geq \lim_{k\to\infty} \mu(x_k - x_0, z, \delta) \text{ and}\\
&\lim_{k\to\infty} \nu'(Tx_k - Tx_0, z, \epsilon) \leq \lim_{k\to\infty} \nu(x_k - x_0, z, \delta)\\
\Rightarrow\ &\lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, \epsilon) \geq 1 \text{ and } \lim_{k\to\infty} \nu'(Tx_k - Tx_0, z, \epsilon) \leq 0\\
\Rightarrow\ &\lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, \epsilon) = 1 \text{ and } \lim_{k\to\infty} \nu'(Tx_k - Tx_0, z, \epsilon) = 0\\
\Rightarrow\ &Tx_k \to Tx_0, \text{ (since, } \epsilon > 0 \text{ is arbitrary).}
\end{aligned}$$

By (4.4), T is sequentially IF-ψ-2-C at x_0. Hence, T is sequentially IF-ψ-2-C on V. □

Conversely, we provide example to prove sequentially IF-ψ-2-C does not imply IF-ψ-2-SC.

Example 4.7. Let $(V, \|\bullet, \bullet\|)$ be 2-normed space. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$.

$$\mu'(x, z, t) = \frac{t}{t + c\|x, z\|}, \quad \nu'(x, z, t) = \frac{c\|x, z\|}{t + c\|x, z\|}, \quad t > \|x, z\| \quad \text{and}$$
$$\mu(x, z, t) = \frac{t}{t + \|x, z\|}, \quad \nu(x, z, t) = \frac{\|x, z\|}{t + \|x, z\|}, \quad t > \|x, z\|.$$

Then $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces.
Consider, $T(x, z, t) = \dfrac{\|x, z\|^4}{1 + \|x, z\|^2}$,
choose sequence $\{x_k\}, k \geq 1$ such that $x_k \to x_0$. Now for all $t > 0$, we have

$$\lim_{k\to\infty} \mu(x_k - x_0, z, t) = 1 \quad \text{and} \quad \lim_{k\to\infty} \nu(x_k - x_0, z, t) = 0$$
$$\Rightarrow \lim_{k\to\infty} \|x_k - x_0\| = 0.$$

Now,

$$\mu'(Tx_k - Tx_0, z, t) = \frac{t}{t + c\|\frac{(x_k,z)^4}{1+(x_k,z)^2} - \frac{(x_0,z)^4}{1+(x_0,z)^2}\|}$$
$$= \frac{t\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\|}{t\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\| + c\|(x_k, z)^4 + (x_k, z)^4(x_0, z)^2 - (x_0, z)^4 - (x_k, z)^2(x_0, z)^4\|}$$
$$= \frac{t\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\|}{t\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\| + c\|(x_k - x_0, z)\|\|(x_k{-}_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|}.$$

Similarly,

$$\nu'(Tx_k - Tx_0, z, t) = \frac{c\left\|\frac{(x_k,z)^4}{1+(x_k,z)^2} - \frac{(x_0,z)^4}{1+(x_0,z)^2}\right\|}{t + c\left\|\frac{(x_k,z)^4}{1+(x_k,z)^2} - \frac{(x_0,z)^4}{1+(x_0,z)^2}\right\|}$$
$$= \frac{c\|(x_k - x_0, z)\|\|(x_k{-}_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|}{t\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\| + c\|(x_k - x_0, z)\|\|(x_k{-}_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|}.$$

Since, $\lim_{k\to\infty} \|x_k - x_0\| = 0$, it follows that,

$$\lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, t) = 1 \ \text{ and } \ \lim_{k\to\infty} \nu'(Tx_k - Tx_0, z, t) = 0, \quad \forall t > 0$$
$$\Rightarrow Tx_k \to Tx_0.$$

Hence, T is Seq-IF-ψ-2-C on V.

