On intuitionistic fuzzy $\psi$-2-normed spaces

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Abstract. The aim of this paper is to study intuitionistic fuzzy $\psi$-2-normed space. We have discussed bounded linear operators, strong boundedness, weak boundedness, continuity in intuitionistic fuzzy $\psi$-2-normed space. Also, we have provided relation between boundedness and continuity. Further we have established some topological results in this new setup.


Keywords: t-norm, t-conorm, Intuitionistic fuzzy 2-normed space, Intuitionistic fuzzy 2-continuity, Intuitionistic fuzzy 2-boundedness, Intuitionistic fuzzy $\psi$-2-normed space.

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1. Introduction

In 1965, L. A. Zadeh[22] introduced the fuzzy set theory. R. Lowen[11] studied the basic results in fuzzy set theory. Fuzzy set theory has wide application in almost all branches of Science and Engineering. A. George and P. V. Veeramani[7] studied fuzzy metric spaces. The concept of intuitionistic fuzzy set was introduced by Atanassov[3]. Many authors[1, 2, 5, 8, 10, 13, 14, 17] have studied topological properties in fuzzy metric spaces, intuitionistic fuzzy metric spaces and related topics. Saadati and Park[15] coined the notion of intuitionistic fuzzy normed space. Continuity, boundedness, completeness and compactness in intuitionistic fuzzy normed spaces are studied in [4, 9, 19, 20, 21]. M. Mursaleen[12] defined the new structure intuitionistic fuzzy 2-normed space and studied some basic results of normed linear spaces. Recently, T.K. Samanta and Sumit Mohinta[16] have introduced the concept of intuitionistic fuzzy $\psi$-normed space and discussed continuity and boundedness in this structure. Some basic results of intuitionistic fuzzy $\psi$-2-normed space are discussed in[6].
In this paper, we have discussed bounded linear operators, strong boundedness, weak boundedness, strong continuity, weak continuity, sequentially continuity in intuitionistic fuzzy \( \psi \)-2-normed space.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

**Definition 2.1** ([18]). A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a continuous t-norm if it satisfies the following conditions:

(a) \( * \) is associative and commutative;
(b) \( * \) is continuous;
(c) \( a * 1 = a \) for all \( a \in [0, 1] \);
(d) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).

**Example 2.2.** Two typical examples of continuous t-norms are \( a \ast b = ab \) and \( a \ast b = \min\{a, b\} \).

**Definition 2.3** ([18]). A binary operation \( \circ : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a continuous t-conorm if it satisfies the following conditions:

(a) \( \circ \) is associative and commutative;
(b) \( \circ \) is continuous;
(c) \( a \circ 0 = a \) for all \( a \in [0, 1] \);
(d) \( a \circ b \leq c \circ d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).

**Example 2.4.** Two typical examples of continuous t-conorms are \( a \circ b = \min\{a + b, 1\} \) and \( a \circ b = \max\{a, b\} \).

**Definition 2.5** ([12]). Let \( V \) be a real vector space of dimension \( d \), where \( 2 \leq d < \infty \). A 2-norm on \( V \) is a function \( \| \cdot \| : V \times V \rightarrow \mathbb{R} \) which satisfies for every \( x, y, z \in V \):

(a) \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent;
(b) \( \|x, y\| = \|y, x\| \);
(c) \( \|\alpha x, y\| = \|x, \alpha y\| \);
(d) \( \|x, y + z\| \leq \|x, y\| + \|y, z\| \).

The pair \( V, \| \cdot, \cdot \| \) is then called a 2-normed space.

As an example of a 2-normed space take \( V = \mathbb{R}^2 \) being equipped with the 2-norm \( \|x, y\| := \text{the area of the parallelogram spanned by the vectors } x \text{ and } y \) which may be given explicitly by the formula \( \|x, y\| = |x_1y_2 - x_2y_1|, x = (x_1, x_2), y = (y_1, y_2) \).

**Definition 2.6** ([12]). The five-tuple \( (V, \mu, \nu, \ast, \circ) \) is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if \( V \) is a vector space over \( F \in \{\mathbb{R}, \mathbb{C}\} \), \( \ast \) is a continuous t-norm, \( \circ \) is a continuous t-conorm, and \( \mu, \nu \) are fuzzy sets on \( V \times V \times (0, \infty) \) satisfying the following conditions. For every \( x, y, z \in V \) and \( s, t > 0 \),

(a) \( \mu(x, y, t) + \nu(x, y, t) \leq 1 \);
(b) \( \mu(x, y, t) > 0 \);
(c) \( \mu(x, y, t) = 1 \) if and only if \( x \) and \( y \) are linearly dependent;
(d) \( \mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|}) \) for each \( \alpha \neq 0 \);
Let \(\lim_{t \to \infty} \mu(x, y, t) = 1\) and \(\lim_{t \to 0} \mu(x, y, t) = 0\);

\((e)\) \(\mu(x, y, t) \ast \mu(x, z, t) \leq \mu(x, y + z, t + s)\);

\((f)\) \(\mu(x, y, \cdot) : (0, \infty) \to [0, 1]\) is continuous;

\((g)\) \(\nu(x, y, t) < 1\);

\((j)\) \(\nu(x, y, t) = 0\) if and only if \(x\) and \(y\) are linearly dependent;

\((k)\) \(\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{|\alpha|})\) for each \(\alpha \neq 0\);

\((l)\) \(\nu(x, y, t) \ast \nu(x, z, s) \geq \nu(x, y + z, t + s)\);

\((m)\) \(\nu(x, y, \cdot) : (0, \infty) \to [0, 1]\) is continuous;

\((n)\) \(\lim_{t \to \infty} \nu(x, y, t) = 0\) and \(\lim_{t \to 0} \nu(x, y, t) = 1\);

\((o)\) \(\nu(x, y, t) = \nu(y, x, t)\).

In this case \((\mu, \nu)_2\) is called an intuitionistic fuzzy 2-norm on \(V\). We denote it by \((\mu, \nu)_2\).

**Example 2.7** ([12]). Let \((V, ||\cdot, \cdot||)\) be 2-normed space over \(F\) and let \(a \ast b = ab\) and \(a \circ b = \min\{a + b, 1\}\), for all \(a, b \in [0, 1]\) and every \(t > 0\), consider \(\mu(x, y, t) = \frac{t}{t + ||x, y||} \nu(x, y, t) = \frac{||x, y||}{t + ||x, y||}\). Then \((V, \mu, \nu, \ast, \circ)\) is an intuitionistic fuzzy 2-normed space.

### 3. Intuitionistic Fuzzy \(\psi\)-2-Normed Space

**Definition 3.1.** Let \(\psi\) be a function defined on the real field \(\mathbb{R}\) into itself satisfying the following properties:

\((a)\) \(\psi(-t) = \psi(t)\) for all \(t \in \mathbb{R}\)

\((b)\) \(\psi(1) = 1\)

\((c)\) \(\psi\) is strictly increasing and continuous on \((0, \infty)\)

\((d)\) \(t \leq \psi(t)\) \(\forall t \in (0, \infty)\)

\((e)\) \(\lim_{t \to 0} \psi(t) = 0\) and \(\lim_{t \to \infty} \psi(t) = \infty\).

**Example 3.2.**

\((1)\) \(\psi(\alpha) = |\alpha|\).

\((2)\) \(\psi(\alpha) = |\alpha|^n, n \in \mathbb{R}^+\).

\((3)\) \(\psi(\alpha) = \frac{2\alpha^{2n}}{|\alpha| + 1}, n \in \mathbb{N}^+\).