Let $\epsilon > 0$ be given. Then

$$\mu'(Tx_k - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta)$$
$$\Rightarrow$$
$$\frac{\epsilon\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\|}{\epsilon\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\| + c\|(x_k - x_0, z)\|\|(x_k{-}_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|}$$
$$\geq \frac{\delta}{\delta + \|x - x_0, z\|}$$

and

$\nu'(Tx_k - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta)$

$$\Rightarrow \frac{c\|(x_k - x_0, z)\|\|(x_k -_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|}{\epsilon\|1 + (x_k, z)^2\|\|1 + (x_0, z)^2\| + c\|(x_k - x_0, z)\|\|(x_k -_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z)\|} \leq \frac{\delta}{\delta + \|x - x_0, z\|}.$$

So

$$\delta \leq \frac{\epsilon\|1 + (x, z)^2\|\|1 + (x_0, z)^2\|}{c\|x + x_0, z\|\|(x, z)^2 + (x_0, z)^2 + (x, z)^2(x_0, z)^2\|}. \tag{4.2}$$

Clearly, T is IF-ψ-2-SC at x_0 if there exists $\delta > 0$ satisfying (4.2) for all $x \neq x_0$. Let

$$\delta_1 = \inf \frac{\|1 + (x, z)^2\|\|1 + (x_0, z)^2\|}{\|x + x_0, z\|\|(x, z)^2 + (x_0, z)^2 + (x, z)^2(x_0, z)^2\|}$$

then $\delta = \frac{\epsilon}{c}\delta_1$ satisfying (4.2). But $\delta_1 = 0$ which is impossible. So, T is not IF-ψ-2-SC at x_0. Hence T is not IF-ψ-2-SC on V.

Theorem 4.8. *If a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-SC then it IF-ψ-2-WC.*

Proof. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ is IF-ψ-2-SC at $x_0 \in V$ then for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta).$$

Let $\mu(x - x_0, z, \delta) \geq \psi(\alpha)$ and $\nu(x - x_0, z, \delta) \leq (1 - \psi(\alpha))$ then

$$\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \geq \psi(\alpha) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta) \leq 1 - \psi(\alpha)$$
$$\Rightarrow \mu'(Tx - Tx_0, z, \epsilon) \geq \psi(\alpha) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq (1 - \psi(\alpha)).$$

So T is IF-ψ-2-WC at x_0. Hence, T is IF-ψ-2-WC on V. □

Theorem 4.9. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-C on V iff it is sequentially IF-ψ-2-C on V.*

Proof. Let a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-C at $x_0 \in V$ then by (4.1), for given $\epsilon > 0$, $\alpha \in (0, 1)$ there exists some $\delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for $x \in V$ and for all nonzero $z \in V$,

$$\mu(x - x_0, z, \delta) > \psi(\beta) \text{ and } \nu(x - x_0, z, \delta) < (1 - \psi(\beta))$$
$$\Rightarrow \mu'(Tx - Tx_0, z, \epsilon) > \psi(\alpha) \text{ and } \nu'(x - x_0, z, \delta) < (1 - \psi(\alpha))$$

Let $\{x_k\},\ k \geq 1$ be a sequence such that $x_k \to x_0$ so there exists positive integer k_0 such that,

$$\begin{aligned}
& \mu(x_k - x_0, z, \delta) > \psi(\beta) \text{ and } \nu(x_k - x_0, z, \delta) < (1 - \psi(\beta)) \\
\Rightarrow\ & \mu'(Tx_k - Tx_0, z, \epsilon) > \psi(\alpha) \text{ and } \nu'(x_k - x_0, z, \delta) < (1 - \psi(\alpha)), \quad \forall k \geq k_0 \\
& \text{which gives} \\
& \lim_{k\to\infty} \mu(x_k - x_0, z, \delta) > \psi(\beta) \text{ and } \lim_{k\to\infty} \nu(x_k - x_0, z, \delta) < (1 - \psi(\beta)) \\
\Rightarrow\ & \lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, \epsilon) > \psi(\alpha) \text{ and } \lim_{k\to\infty} \nu'(x_k - x_0, z, \delta) < (1 - \psi(\alpha)), \quad \forall k \geq k_0 \\
& \text{which gives} \\
& \lim_{k\to\infty} \mu'(Tx_k - Tx_0, z, \epsilon) = 1 \text{ and } \lim_{k\to\infty} \nu'(x_k - x_0, z, \delta) = 0, \quad \forall k \geq k_0 \\
\Rightarrow\ & Tx_k \to Tx_0, \quad \text{(since } \epsilon > 0 \text{ is arbitrary).}
\end{aligned}$$

By (4.4), we get T is sequentially IF-ψ-2-C at x_0. Hence, T is sequentially IF-ψ-2-C on V.

Conversely, let T is sequentially IF-ψ-2-C on V so T is sequentially IF-ψ-2-C at x_0. If possible T is not IF-ψ-2-C at x_0 then for given $\epsilon > 0, \alpha \in (0,1)$ there exists some $\delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for $y \in V$ and for all nonzero $z \in V$,

$$\mu(x_0 - y, z, \delta) > \psi(\beta) \text{ and } \nu(x_0 - y, z, \delta) < (1 - \psi(\beta)) \tag{4.3}$$

$$\text{but} \quad \mu'(Tx_0 - Ty, z, \epsilon) \leq \psi(\alpha) \text{ and } \nu'(Tx_0 - Ty, z, \epsilon) \geq (1 - \psi(\alpha)). \tag{4.4}$$

Thus for $\beta = \frac{k}{k+1},\ \ \delta = \frac{1}{k+1},\ \ k = 1, 2.\cdots$ there exists $\{y_k\}$ such that,

$$\begin{aligned}
& \mu(x_0 - y_k, z, \frac{1}{k+1}) > \psi(\frac{k}{k+1}) \text{ and } \nu(x_0 - y_k, z, \frac{1}{k+1}) < (1 - \psi(\frac{k}{k+1})) \\
\text{but} \quad & \mu'(Tx_0 - Ty_k, z, \epsilon) \leq \psi(\alpha) \text{ and } \nu'(Tx_0 - Ty_k, z, \epsilon) \geq (1 - \psi(\alpha)).
\end{aligned}$$

Taking $\delta > 0$ there exists k_0 such that $\frac{1}{k+1} > \delta$ for all $k \geq k_0$ then

$$\begin{aligned}
& \mu(x_0 - y_k, z, \delta) \geq \mu(x_0 - y_k, z, \frac{1}{k+1}) > (\psi(\frac{k}{k+1})) \ \text{ and} \\
& \nu(x_0 - y_k, z, \delta) \leq \nu(x_0 - y_k, z, \frac{1}{k+1}) < (1 - \psi(\frac{k}{k+1})) \\
\Rightarrow\ & \lim_{k\to\infty} \mu(x_0 - y_k, z, \delta) \geq \lim_{k\to\infty} \mu(x_0 - y_k, z, \frac{1}{k+1}) > \lim_{k\to\infty} (\psi(\frac{k}{k+1})) = 1 \ \text{ and} \\
& \lim_{k\to\infty} \nu(x_0 - y_k, z, \delta) \leq \lim_{k\to\infty} \nu(x_0 - y_k, z, \frac{1}{k+1}) < \lim_{k\to\infty} (1 - \psi(\frac{k}{k+1})) = 0 \\
\Rightarrow\ & \lim_{k\to\infty} \mu(x_0 - y_k, z, \delta) = 1 \ \text{ and } \ \lim_{k\to\infty} \nu(x_0 - y_k, z, \delta) = 0 \\
\Rightarrow\ & \{y_k\} \to x_0,
\end{aligned}$$

from (4.3), $\lim_{k\to\infty} \mu'(Tx_0 - Ty_k, z, \epsilon) \leq \psi(\alpha)$ and $\lim_{k\to\infty} \nu'(Tx_0 - Ty_k, z, \epsilon) \geq (1 - \psi(\alpha))$

$$\therefore \lim_{k\to\infty} \mu'(Tx_0 - Ty_k, z, \epsilon) \neq 1 \ \text{ and } \ \lim_{k\to\infty} \nu'(Tx_0 - Ty_k, z, \epsilon) \neq 0.$$

$\therefore$ Ty_k does not converge to Tx_0 but $\{y_k\} \to x_0$, which is contradiction to our assumption that T is sequentially IF-ψ-2-C at x_0. Hence, T is IF-ψ-2-C at x_0 so T is IF-ψ-2-C on V. □

Theorem 4.10. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *', \diamond')$ is IF-ψ-2-SC on V then it is IF-ψ-2-C on V but converse need not true.*

Proof. Let T is IF-ψ-2-SC on V then by using (4.6) and (4.9) it is IF-ψ-2-C on V. Converse easily disproved by using theorem (4.9) and example (4.7). □