**Definition 3.3.** The five-tuple \((V, \mu, \nu, \ast, \circ)\) is said to be an intuitionistic fuzzy \(\psi\)-2-normed space, if \(V\) is a vector space over \(F \in \{\mathbb{R}, \mathbb{C}\}\), \(\ast\) is a continuous t-norm, \(\circ\) is a continuous t-conorm, and \(\mu, \nu\) are fuzzy sets on \(V \times V \times (0, \infty)\) satisfying the following conditions. For every \(x, y, z \in V\) and \(s, t > 0\),

\((a)\) \(\mu(x, y, t) + \nu(x, y, t) \leq 1\);

\((b)\) \(\mu(x, y, t) > 0\);

\((c)\) \(\mu(x, y, t) = 1\) if and only if \(x\) and \(y\) are linearly dependent;

\((d)\) \(\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})\) for each \(\alpha \neq 0\);

\((e)\) \(\mu(x, y, t) \ast \mu(x, z, s) \leq \mu(x, y + z, t + s)\);

\((f)\) \(\mu(x, y, \cdot) : (0, \infty) \to [0, 1]\) is continuous;

\((g)\) \(\lim_{t \to \infty} \mu(x, y, t) = 1\) and \(\lim_{t \to 0} \mu(x, y, t) = 0\);

\((h)\) \(\mu(x, y, t) = \mu(y, x, t)\);

\((i)\) \(\nu(x, y, t) < 1\);
3.10 In intuitionistic fuzzy
3.9 Let $(\mu, \nu, \ast, \circ)$ be an intuitionistic fuzzy
3.8 sequence.

\begin{align*}
(\ast) \nu(x, y, t) &= 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(\ast) \nu(\alpha x, y, t) &= \nu(x, y, \frac{t}{1-\alpha}) \text{ for each } \alpha \neq 0; \\
(\ast) \nu(x, y, t) \circ \nu(x, z, s) \geq \nu(x, y + z, t + s); \\
(\ast) \nu(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous}; \\
(\ast) \lim_{t \to \infty} \nu(x, y, t) = 0 \text{ and } \lim_{t \to 0} \nu(x, y, t) = 1; \\
(\ast) \nu(x, y, t) &= \nu(y, x, t).
\end{align*}

In this case $(\mu, \nu)_2$ is called an intuitionistic fuzzy $\psi$-2-norm on $V$.

**Definition 3.4.** Let $(V, \mu, \nu, \ast, \circ)$ be an intuitionistic fuzzy $\psi$-2-normed space. A sequence $\{x_n\}$ is said to be convergent to $x \in V$ with respect to the intuitionistic fuzzy $\psi$-2-norm $(\mu, \nu)_2$, if for every $r > 0$ and $t > 0$, $r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$ for all $n \geq n_0$ and for all $z \in V$.

**Definition 3.5.** Let $(V, \mu, \nu, \ast, \circ)$ be an intuitionistic fuzzy $\psi$-2-normed space. A sequence $\{x_n\}$ in $V$ is said to be Cauchy if for each $r > 0$ and each $t > 0, r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, z, t) > 1 - r$ and $\nu(x_n - x_m, z, t) < r$ for all $n, m \geq n_0$ and for all $z \in V$.

**Definition 3.6.** Let $(V, \mu, \nu, \ast, \circ)$ be an intuitionistic fuzzy $\psi$-2-normed space and let $r \in (0, 1), t > 0$ and $x \in X$. The set $B(x, r, t) = \{y \in V : \mu(y - x, z, t) > 1 - r, \nu(y - x, z, t) < r, \forall z \in V\}$ is called the open ball with center $x$ and radius $r$ with respect to $t$.

**Definition 3.7.** Let $(V, \mu, \nu, \ast, \circ)$ be an intuitionistic fuzzy $\psi$-2-normed space. A set $U \subset V$ is said to be open if each of its points is the centre of some open ball contained in $U$. The open set in an intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, \ast, \circ)$ is denoted by $U$.

**Theorem 3.8.** In intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, \ast, \circ)$. A sequence $\{x_n\}$ converges to $x$ if and only if $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$.

**Theorem 3.9.** The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, \ast, \circ)$.

**Theorem 3.10.** In IF $\psi$-2-NS $(V, \mu, \nu, \ast, \circ)$. Every convergent sequence is a cauchy sequence.

**Theorem 3.11.** In IF $\psi$-2-NS $(V, \mu, \nu, \ast, \circ)$. A sequence $\{x_n\}$ is a cauchy sequence if and only if $\mu(x_{n+p} - x, z, t) \to 1$ and $\nu(x_{n+p} - x, z, t) \to 0$ as $n \to \infty$.

**Definition 3.12.** An intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, \ast, \circ)$ is said to be complete if every cauchy sequence in IF $\psi$-2-NS $(V, \mu, \nu, \ast, \circ)$ is convergent.

**Theorem 3.13.** Let $(V, \mu, \nu, \ast, \circ)$ be a IF $\psi$-2-NS. A sufficient condition for the IF $\psi$-2-NS $(V, \mu, \nu, \ast, \circ)$ to be complete is that every Cauchy sequence in $(V, \mu, \nu, \ast, \circ)$ has a convergent subsequence.

The proofs of theorems 3.8, 3.9, 3.10, 3.11, 3.13 are easily obtained using [6].

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Definition 3.14. Let \((V, \mu, \nu, *, \circ)\) be an intuitionistic fuzzy \(\psi\)-2-normed space. A subset \(A\) of \(V\) is said to be \(IF_2\)-bounded if there exist \(t > 0\) and \(0 < r < 1\) such that 
\[
\mu(y - x, z, t) > 1 - r \quad \text{and} \quad \nu(y - x, z, t) < r \quad \text{for all} \ x, y \in A \quad \text{and for all} \ z \in V.
\]

Theorem 3.15. In an intuitionistic fuzzy \(\psi\)-2-normed space every compact set is closed and \(IF_2\)-bounded.

Lemma 3.16. A subset \(A\) of \(\mathbb{R}\) is \(IF_2\)-bounded in \((\mathbb{R}, \mu, \nu, *, \circ)\) if and only if it is bounded in \(\mathbb{R}\).

Lemma 3.17. A sequence \(\{\beta_n\}\) is convergent in the intuitionistic fuzzy \(\psi\)-2-normed space \((V, \mu, \nu, *, \circ)\) if and only if it is convergent in \((\mathbb{R}, | \cdot |)\).

Corollary 3.18. If the real sequence \(\{\beta_n\}\) is \(IF_2\)-bounded, then it has at least one limit point.

The proofs of theorem 3.15, lemmas 3.16, 3.17 and corollary 3.18 are easy so omitted.

Note that if \(t_1 \leq t_2\) then \(\psi(t_1) \leq \psi(t_2)\) which gives \(\frac{t}{\psi(t_1)} \geq \frac{t}{\psi(t_2)}\). Now 
\[
\mu(x_1, z, t) = \mu(x, z, \frac{t}{\psi(t_1)}) \geq \mu(x, z, \frac{t}{\psi(t_2)}) = \mu(x_1, z, t).
\]

Hence,
\[
(3.1) \quad \mu(x_1, z, t) \geq \mu(x_2, z, t) \quad \text{if} \quad t_1 \leq t_2.
\]

Theorem 3.19. Let \(\{x_1, \ldots, x_n\}\) be a linearly independent set of vectors in vector space \(V\) and \((V, \mu, \nu, *, \circ)\) be an intuitionistic fuzzy \(\psi\)-2-normed space. Then there are numbers \(c, d \neq 0\) and an intuitionistic fuzzy \(\psi\)-2-normed space \((\mathbb{R}, \mu, \nu, *, \circ)\) such that for every choice of real scalars \(\alpha_1, \ldots, \alpha_n\), we have,
\[
(3.2) \quad \mu(\alpha_1 x_1 + \cdots + \alpha_n x_n, z, t) \leq \mu_0(c |\psi(\alpha_1) + \cdots + \psi(\alpha_n)|, z, t)
\]

and
\[
(3.3) \quad \nu(\alpha_1 x_1 + \cdots + \alpha_n x_n, t) \geq \mu_0(d |\psi(\alpha_1) + \cdots + \psi(\alpha_n)|, z, t).
\]

Proof. Put \(s = |\alpha_1| + \cdots + |\alpha_n|\). If \(s = 0\), all \(\alpha_j\)'s must be zero so all \(\psi(\alpha_j)\) must be zero, hence (3.2) and (3.3) holds for any \(c, d \neq 0\). If \(s > 0\), by using definition 3.1, we have,
\[
\psi(\alpha_1) \geq |\alpha_1|, \ldots, \psi(\alpha_n) \geq |\alpha_n|
\]
\[
|\psi(\alpha_1) + \cdots + \psi(\alpha_n)| \geq |\alpha_1| + \cdots + |\alpha_n|.
\]

By using (3.1), we get,
\[
\mu_0(c |\psi(\alpha_1) + \cdots + \psi(\alpha_n)|, z, t) \leq \mu_0(c |\alpha_1| + \cdots + |\alpha_n|, z, t).
\]

Hence, (3.2) takes the form,
\[
\mu(\alpha_1 x_1 + \cdots + \alpha_n x_n, z, t) \leq \mu_0(c |\alpha_1| + \cdots + |\alpha_n|, z, t).
\]

Dividing by \(s\), we get,
\[
\mu \left( \frac{\alpha_1 x_1}{s} + \cdots + \frac{\alpha_n x_n}{s}, z, \frac{t}{s} \right) \leq \mu_0 \left( c \frac{|\alpha_1| + \cdots + |\alpha_n|}{|\alpha_1| + \cdots + |\alpha_n|}, z, \frac{t}{s} \right)
\]
\[
\leq \mu_0 \left( c, z, \frac{t}{s} \right).
\]

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Putting $\frac{\alpha_j}{s} = \beta_j$,

$$(3.4) \ \mu(\beta_1x_1 + \cdots + \beta_nx_n, z, t') \leq \mu_0(c, z, t'), \quad \text{where} \quad t' = \frac{t}{s}, \quad \sum_{j=1}^{n} |\beta_j| = 1. \tag{3.4}$$

Similarly,

$$(3.5) \ \nu(\beta_1x_1 + \cdots + \beta_nx_n, z, t') \geq \nu_0(d, z, t'), \quad \text{where} \quad t' = \frac{t}{s}, \quad \sum_{j=1}^{n} |\beta_j| = 1. \tag{3.5}$$

Hence, it suffices to prove the existence of $c, d \neq 0$ and an intuitionistic fuzzy-$\psi$-2-norm $(\mu_0, \nu_0)_2$ such that (3.4) and (3.5) holds.

Suppose that this is not true. Then there exists a sequence $\{y_m\}$ of vectors

$$y_m = \beta_{1,m}x_1 + \cdots + \beta_{n,m}x_n, \quad \left(\sum_{j=1}^{n} |\beta_{j,m}| = 1\right),$$

such that $\mu(y_m, z, t) \to 1$ and $\nu(y_m, z, t) \to 0$ as $m \to \infty$ for every $t > 0$. Since $\sum_{j=1}^{n} |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \leq 1$ and then by Lemma 3.16, the sequence of $\{\beta_{j,m}\}$ is IF-bounded. In according to Corollary 3.18, $\{\beta_{1,m}\}$ has a convergent subsequence, let $\beta$ denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding sequence of real scalars $\{\beta_{2,m}\}$ converges to some $\beta_2$.

Continuing this process, after $n$ steps we obtain a subsequence $\{y_{n,m}\}$ of $\{y_m\}$ such that

$$y_{n,m} = \sum_{j=1}^{n} \gamma_{j,m}x_j, \quad \left(\sum_{j=1}^{n} |\gamma_{j,m}| = 1\right) \quad \text{and} \quad \gamma_{j,m} \to \beta_j \quad \text{as} \quad m \to \infty.$$ 

Since,

$$\lim_{m} \mu(y_{n,m} - \sum_{j=1}^{n} \beta_jx_j, z, t) = \lim_{m} \mu\left(\sum_{j=1}^{n} (\gamma_{j,m} - \beta_j)x_j, z, t\right) \geq \lim_{m} \left(\mu((\gamma_{1,m} - \beta_1)x_1, z, \frac{t}{n}) \cdots \mu((\gamma_{n,m} - \beta_n)x_n, z, \frac{t}{n})\right) = 1$$

and

$$\lim_{m} \nu(y_{n,m} - \sum_{j=1}^{n} \beta_jx_j, z, t) = \lim_{m} \nu\left(\sum_{j=1}^{n} (\gamma_{j,m} - \beta_j)x_j, z, t\right) \leq \lim_{m} \left(\nu((\gamma_{1,m} - \beta_1)x_1, z, \frac{t}{n}) \cdots \nu((\gamma_{n,m} - \beta_n)x_n, z, \frac{t}{n})\right) = 0.$$ 

We have, $\lim_{m \to \infty} y_{n,m} = \sum_{j=1}^{n} \beta_jx_j$, $\left(\sum_{j=1}^{n} |\beta_j| = 1\right)$, so that not all $\beta_j$ can be zero.
Put \( y = \sum_{j=1}^{n} \beta_j x_j \) then \( y \neq 0 \) because \( \{x_1, \ldots, x_n\} \) is a linearly independent set. Since \( \mu(y_m, z, t) \to 1 \) and \( \nu(y_m, z, t) \to 0 \) by assumption, then we have \( \mu(y_n, z, t) \to 1 \) and \( \nu(y_n, z, t) \to 0 \). Hence
\[
\mu(y, z, t) = \mu(\langle y - y_n \rangle + y_n, z, t) \geq \left( \mu(y - y_n, z, t) \frac{t}{2} \right) \ast \mu(y_n, z, t) \to 1
\]
and
\[
\nu(y, z, t) = \nu(\langle y - y_n \rangle + y_n, z, t) \leq \left( \nu(y - y_n, z, t) \frac{t}{2} \right) \ast \nu(y_n, z, t) \to 0.
\]
Thus \( y = 0 \) which is a contradiction. □

**Definition 3.20.** Let \( (V, \mu, \nu, \ast, \diamond) \) and \( (V, \mu', \nu', \ast', \diamond') \) be intuitionistic fuzzy-\( \psi \)-2-normed spaces then intuitionistic fuzzy-\( \psi \)-2-norms \( (\mu, \nu)_2 \) and \( (\mu', \nu')_2 \) are said to be equivalent whenever \( x_n \to (\mu, \nu) \) \( x \) in \( (V, \mu, \nu, \ast, \diamond) \) and only if \( x_n \to (\mu', \nu') \) \( x \) in \( (V, \mu', \nu', \ast', \diamond') \).

**Theorem 3.21.** On a finite dimensional vector space \( V \), every two intuitionistic fuzzy-\( \psi \)-2-norms \((\mu, \nu)_2\) and \((\mu', \nu')_2\) are equivalent.