5. Bounded linear operators

Definition 5.1. Let $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ is said to be (IF-ψ-2-SB) intuitionistic fuzzy-ψ-2-strongly bounded if there exist constants $h, k \in \mathbb{R} - \{0\}$ such that for every $x, z(nonzero) \in V$ and for every $t > 0$,

$$\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t).$$

Definition 5.2. Let $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ is said to be (IF-ψ-2-WB) intuitionistic fuzzy-ψ-2-weakly bounded if there exist $l > 0$ such that for all $x, z(nonzero) \in V$ and for every $t > 0$,

$$\begin{aligned} &\mu(lx, z, t) \geq \alpha \quad \text{and} \quad \nu(lx, z, t) \leq 1 - \alpha \\ \text{i.e.} \quad &\mu(x, z, \frac{t}{\psi(l)}) \geq \alpha \quad \text{and} \quad \nu(x, z, \frac{t}{\psi(l)}) \leq 1 - \alpha \\ \Rightarrow &\mu'(Tx, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx, z, t) \leq 1 - \alpha. \end{aligned}$$

Theorem 5.3. *Let $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. If a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ is IF-ψ-2-SB then it is IF-ψ-2-WB but converse is not true.*

Proof. Let a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ is IF-ψ-2-SB. Then by (5.1) there exists constants $h, k \in \mathbb{R} - \{0\}$ such that for every $x, z(nonzero) \in V$ and for every $t > 0$,

$$\begin{aligned} &\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t) \\ \text{i.e.} \quad &\mu'(Tx, z, t) \geq \mu(x, z, \frac{t}{\psi(h)}) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(x, z, \frac{t}{\psi(k)}) \\ \Rightarrow \quad &\mu'(Tx, z, t) \geq \mu(x, z, \frac{t}{\psi(h)}) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(x, z, \frac{t}{\psi(h)}). \end{aligned}$$

Thus for any $\alpha \in (0, 1)$ there exists $l > 0$ such that,

$$\begin{aligned} &\mu(lx, z, t) \geq \alpha \quad \text{and} \quad \nu(lx, z, t) \leq 1 - \alpha \\ \text{i.e.} \quad &\mu(x, z, \frac{t}{\psi(l)}) \geq \alpha \quad \text{and} \quad \nu(x, z, \frac{t}{\psi(l)}) \leq 1 - \alpha \\ \Rightarrow \quad &\mu'(Tx, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx, z, t) \leq 1 - \alpha. \end{aligned}$$

Hence by (5.2) a linear operator T is IF-ψ-2-WB. □

We provide following example to show converse is not true.

Example 5.4. Let $(V, \|\bullet, \bullet\|)$ be 2-normed space. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$.

$$\mu(x, z, t) = \frac{t^2 - \|x, z\|^2}{t^2 + \|x, z\|^2}, \quad \nu(x, z, t) = \frac{2\|x, z\|}{t^2 + \|x, z\|^2}, \quad t > \|x, z\| \quad \text{and}$$
$$\mu'(x, z, t) = \frac{t}{t + \|x, z\|}, \quad \nu'(x, z, t) = \frac{\|x, z\|}{t + \|x, z\|}, \quad t > \|x, z\|.$$

Then $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. Now, for $t > \|x, z\|$, $\quad \mu(lx, z, t) \geq \alpha$

$$\begin{aligned}
\Rightarrow \quad & \mu(x, z, \frac{t}{\psi(l)}) \geq \alpha \\
& \Rightarrow \quad \frac{\frac{t^2}{\psi(l)^2} - \|x, z\|^2}{\frac{t^2}{\psi(l)^2} + \|x, z\|^2} \geq \alpha \\
& \Rightarrow \quad \frac{t^2 - \psi(l)^2\|x, z\|^2}{t^2 + \psi(l)^2\|x, z\|^2} \geq \alpha \\
& \Rightarrow \quad t^2 - \psi(l)^2\|x, z\|^2 \geq \alpha t^2 + \alpha\psi(l)^2\|x, z\|^2 \\
& \Rightarrow \quad t^2 - \alpha t^2 \geq \alpha\psi(l)^2\|x, z\|^2 + \psi(l)^2\|x, z\|^2 \\
& \Rightarrow \quad t^2(1 - \alpha) \geq \|x, z\|^2\psi(l)^2(1 + \alpha) \\
& \Rightarrow \quad \|x, z\|^2 \leq \frac{t^2(1 - \alpha)}{\psi(l)^2(1 + \alpha)} \\
& \Rightarrow \quad \|x, z\| \leq \frac{t(1 - \alpha)^{\frac{1}{2}}}{\psi(l)(1 + \alpha)^{\frac{1}{2}}} \\
& \Rightarrow \quad \|x, z\| + t \leq \frac{t(1 - \alpha)^{\frac{1}{2}} + t\psi(l)(1 + \alpha)^{\frac{1}{2}}}{\psi(l)(1 + \alpha)^{\frac{1}{2}}} \\
& \Rightarrow \quad \frac{t}{\|x, z\| + t} \geq \frac{\psi(l)(1 + \alpha)^{\frac{1}{2}}}{(1 - \alpha)^{\frac{1}{2}} + \psi(l)(1 + \alpha)^{\frac{1}{2}}} \geq \alpha \\
& \Rightarrow \quad \mu'(Tx, z, t) \geq \alpha.
\end{aligned}$$