**Proof.** Let \( dim V = n \) and \( \{v_1, \ldots, v_n\} \) be a basis for \( V \). Then every \( x \in V \) has a unique representation \( x = \sum_{j=1}^{n} \alpha_j v_j \). Let \( x_n \to (\mu, \nu) \) \( x \) in \( (V, \mu, \nu, \ast, \diamond) \) but for each \( m \in N \), \( x_m \) has a unique representation, i.e. \( x_m = \alpha_1 v_1 + \cdots + \alpha_n v_n \). By theorem 3.19, there are \( c, d \neq 0 \) and an intuitionistic fuzzy-\( \psi \)-2-norm \((\mu_0, \nu_0)_2\) such that (3.2) and (3.3) hold. So
\[
\mu(x_m - x, z, t) = \mu\left( (\alpha_1 - \alpha_1) v_1 + (\alpha_2 - \alpha_2) v_2 + \cdots + (\alpha_n - \alpha_n) v_n \right)
\]
\[
\leq \mu_0\left( c \left[ \psi(\alpha_1 - \alpha_1) + \psi(\alpha_2 - \alpha_2) + \cdots + \psi(\alpha_n - \alpha_n) \right] \right)
\]
\[
\leq \mu_0(c \psi(\alpha_j - \alpha_j), z, t)
\]
and
\[
\nu(x_m - x, z, t) = \nu_0(d \psi(\alpha_j - \alpha_j), z, t).
\]
Now, if \( m \to \infty \) then \( \mu(x_m - x, z, t) \to 1 \) and \( \nu(x_m - x, z, t) \to 0 \) for every \( t > 0 \) and hence \( \psi(\alpha_j - \alpha_j) \to 0 \) in \( \mathbb{R} \).

On the other hand,
\[
\mu'(x_m - x, z, t) \geq \mu'(\langle \alpha_1 - \alpha_1 \rangle v_1, z, t) \ast \mu'(\langle \alpha_n - \alpha_n \rangle v_n, z, t) = \mu'(v_1, z, n \psi(\alpha_1 - \alpha_1), t) \ast \mu'(v_n, z, n \psi(\alpha_n - \alpha_n), t)
\]
and
\[
\nu'(x_m - x, z, t) \leq \nu'(\langle \alpha_1 - \alpha_1 \rangle v_1, z, t) \ast \nu'(\langle \alpha_n - \alpha_n \rangle v_n, z, t) = \nu'(v_1, z, n \psi(\alpha_1 - \alpha_1), t) \ast \nu'(v_n, z, n \psi(\alpha_n - \alpha_n), t)
\]
Since \( \psi(\alpha_j - \alpha_j) \to 0, n \psi(\alpha_j - \alpha_j) \to \infty \) and thus we have \( \mu'(v_1, z, n \psi(\alpha_1 - \alpha_1), t) \to 1 \) and \( \nu'(v_1, z, n \psi(\alpha_1 - \alpha_1), t) \to 0 \) then \( x_m \to (\mu', \nu') \) \( x \) in \((V, \mu', \nu', \ast', \diamond')\). With the same
argument, $x_m \rightarrow_{(\mu', \nu')_2} x$ in $(V, \mu', \nu', s', o')$ implies $x_m \rightarrow_{(\mu, \nu)_2} x$ in $(V, \mu, \nu, s, o)$.

4. Continuous linear operators

**Definition 4.1.** Let $(V, \mu, \nu, s, o), (V', \mu', \nu', s', o')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. A linear operator $T : (V, \mu, \nu, s, o) \rightarrow (V', \mu', \nu', s', o')$ is said to be intuitionistic fuzzy-$\psi$-2-continuous (IF-$\psi$-2-C) at $x_0 \in V$ if for given $\epsilon > 0, \alpha \in (0, 1)$ there exists some $\delta = \delta(\alpha, \epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$
\mu(x - x_0, z, \delta) > \psi(\delta) \text{ and } \nu(x - x_0, z, \delta) < (1 - \psi(\delta))
$$

\[ \Rightarrow \] $\mu'(Tx - Tx_0, z, \epsilon) > \psi(\alpha)$ and $\nu'(Tx - Tx_0, z, \epsilon) < (1 - \psi(\alpha))$.

If $T$ is IF-$\psi$-2-C at each point of $V$ then it is said to be IF-$\psi$-2-C on $V$.

**Definition 4.2.** Let $(V, \mu, \nu, s, o), (V', \mu', \nu', s', o')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. A linear operator $T : (V, \mu, \nu, s, o) \rightarrow (V', \mu', \nu', s', o')$ is said to be intuitionistic fuzzy-$\psi$-2-strongly continuous (IF-$\psi$-2-SC) at $x_0 \in V$ if for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta).
$$

If $T$ is IF-$\psi$-2-SC at each point of $V$ then $T$ is said to be IF-$\psi$-2-SC on $V$.

**Definition 4.3.** Let $(V, \mu, \nu, s, o), (V', \mu', \nu', s', o')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. A linear operator $T : (V, \mu, \nu, s, o) \rightarrow (V', \mu', \nu', s', o')$ is said to be intuitionistic fuzzy-$\psi$-2-weakly continuous (IF-$\psi$-2-WC) at $x_0 \in V$ if for given $\epsilon > 0, \alpha \in (0, 1)$ there exists some $\delta = \delta(\alpha, \epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

$$
\mu(x - x_0, z, \delta) \geq \psi(\alpha) \text{ and } \nu(x - x_0, z, \delta) \leq (1 - \psi(\alpha))
$$

\[ \Rightarrow \] $\mu'(Tx - Tx_0, z, \epsilon) \geq \psi(\alpha)$ and $\nu'(Tx - Tx_0, z, \epsilon) \leq (1 - \psi(\alpha))$.

**Definition 4.4.** Let $(V, \mu, \nu, s, o), (V', \mu', \nu', s', o')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. A linear operator $T : (V, \mu, \nu, s, o) \rightarrow (V', \mu', \nu', s', o')$ is said to be intuitionistic fuzzy-$\psi$-2-sequentially continuous (Seq-IF-$\psi$-2-C) at $x_0 \in V$ if for any sequence $\{x_k\}, \ k \geq 1$ with $x_k \rightarrow x_0$ implies $Tx_k \rightarrow Tx_0$.

\[ \text{i.e. } \lim_{k \to \infty} \mu(x_k - x_0, z, t) = 1 \text{ and } \lim_{k \to \infty} \nu(x_k - x_0, z, t) = 0 \ \forall t > 0 \]

\[ \Rightarrow \] $\lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, t) = 1$ and $\lim_{k \to \infty} \nu'(Tx_k - Tx_0, z, t) = 0 \ \forall t > 0$.

If $T$ is sequentially IF-$\psi$-2-C at each point of $V$ then $T$ is said to be sequentially IF-$\psi$-2-C on $V$.

**Theorem 4.5.** Let $(V, \mu, \nu, s, o), (V', \mu', \nu', s', o')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. Let $T : (V, \mu, \nu, s, o) \rightarrow (V', \mu', \nu', s', o')$ be a linear operator.

(a) If $T$ is IF-$\psi$-2-SC on $V$ iff it is IF-$\psi$-2-SC at point $x_0$.

(b) If $T$ is IF-$\psi$-2-WC on $V$ iff it is IF-$\psi$-2-WC at point $x_0$.

(c) If $T$ is Seq-IF-$\psi$-2-C on $V$ iff it is Seq-IF-$\psi$-2-C at point $x_0$. 350
(d) If $T$ is IF-ψ-2-C on $V$ iff it is IF-ψ-2-C at point $x_0$.

Proof. (a) Let $T$ be IF-ψ-2-SC at $x_0 \in V$ then for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

\begin{equation}
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta).
\end{equation}

Take any $y \in V$ and replace $x$ by $x + x_0 - y$, by (4.1), we get

\[
\mu'(Tx + x_0 - Ty - Tx_0, z, \epsilon) \geq \mu(x + x_0 - y - x_0, z, \delta) \text{ and } \\
\nu'(Tx + x_0 - Ty - Tx_0, z, \epsilon) \leq \nu(x + x_0 - y - x_0, z, \delta) \\
\Rightarrow \mu'(Tx - Ty, z, \epsilon) \geq \mu(x - y, z, \delta) \text{ and } \\
\nu'(Tx - Ty, z, \epsilon) \leq \nu(x - y, z, \delta) \\
\Rightarrow T \text{ is IF-ψ-2-SC at } y.
\]

Since, $y$ is arbitrary, hence $T$ is IF-ψ-2 SC on $V$.

Conversely, if $T$ is IF-ψ-2 SC on $V$ then by (4.2), $T$ is IF-ψ-2 SC at any $x_0 \in V$. The Proof of (b) follows easily by using (4.3) and (4.5)(a).

The Proof of (c) follows easily by using (4.4) and (4.5)(a).

The Proof of (d) follows easily by using (4.1) and (4.5)(a). □

Theorem 4.6. If a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', \ast', \lozenge')$ is IF-ψ-2-SC then it sequentially IF-ψ-2-C but converse need not true.

Proof. Let $\{x_k\}$, $k \geq 1$ be a sequence such that $x_k \to x_0$.

i.e. $\lim_{k \to \infty} \mu(x_k - x_0, z, t) = 1$ and $\lim_{k \to \infty} \nu(x_k - x_0, z, t) = 0$, $\forall t > 0$.

Let $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', \ast', \lozenge')$ is IF-ψ-2-SC at $x_0 \in V$ if for given $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in V$ and for all nonzero $z \in V$,

\[
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta) \\
\Rightarrow \mu'(Tx_k - Tx_0, z, \epsilon) \geq \mu(x_k - x_0, z, \delta) \text{ and } \\
\nu'(Tx_k - Tx_0, z, \epsilon) \leq \nu(x_k - x_0, z, \delta) \\
\Rightarrow \lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, \epsilon) \geq \lim_{k \to \infty} \mu(x_k - x_0, z, \delta) \text{ and } \\
\lim_{k \to \infty} \nu'(Tx_k - Tx_0, z, \epsilon) \leq \lim_{k \to \infty} \nu(x_k - x_0, z, \delta) \\
\Rightarrow \lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, \epsilon) \geq 1 \text{ and } \lim_{k \to \infty} \nu'(Tx_k - Tx_0, z, \epsilon) \leq 0 \\
\Rightarrow \lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, \epsilon) = 1 \text{ and } \lim_{k \to \infty} \nu'(Tx_k - Tx_0, z, \epsilon) = 0 \\
\Rightarrow Tx_k \to Tx_0, \text{ (since, } \epsilon > 0 \text{ is arbitrary).}
\]

By (4.4), $T$ is sequentially IF-ψ-2-C at $x_0$. Hence, $T$ is sequentially IF-ψ-2-C on $V$.

Conversely, we provide example to prove sequentially IF-ψ-2-C does not imply IF-ψ-2-SC.
Example 4.7. Let \((V, \|\cdot\|)\) be 2-normed space. Define \(a \ast b = \min\{a, b\}\) and 
\(a \circ b = \max\{a, b\}\) for all \(a, b \in [0, 1]\).

\[
\mu'(x, z, t) = \frac{t}{t + c\|x, z\|}, \quad \nu'(x, z, t) = \frac{c\|x, z\|}{t + c\|x, z\|}, \quad t > \|x, z\| \quad \text{and} \quad \mu(x, z, t) = \frac{t}{t + \|x, z\|}, \quad \nu(x, z, t) = \frac{\|x, z\|}{t + \|x, z\|}, \quad t > \|x, z\|.
\]

Then \((V, \mu, \nu, \ast, \circ)\), \((V, \mu', \nu', \ast', \circ')\) are intuitionistic fuzzy-\(\psi\)-2-normed spaces.

Consider, \(T(x, z, t) = \frac{\|x, z\|^4}{1 + \|x, z\|^2}\), choose sequence \(\{x_k\}, k \geq 1\) such that \(x_k \to x_0\). Now for all \(t > 0\), we have

\[
\lim_{k \to \infty} \mu(x_k - x_0, z, t) = 1 \quad \text{and} \quad \lim_{k \to \infty} \nu(x_k - x_0, z, t) = 0
\]

\[
\Rightarrow \lim_{k \to \infty} \|x_k - x_0\| = 0.
\]

Now,

\[
\mu'(Tx_k - Tx_0, z, t) = \frac{t}{t + c\|x, z\|^4 - \|x, z\|^2 - (x_0 - z)^2} = \frac{t}{t + \|x, z\|^4 - \|x, z\|^2 - (x_0 - z)^2}
\]

\[
\nu'(Tx_k - Tx_0, z, t) = \frac{c\|x, z\|^4 - \|x, z\|^2 - (x_0 - z)^2}{t + c\|x, z\|^4 - \|x, z\|^2 - (x_0 - z)^2}
\]

Similarly,

\[
\lim_{k \to \infty} \|x_k - x_0\| = 0, \quad \text{it follows that,}
\]

\[
\lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, t) = 1 \quad \text{and} \quad \lim_{k \to \infty} \nu'(Tx_k - Tx_0, z, t) = 0, \quad \forall t > 0
\]

\[
\Rightarrow Tx_k \to Tx_0.
\]

Hence, \(T\) is Seq-IF-\(\psi\)-2-C on \(V\).

Let \(\epsilon > 0\) be given. Then

\[
\mu'(Tx_k - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta)
\]

\[
\Rightarrow \frac{\epsilon(1 + (x_k, z)^2)(1 + (x_0, z)^2)}{\epsilon(1 + (x_k, z)^2)(1 + (x_0, z)^2) + c((x_k - x_0, z)(x_k + x_0, z)^2 + (x_k, z)^2 - (x_0, z)^2(x_k - x_0, z))^2}
\]

\[
\geq \frac{\delta}{\delta + \|x - x_0, z\|}
\]

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\[
\nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta)
\]

\[
\Rightarrow \frac{c \|(x - x_0, z)\|(x + x_0, (x + x_0)^2 - (x_0, z)^2)}{\epsilon 1 + (x, z)^2(1 + (x, z)^2 + c \|(x - x_0, z)\|(x + x_0, (x + x_0)^2 - (x_0, z)^2)} \leq \frac{\delta}{\delta + \|x - x_0, z\|}.
\]

So

\[
(4.2) \quad \delta \leq \frac{c \|x + x_0, z\|(x, z)^2}{\epsilon 1 + (x, z)^2(1 + (x, z)^2 + c \|x - x_0, z\| (x + x_0, (x + x_0)^2 - (x_0, z)^2)}
\]

Clearly, \(T\) is IF-\(\psi\)-2-SC at \(x_0\) if there exists \(\delta > 0\) satisfying (4.2) for all \(x \neq x_0\).

Let

\[
\delta_1 = \inf \frac{\|1 + (x, z)^2\|(1 + (x, z)^2)}{\|x + x_0, z\| (x, z)^2 + (x, z)^2 (x_0, z)^2}
\]

then \(\delta = \frac{\epsilon}{\epsilon} \delta_1\) satisfying (4.2). But \(\delta_1 = 0\) which is impossible. So, \(T\) is not IF-\(\psi\)-2-SC on \(V\).

**Theorem 4.8.** If a linear operator \(T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond')\) is IF-\(\psi\)-2-SC then it IF-\(\psi\)-2-WC.