On similar lines, we can prove $\nu(hx,z,t) \leq 1-\alpha \Rightarrow \nu'(Tx,z,t) \leq 1-\alpha$. By (5.2) linear operator T is IF-ψ-2-WB. Conversely, for $t > \|z,x\|$,

$$\begin{aligned}
\mu'(Tx,z,t) \geq \mu(hx,z,t) \Leftrightarrow \quad & \mu'(Tx,z,t) \geq \mu(x,z,\frac{t}{\psi(h)}) \\
\Leftrightarrow \quad & \frac{t}{t+\|x,z\|} \geq \frac{\frac{t^2}{(\psi(h))^2} - \|x,z\|^2}{\frac{t^2}{(\psi(h))^2} + \|x,z\|^2} \\
\Leftrightarrow \quad & \frac{t}{t+\|x,z\|} \geq \frac{t^2 - (\psi(h))^2\|x,z\|^2}{t^2 + (\psi(h))^2\|x,z\|^2} \\
\Leftrightarrow \quad & t^3 + (\psi(h))^2 t\|x,z\|^2 \geq t^3 - (\psi(h))^2 t\|x,z\|^2 + t^2\|x,z\|^2 - (\psi(h))^2\|x,z\|^3 \\
\Leftrightarrow \quad & (\psi(h))^2 t\|x,z\|^2 + (\psi(h))^2 t\|x,z\|^2 + (\psi(h))^2\|x,z\|^3 \geq t^2\|x,z\|^2 \\
\Leftrightarrow \quad & (\psi(h))^2(2t\|x,z\|^2 + \|x,z\|^3) \geq t^2\|x,z\|^2 \\
\Leftrightarrow \quad & (\psi(h))^2 \geq \frac{t^2\|x,z\|^2}{2t\|x,z\|^2 + \|x,z\|^3} \\
\Leftrightarrow \quad & \psi(h) \geq \Big(\frac{t^2\|x,z\|^2}{2t\|x,z\|^2 + \|x,z\|^3}\Big)^{\frac{1}{2}} \\
\Leftrightarrow \quad & \psi(h) = \infty \quad \text{as} \quad t \to \infty.
\end{aligned}$$

Hence, T is not IF-ψ-2-SB.

Theorem 5.5. *Let $(V,\mu,\nu,*,\diamond), (V,\mu',\nu',*',\diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. Let $T : (V,\mu,\nu,*,\diamond) \to (V,\mu',\nu',*',\diamond')$ be a linear operator then T is strongly IF-ψ-2-C if and only if T is strongly IF-ψ-2-B.*

Proof. Let T be a strongly IF-ψ-2-B then by (5.1) then there exist constants $h,k \in \mathbb{R} - \{0\}$ such that for every $x, z(nonzero) \in V$ and for every $t > 0$,

$$\begin{aligned}
&\mu'(Tx,z,t) \geq \mu(hx,z,t) \quad \text{and} \quad \nu'(Tx,z,t) \leq \nu(kx,z,t) \\
\Rightarrow &\mu'(Tx,z,t) \geq \mu\Big(x,z,\frac{t}{\psi(h)}\Big) \quad \text{and} \quad \nu'(Tx,z,t) \leq \nu\Big(x,z,\frac{t}{\psi(k)}\Big) \\
\Rightarrow &\mu'(Tx - Tx_o,z,t) \geq \mu\Big(x - x_0,z,\frac{t}{\psi(h)}\Big) \quad \text{and} \quad \nu'(Tx - Tx_0,z,t) \leq \nu\Big(x - x_0,z,\frac{t}{\psi(k)}\Big) \\
\Rightarrow &\mu'(Tx - Tx_o,z,t) \geq \mu\Big(h(x - x_0),z,t\Big) \quad \text{and} \quad \nu'(Tx - Tx_0,z,t) \leq \nu\Big(k(x - x_0),z,t\Big) \\
\Rightarrow &T \;\; be \;\; a \;\; strongly \;\; IF - \psi - 2 - C.
\end{aligned}$$

Conversely, Let T be a strongly IF-ψ-2-C then by (4.2) if for given ϵ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in X$ and for all nonzero $z \in X$,

$$\mu'(Tx - Tx_0,z,\epsilon) \geq \mu(x - x_0,z,\delta) \quad \text{and} \quad \nu'(Tx - Tx_0,z,\epsilon) \leq \nu(x - x_0,z,\delta).$$

Suppose that $x \neq x_0$ and $t > 0$ putting $U = \frac{x}{\psi(\frac{1}{t})}$ then

$$\begin{aligned}
\mu'(Tx, z, t) &= \mu'(\psi(\frac{1}{t})TU, z, t) \\
&= \mu'(TU, z, \frac{t}{\psi(\psi(\frac{1}{t}))}) \\
&= \mu'(TU, z, \epsilon) \\
&\geq \mu(U, z, \delta) \\
&= \mu(x\psi^{-1}(\frac{1}{t}), z, \delta) \\
&= \mu(x, z, t\delta) \\
&= \mu(x, z, \frac{t}{\psi^{-1}(h)}), \text{where,} \psi^{-1}(h) = \frac{1}{\delta} \\
&= \mu(hx, z, t).
\end{aligned}$$

Similarly, we obtain $\nu'(Tx, z, t) \leq \nu(hx, z, t)$. Hence, T is strongly IF-ψ-2-B. □

Theorem 5.6. *Let $(V, \mu, \nu, *, \diamond), (V, \mu', \nu', *', \diamond')$ are intuitionistic fuzzy-ψ-2-normed spaces. Let $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *', \diamond')$ be a linear operator then T is IF-ψ-2-WC if and only if T is IF-ψ-2-WB.*

Proof. Let T be a IF-ψ-2-WB then by (5.2) then there exist constants $l > 0$ such that for every $x, z(nonzero) \in V$ and for every $t > 0$,

$$\begin{aligned}
& \mu(lx, z, t) \geq \alpha \quad \text{and} \quad \nu(lx, z, t) \leq 1 - \alpha \\
\Rightarrow \quad & \mu'(Tx, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx, z, t) \leq 1 - \alpha,
\end{aligned}$$

then

$$\begin{aligned}
& \mu(l(x - x_0), z, t) \geq \alpha \quad \text{and} \quad \nu(l(x - x_0), z, t) \leq 1 - \alpha \\
\Rightarrow \quad & \mu'(Tx - Tx_0, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx - Tx_0, z, t) \leq 1 - \alpha \\
\Rightarrow \quad & \text{T be a } IF - \psi - 2 - WC.
\end{aligned}$$

Conversely,

Let T be a IF-ψ-2-WC at $x = x_0$ then by (4.3) if for given $\epsilon = 1, \alpha \in (0, 1)$ there exists some $\delta = \delta(\alpha, \epsilon) > 0$ such that for $x \in X$ and for all nonzero $z \in X$,

$$\begin{aligned}
& \mu(x - x_0, z, \delta) \geq \alpha \quad \text{and} \quad \nu(x - x_0, z, \delta) \leq (1 - \alpha) \\
\Rightarrow & \mu'(Tx - Tx_0, z, 1) \geq \alpha \quad \text{and} \quad \nu'(Tx - Tx_0, z, 1) \leq (1 - \alpha).
\end{aligned}$$

If $x \neq x_0$ and $t > 0$. Take $x = U\psi^{-1}(\frac{1}{t})$.

We have

$$\mu(x, z, \delta) \geq \alpha \Rightarrow \mu\Big(U\psi^{-1}(\frac{1}{t}), z, \delta\Big) \geq \alpha \Rightarrow \mu'\Big(TU\psi^{-1}(\frac{1}{t}), z, t1\Big) \geq \alpha$$

$$\therefore \quad \mu(U, z, t\delta) \geq \alpha \Rightarrow \mu'(TU, z, t) \geq \alpha$$

$$\therefore \quad \mu(U, z, \frac{t}{\psi(l)}) \geq \alpha \Rightarrow \mu'(TU, z, t) \geq \alpha, \quad \text{where} \quad \delta = \frac{1}{\psi(l)}$$

$$\therefore \quad \mu(lU, z, t) \geq \alpha \Rightarrow \mu'(TU, z, t) \geq \alpha.$$

Similarly,

$$\nu\Big(lU, z, t\Big) \leq 1-\alpha \Rightarrow \nu'\Big(TU, z, t\Big) \leq 1-\alpha,$$

so T is IF-ψ-2-WB.