**Proof.** A linear operator \(T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond')\) is IF-\(\psi\)-2-SC at \(x_0 \in V\) then for given \(\epsilon > 0\) there exists some \(\delta = \delta(\epsilon) > 0\) such that for \(x \in V\) and for all nonzero \(z \in V\),

\[
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta).
\]

Let \(\mu(x - x_0, z, \delta) \geq \psi(\alpha)\) and \(\nu(x - x_0, z, \delta) \leq (1 - \psi(\alpha))\) then

\[
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \geq \psi(\alpha) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta) \leq (1 - \psi(\alpha)).
\]

\[
\Rightarrow \mu'(Tx - Tx_0, z, \epsilon) \geq \psi(\alpha) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) \leq (1 - \psi(\alpha)).
\]

So \(T\) is IF-\(\psi\)-2-WC at \(x_0\). Hence, \(T\) is IF-\(\psi\)-2-WC on \(V\).

**Theorem 4.9.** A linear operator \(T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond')\) is IF-\(\psi\)-2-C on \(V\) iff it is sequentially IF-\(\psi\)-2-C on \(V\).

**Proof.** Let a linear operator \(T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond')\) is IF-\(\psi\)-2-C at \(x_0 \in V\) then by (4.1), for given \(\epsilon > 0, \alpha \in (0, 1)\) there exists some \(\delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0, 1)\) such that for \(x \in V\) and for all nonzero \(z \in V\),

\[
\mu(x - x_0, z, \delta) > \psi(\beta) \quad \text{and} \quad \nu(x - x_0, z, \delta) < (1 - \psi(\beta))
\]

\[
\Rightarrow \mu'(Tx - Tx_0, z, \epsilon) > \psi(\alpha) \quad \text{and} \quad \nu'(x - x_0, z, \delta) < (1 - \psi(\alpha))
\]

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Let \( \{x_k\} \), \( k \geq 1 \) be a sequence such that \( x_k \to x_0 \) so there exists positive integer \( k_0 \) such that,

\[
\mu(x_k - x_0, z, \delta) > \psi(\beta) \quad \text{and} \quad \nu(x_k - x_0, z, \delta) < (1 - \psi(\beta))
\]

\[\Rightarrow \mu'(Tx_k - Tx_0, z, \epsilon) > \psi(\alpha) \quad \text{and} \quad \nu'(x_k - x_0, z, \delta) < (1 - \psi(\alpha)), \quad \forall k \geq k_0
\]

which gives

\[
\lim_{k \to \infty} \mu(x_k - x_0, z, \delta) > \psi(\beta) \quad \text{and} \quad \lim_{k \to \infty} \nu(x_k - x_0, z, \delta) < (1 - \psi(\beta))
\]

\[\Rightarrow \lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, \epsilon) > \psi(\alpha) \quad \text{and} \quad \lim_{k \to \infty} \nu'(x_k - x_0, z, \delta) < (1 - \psi(\alpha)), \quad \forall k \geq k_0
\]

which gives

\[
\lim_{k \to \infty} \mu'(Tx_k - Tx_0, z, \epsilon) = 1 \quad \text{and} \quad \lim_{k \to \infty} \nu'(x_k - x_0, z, \delta) = 0, \quad \forall k \geq k_0
\]

\[\Rightarrow Tx_k \to Tx_0, \quad (\text{since} \; \epsilon > 0 \; \text{is arbitrary}).
\]

By (4.4), we get \( T \) is sequentially IF-\( \psi \)-2-C at \( x_0 \). Hence, \( T \) is sequentially IF-\( \psi \)-2-C on \( V \).

Conversely, let \( T \) is sequentially IF-\( \psi \)-2-C on \( V \) so \( T \) is sequentially IF-\( \psi \)-2-C at \( x_0 \). If possible \( T \) is not IF-\( \psi \)-2-C at \( x_0 \) then for given \( \epsilon > 0, \alpha \in (0, 1) \) there exists some \( \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0, 1) \) such that for \( y \in V \) and for all nonzero \( z \in V \),

\[
(3.1) \quad \mu(x_0 - y, z, \delta) > \psi(\beta) \quad \text{and} \quad \nu(x_0 - y, z, \delta) < (1 - \psi(\beta)) \quad \text{and for all nonzero} \quad y \in V,
\]

\[
(4.4) \quad \mu'(Tx_0 - Ty, z, \epsilon) \leq \psi(\alpha) \quad \text{and} \quad \nu'(Tx_0 - Ty, z, \epsilon) \geq (1 - \psi(\alpha)).
\]

Thus for \( \beta = \frac{k}{k+1}, \; \delta = \frac{1}{k+1}, \; k = 1, 2, \ldots \) there exists \( \{y_k\} \) such that,

\[
\mu(x_0 - y_k, z, \frac{1}{k+1}) > \psi(\frac{k}{k+1}) \quad \text{and} \quad \nu(x_0 - y_k, z, \frac{1}{k+1}) < (1 - \psi(\frac{k}{k+1}))
\]

but \( \mu'(Tx_0 - Ty_k, z, \epsilon) \leq \psi(\alpha) \) and \( \nu'(Tx_0 - Ty, z, \epsilon) \geq (1 - \psi(\alpha)) \).

Taking \( \delta > 0 \) there exists \( k_0 \) such that \( \frac{1}{k+1} > \delta \) for all \( k \geq k_0 \) then

\[
\mu(x_0 - y_k, z, \delta) \geq \mu(x_0 - y_k, z, \frac{1}{k+1}) > (\psi(\frac{k}{k+1})) \quad \text{and} \quad \nu(x_0 - y_k, z, \delta) \leq \nu(x_0 - y_k, z, \frac{1}{k+1}) < (1 - \psi(\frac{k}{k+1}))
\]

\[\Rightarrow \lim_{k \to \infty} \mu(x_0 - y_k, z, \delta) \geq \lim_{k \to \infty} \mu(x_0 - y_k, z, \frac{1}{k+1}) > \lim_{k \to \infty} (\psi(\frac{k}{k+1})) = 1 \quad \text{and}
\]

\[\lim_{k \to \infty} \nu(x_0 - y_k, z, \delta) \leq \lim_{k \to \infty} \nu(x_0 - y_k, z, \frac{1}{k+1}) < \lim_{k \to \infty} (1 - \psi(\frac{k}{k+1})) = 0
\]

\[\Rightarrow \lim_{k \to \infty} \mu(x_0 - y_k, z, \delta) = 1 \quad \text{and} \quad \lim_{k \to \infty} \nu(x_0 - y_k, z, \delta) = 0
\]

\[\Rightarrow \{y_k\} \to x_0,
\]

from (4.3), \( \lim_{k \to \infty} \mu'(Tx_0 - Ty_k, z, \epsilon) \leq \psi(\alpha) \) and \( \lim_{k \to \infty} \nu'(Tx_0 - Ty_k, z, \epsilon) \geq (1 - \psi(\alpha)) \)

\[\therefore \lim_{k \to \infty} \mu'(Tx_0 - Ty_k, z, \epsilon) \neq 1 \quad \text{and} \quad \lim_{k \to \infty} \nu'(Tx_0 - Ty_k, z, \epsilon) \neq 0.
\]
A linear operator

Let a linear operator

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Let

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Let

\( (5.1) \)

Hence by \((5.1)\) a linear operator \( T \) is IF-\( \psi \)-2-SC on \( V \) but converse need not true.

**Theorem 4.10.** A linear operator \( T : (V, \mu, \nu, * , \diamond) \rightarrow (V', \mu', \nu', * , \diamond') \) is IF-\( \psi \)-2-SC on \( V \) then it is IF-\( \psi \)-2-C on \( V \) but converse need not true.

**Proof.** Let \( T \) is IF-\( \psi \)-2-SC on \( V \) then by using \((4.6)\) and \((4.9)\) it is IF-\( \psi \)-2-C on \( V \). Converse easily disproved by using theorem \((4.9)\) and example \((4.7)\). \( \Box \)

5. Bounded linear operators

**Definition 5.1.** Let \((V, \mu, \nu, * , \diamond)\), \((V', \mu', \nu', * , \diamond')\) are intuitionistic fuzzy-\( \psi \)-2-normed spaces. A linear operator \( T : (V, \mu, \nu, * , \diamond) \rightarrow (V', \mu', \nu', * , \diamond') \) is said to be (IF-\( \psi \)-2-SB) intuitionistic fuzzy-\( \psi \)-2-strongly bounded if there exist constants \( h, k \in \mathbb{R} - \{0\} \) such that for every \( x, z(\text{nonzero}) \in V \) and for every \( t > 0 \),

\[
\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t).
\]

**Definition 5.2.** Let \((V, \mu, \nu, * , \diamond)\), \((V', \mu', \nu', * , \diamond')\) are intuitionistic fuzzy-\( \psi \)-2-normed spaces. A linear operator \( T : (V, \mu, \nu, * , \diamond) \rightarrow (V', \mu', \nu', * , \diamond') \) is said to be (IF-\( \psi \)-2-WB) intuitionistic fuzzy-\( \psi \)-2-weakly bounded if there exist \( l > 0 \) such that for all \( x, z(\text{nonzero}) \in V \) and for every \( t > 0 \),

\[
\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t).
\]

**Theorem 5.3.** Let \((V, \mu, \nu, * , \diamond)\), \((V', \mu', \nu', * , \diamond')\) are intuitionistic fuzzy-\( \psi \)-2-normed spaces. If a linear operator \( T : (V, \mu, \nu, * , \diamond) \rightarrow (V', \mu', \nu', * , \diamond') \) is IF-\( \psi \)-2-SB then it is IF-\( \psi \)-2-WB but converse is not true.

**Proof.** Let a linear operator \( T : (V, \mu, \nu, * , \diamond) \rightarrow (V', \mu', \nu', * , \diamond') \) is IF-\( \psi \)-2-SB. Then by \((5.1)\) there exists constants \( h, k \in \mathbb{R} - \{0\} \) such that for every \( x, z(\text{nonzero}) \in V \) and for every \( t > 0 \),

\[
\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t).
\]

i.e.

\[
\mu'(Tx, z, t) \geq \mu(x, z, \frac{t}{\psi(h)}) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(x, z, \frac{t}{\psi(k)}).
\]

Thus for any \( \alpha \in (0, 1) \) there exists \( t > 0 \) such that,

\[
\mu(x, z, \frac{t}{\psi(h)}) \geq \alpha \quad \text{and} \quad \nu(x, z, \frac{t}{\psi(h)}) \leq 1 - \alpha.
\]

i.e.

\[
\mu'(Tx, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx, z, t) \leq 1 - \alpha.
\]

Hence by \((5.2)\) a linear operator \( T \) is IF-\( \psi \)-2-WB. \( \Box \)
We provide following example to show converse is not true.

**Example 5.4.** Let \((V, \bullet, \bullet)\) be \(2\)-normed space. Define \(a \ast b = \min\{a, b\}\) and \(a \circ b = \max\{a, b\}\) for all \(a, b \in [0, 1]\).

\[
\mu(x, z, t) = \frac{t^2 - \|x, z\|^2}{t^2 + \|x, z\|^2}, \quad \nu(x, z, t) = \frac{2\|x, z\|}{t^2 + \|x, z\|^2}, \quad t > \|x, z\| \quad \text{and}
\]
\[
\mu'(x, z, t) = \frac{t}{t + \|x, z\|}, \quad \nu'(x, z, t) = \frac{\|x, z\|}{t + \|x, z\|}, \quad t > \|x, z\|.
\]

Then \((V, \mu, \nu, \ast, \circ), (V, \mu', \nu', \ast', \circ')\) are intuitionistic fuzzy-\(\psi\)-\(2\)-normed spaces. Now, for \(t > \|x, z\|, \quad \mu'(Tx, z; t) \geq \alpha\)

\[
\Rightarrow \mu(x, z, \frac{t}{\psi(l)}) \geq \alpha
\]

\[
\Rightarrow \frac{t^2 - \|x, z\|^2}{\psi(l)^2} - \frac{\|x, z\|^2}{\psi(l)^2} \geq \alpha
\]

\[
\Rightarrow \frac{t^2 - \psi(l)^2\|x, z\|^2}{\psi(l)^2} \geq \alpha
\]

\[
\Rightarrow \frac{t^2}{\psi(l)^2\|x, z\|^2} \geq \alpha
\]

\[
\Rightarrow t^2 - \psi(l)^2\|x, z\|^2 \geq \alpha t^2 + \alpha \psi(l)^2\|x, z\|^2
\]

\[
\Rightarrow t^2 - \alpha t^2 \geq \alpha \psi(l)^2\|x, z\|^2 + \psi(l)^2\|x, z\|^2
\]

\[
\Rightarrow t^2(1 - \alpha) \geq \|x, z\|^2\psi(l)^2(1 + \alpha)
\]

\[
\Rightarrow \|x, z\|^2 \leq \frac{t^2(1 - \alpha)}{\psi(l)^2(1 + \alpha)}
\]

\[
\Rightarrow \|x, z\| \leq \frac{t(1 - \alpha)^{\frac{1}{2}}}{\psi(l)(1 + \alpha)^{\frac{1}{2}}}
\]

\[
\Rightarrow \|x, z\| + t \leq \frac{t(1 - \alpha)^{\frac{1}{2}} + t\psi(l)(1 + \alpha)^{\frac{1}{2}}}{\psi(l)(1 + \alpha)^{\frac{1}{2}}}
\]

\[
\Rightarrow \frac{t}{\psi(l)(1 + \alpha)^{\frac{1}{2}}} \geq \frac{\psi(l)(1 + \alpha)^{\frac{1}{2}}}{(1 - \alpha)^{\frac{1}{2}} + \psi(l)(1 + \alpha)^{\frac{1}{2}}} \geq \alpha
\]

\[
\Rightarrow \mu'(Tx, z, t) \geq \alpha.
\]
On similar lines, we can prove $\nu(hx, z, t) \leq 1 - \alpha \Rightarrow \nu'(Tx, z, t) \leq 1 - \alpha$. By (5.2) linear operator $T$ is IF-$\psi$-2-WB. Conversely, for $t > \|z, z\|,$

\[
\mu'(Tx, z, t) \geq \mu(hx, z, t) \iff \mu'(Tx, z, t) \geq \mu(x, z, \frac{t}{\psi(h)})
\]

\[
\Rightarrow \frac{t}{t + \|x, z\|} \geq \frac{t^2}{(\psi(h))^2} - \|x, z\|^2
\]

\[
\Rightarrow \frac{t}{t + \|x, z\|} \geq \frac{t^2 - (\psi(h))^2 \|x, z\|^2}{t^2 + (\psi(h))^2 \|x, z\|^2}
\]

\[
\Rightarrow t^3 + (\psi(h))^2 t \|x, z\|^2 \geq t^3 - (\psi(h))^2 t \|x, z\|^2 + t^2 \|x, z\|^2 - (\psi(h))^2 \|x, z\|^3
\]

\[
\Rightarrow (\psi(h))^2 (2t \|x, z\|^2 + \|x, z\|^3) \geq t^2 \|x, z\|^2
\]

\[
\Rightarrow (\psi(h))^2 \geq \frac{t^2 \|x, z\|^2}{2t \|x, z\|^2 + \|x, z\|^3}
\]

\[
\Rightarrow \psi(h) \geq \left(\frac{2t \|x, z\|^2 + \|x, z\|^3}{t^2 \|x, z\|^2}\right)^\frac{1}{2}
\]

\[
\Rightarrow \psi(h) = \infty \text{ as } t \to \infty.
\]

Hence, $T$ is not IF-$\psi$-2-SB.