If $x = x_0$ then $\mu\Big(x - x_0, z, t\Big) = \mu'\Big(Tx - Tx_0, z, t\Big) = 1$ and $\nu\Big(x - x_0, z, t\Big) = \nu'\Big(Tx - Tx_0, z, t\Big) = 0$,
so $\mu\Big(lU, z, t\Big) \geq \alpha \Rightarrow \mu'\Big(TU, z, t\Big) \geq \alpha$ and $\nu\Big(lU, z, t\Big) \leq 1-\alpha \Rightarrow \nu'\Big(TU, z, t\Big) \leq 1-\alpha$,
so T is IF-ψ-2-WB. Hence, T is IF-ψ-2-WB. $\square$

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References

[1] M. Amini and R. Saadati, Some properties of continuous t-norms and s-norms, Int. J. Pure Appl. Math. 16 (2004) 157–164.

[2] M. Amini and R. Saadati, Topics in fuzzy metric space, J. Fuzzy Math. 4 (2003) 765–768.

[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[4] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, Ann. Fuzzy Math. Inform. 6 (2) (2013) 271–283.

[5] S. G. Dapke and C. T. Aage, J. N. Salunke, A new form of fuzzy compactness, Journal of contemporary applied mathematics 2 (2) (2012) 107–113.

[6] S. G. Dapke, C. T. Aage and J. N. Salunke, Intuitionistic fuzzy generalized normed spaces, Ann. Fuzzy Math. Inform. 7 (3) (2014) 369–383.

[7] A. George and P. V. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and Systems 64 (1994) 395–399.

[8] V. Gregori, S. Romaguera and P. V. Veeramani, A note on intuitionistic fuzzy metric space, Chaos Solitons Fractals 28 (2006) 902–905.

[9] Hai-Yan Si, Huai-Xin Cao and Ping Yang, Continuity in an intuitinistic fuzzy normed space, Fuzzy Systems and Knowledge Discovery (FSKD), Seventh International Conference 1 (2010) 144–148.

[10] H. W. Kang, J. G. Lee and K. Hur, Intuitionistic fuzzy mappings and intuitionistic fuzzy equivalence relations, Ann. Fuzzy Math. Inform. 3 (1) (2012) 61–87.

[11] R. Lowen, Fuzzy set theory, Kluwer Academic Publishers, Dordrecht 1996.

[12] M. Mursaleen and Q. M. Danish Lohani, Intuitionistic fuzzy 2-normed space and some related concepts, Chaos Solitons Fractals 42 (2009) 224–234.

[13] A. Narayanan, S. Vijayabalaji and N. Thillaigovindan, Intuitionistic fuzzy bounded linear operators, Iran. J. Fuzzy Syst. 4 (1) (2007) 89–101.

[14] J. H. Park, Intuitionistic fuzzy metric space, Chaos Solitons Fractals 22 (2004) 1039–1046.

[15] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27 (2006) 331–344.

[16] T. K. Samanta and S. Mohinta, A note on generalized intuitionistic fuzzy ψ normed linear space, Global Journal Of Science Frontier Research 11 (2011) 23–33.

[17] T. K. Samanta and I. H. Jebril, Finite dimensional intuitionistic fuzzy normed linear space, Int. J. Open Probl. Comput. Sci. Math. 2 (4) (2009) 574–591.

[18] B. Schweizer and A. Sklar, Statistical metric space, Pacific J. Math. 10 (1960) 314–334.

[19] Thangaraj Beaula and D. Lilly Esther Rani, Intuitionistic 2-fuzzy strong, weak continuity and boundedness, Ann. Fuzzy Math. Inform. 7 (6) (2014) 901–906.

[20] N. Thillaigovindan and S. Anita Shanti, On intuitionistic fuzzy compact linear operators, Ann. Fuzzy Math. Inform. 3 (1) (2012) 133–149.

[21] Upasana Samanta and T. Bag, Completeness and compactness of finite dimensional fuzzy n-normed linear spaces, Ann. Fuzzy Math. Inform. 7 (5) (2014) 837–850.
[22] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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