**Theorem 5.5.** Let $(V, \mu, \nu, *, \circ), (V, \mu', \nu', *, \circ')$ are intuitionistic fuzzy-$\psi$-2-normed spaces. Let $T : (V, \mu, \nu, *, \circ) \to (V, \mu', \nu', *, \circ')$ be a linear operator then $T$ is strongly IF-$\psi$-2-C if and only if $T$ is strongly IF-$\psi$-2-B.

**Proof.** Let $T$ be a strongly IF-$\psi$-2-B then by (5.1) there exist constants $h, k \in \mathbb{R} - \{0\}$ such that for every $x, z (\text{nonzero}) \in V$ and for every $t > 0$,

\[
\mu'(Tx, z, t) \geq \mu(hx, z, t) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(kx, z, t)
\]

\[
\Rightarrow \mu'(Tx, z, t) \geq \mu(x, z, \frac{t}{\psi(h)}) \quad \text{and} \quad \nu'(Tx, z, t) \leq \nu(x, z, \frac{t}{\psi(h)})
\]

\[
\Rightarrow \mu'(Tx - Tx_0, z, t) \geq \mu(x - x_0, z, \frac{t}{\psi(h)}) \quad \text{and} \quad \nu'(Tx - Tx_0, z, t) \leq \nu(x - x_0, z, \frac{t}{\psi(h)})
\]

\[
\Rightarrow \mu'(Tx - Tx_0, z, t) \geq \mu(h(x - x_0), z, t) \quad \text{and} \quad \nu'(Tx - Tx_0, z, t) \leq \nu(k(x - x_0), z, t)
\]

\[
\Rightarrow T \text{ be a strongly IF- } \psi - 2 - C.
\]

Conversely, Let $T$ be a strongly IF-$\psi$-2-C then by (4.2) if for given $\epsilon$ there exists some $\delta = \delta(\epsilon) > 0$ such that for $x \in X$ and for all nonzero $z \in X$,

\[
\mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) \leq \nu(x - x_0, z, \delta).
\]
Suppose that \( x \neq x_0 \) and \( t > 0 \) putting \( U = \frac{x}{\psi(t)} \) then
\[
\mu'(Tx, z, t) = \mu' \left( \frac{1}{t}TU, z, t \right)
\]
\[
= \mu' \left( TU, z, \frac{t}{\psi \left( \frac{1}{t} \right)} \right)
\]
\[
= \mu' \left( TU, z, \epsilon \right)
\]
\[
\geq \mu(U, z, \delta)
\]
\[
= \mu(x, \psi^{-1}(\frac{1}{t}), z, \delta)
\]
\[
= \mu(x, z, t\delta)
\]
\[
= \mu(x, z, \frac{t}{\psi^{-1}(\delta)}), \text{where} \psi^{-1}(\delta) = \frac{1}{\delta}
\]
\[
= \mu(hx, z, t).
\]
Similarly, we obtain \( \nu'(Tx, z, t) \leq \nu(hx, z, t) \). Hence, \( T \) is strongly \( IF-\psi-2-B. \)

**Theorem 5.6.** Let \((V, \mu, \nu, *, \circ), (V, \mu', \nu', *, \circ')\) are intuitionistic fuzzy-\( \psi-2 \)-normed spaces. Let \( T : (V, \mu, \nu, *, \circ) \rightarrow (V, \mu', \nu', *, \circ')\) be a linear operator then \( T \) is \( IF-\psi-2-WC \) if and only if \( T \) is \( IF-\psi-2-WB \).

**Proof.** Let \( T \) be a \( IF-\psi-2-WB \) then by (5.2) then there exist constants \( l > 0 \) such that for every \( x, z \) (nonzero) \( \in V \) and for every \( t > 0 \),
\[
\mu(lx, z, t) \geq \alpha \quad \text{and} \quad \nu(lx, z, t) \leq 1 - \alpha
\]
\[\Rightarrow\]
\[
\mu'(Tx, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx, z, t) \leq 1 - \alpha,
\]
then
\[
\mu(l(x - x_0), z, t) \geq \alpha \quad \text{and} \quad \nu(l(x - x_0), z, t) \leq 1 - \alpha
\]
\[\Rightarrow\]
\[
\mu'(Tx - Tx_0, z, t) \geq \alpha \quad \text{and} \quad \nu'(Tx - Tx_0, z, t) \leq 1 - \alpha
\]
\[\Rightarrow\]
\[
T \text{ be a } IF-\psi-2-WC.
\]
Conversely,
Let \( T \) be a \( IF-\psi-2-WC \) at \( x = x_0 \) then by (4.3) if for given \( \epsilon = 1, \alpha \in (0, 1) \) there exists some \( \delta = \delta(\alpha, t) > 0 \) such that for \( x \in X \) and for all nonzero \( z \in X \),
\[
\mu(x - x_0, z, \delta) \geq \alpha \quad \text{and} \quad \nu(x - x_0, z, \delta) \leq (1 - \alpha)
\]
\[\Rightarrow\]
\[
\mu'(Tx - Tx_0, z, 1) \geq \alpha \quad \text{and} \quad \nu'(Tx - Tx_0, z, 1) \leq (1 - \alpha).
\]
If \( x \neq x_0 \) and \( t > 0 \). Take \( x = U\psi^{-1}(\frac{1}{t}) \).
We have
\[
\mu(x, z, \delta) \geq \alpha \Rightarrow \mu \left( U\psi^{-1}(\frac{1}{t}), z, \delta \right) \geq \alpha \Rightarrow \mu' \left( TU\psi^{-1}(\frac{1}{t}), z, t1 \right) \geq \alpha
\]
\[\therefore\]
\[
\mu(U, z, t\delta) \geq \alpha \Rightarrow \mu' \left( TU, z, t \right) \geq \alpha
\]
\[\therefore\]
\[
\mu(U, z, \frac{t}{\psi(t)}) \geq \alpha \Rightarrow \mu' \left( TU, z, t \right) \geq \alpha, \quad \text{where} \quad \delta = \frac{1}{\psi(t)}
\]
\[\therefore\]
\[
\mu(lU, z, t) \geq \alpha \Rightarrow \mu' \left( TU, z, t \right) \geq \alpha.
\]
Similarly,
\[ \nu(IU, z, t) \leq 1 - \alpha \Rightarrow \nu'(TU, z, t) \leq 1 - \alpha, \]
so \( T \) is IF-\( \psi \)-2-WB.

If \( x = x_0 \) then \( \mu(x - x_0, z, t) = \mu'(Tx - Tx_0, z, t) = 1 \) and \( \nu(x - x_0, z, t) = \nu'(Tx - Tx_0, z, t) = 0 \),
so \( \mu(IU, z, t) \geq \alpha \Rightarrow \mu'(TU, z, t) \geq \alpha \) and \( \nu(IU, z, t) \leq 1 - \alpha \Rightarrow \nu'(TU, z, t) \leq 1 - \alpha \),
so \( T \) is IF-\( \psi \)-2-WB. Hence, \( T \) is IF-\( \psi \)-2-WB. \( \square \)

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