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A note on the lattice of TL-submodules of a module

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Abstract. The main goal of this paper is to study the lattice of TL-submodules of a module. It is well-known that the lattice of submodules of a module is modular. In this study, we prove an analogous result for L-sets that is the lattice of L-submodules of a module is modular for an infinitely $\vee$-distributive lattice.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [23] in 1965 and after that Rosenfeld [20] developed the theory of fuzzy subgroups. To the present day, fuzzy subalgebraic structures were developed and many significant results were obtained. Fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, hyperstructure theory, real analysis, measure theory etc.

The join structure of two fuzzy algebraic substructures with identical tips play a key role in development of most of the fuzzy algebraic substructures. If the set product of fuzzy subgroups is a fuzzy subgroup, then their set product is the join of this fuzzy subgroups if and only if their tips are equal. However when the tips differ, even then the set product of the tip extended pair μ' and ν' of fuzzy subgroups μ and ν provides the join of μ and ν as demonstrated by Prof. Tom Head [11]. A similar technique is used by I. Jahan [13] for constructing the join of two L-ideals of a ring. That is for two L-ideals μ and ν of a ring R, the sum $\mu^\nu + \nu^\mu$ of their tip extended pair of L-ideals μ^ν and ν^μ is the join of μ and ν. Since the characterization of the join of TL-submodules cannot be obtained by this method, in this paper, another characterization of the join of TL-submodules is given.

The lattice theoretic aspects of the sets of algebraic substructures and L-algebraic substructures have been a topic of discussion in the literature for quite some time [2, 3, 5, 14, 16, 19, 21, 22]. In particular, the modularity and the distributivity of these lattices have been investigated. The construction of various types of lattices and sublattices of fuzzy subgroups in a systematic and organized way were initiated by Ajmal and Thomas. In [6], it is proved that the lattice L_{nt} of all fuzzy normal subgroups with the same tip 't' is modular. In [4], it was established that the lattice L_{fnt} of all fuzzy normal subgroups of a group G with the same tip having finite range is modular. In fact,

$$L_{fnt} \subseteq L_{nt} \subseteq L_n$$

where L_n denotes the lattice of all fuzzy normal subgroups of G. Finally, it was shown [1] that the lattice L_n is modular. Head [10, 11] also discussed modularity of the lattice of all fuzzy normal subgroups, wherein he formulated the well-known metatheorem and the subdirect product theorem. Tarnauceanu [18] characterized distributivity of the lattice of the fuzzy subgroups of a finite group. Recently in [13], the modularity of L-ideals of a ring is established where the subdirect product theorem of Tom Head does not apply. Also in [7], the modularity of the lattice of all fuzzy submodules with same tip 't' , of a module is obtained. We prove that the lattice of L-submodules of a module is modular for an infinitely $\vee$-distributive lattice L.

2. Preliminaries

In this section, we briefly recall some basic concepts of lattices, L-subsets, modules and t-norms from [8, 12, 13, 15, 17]. Throughout this paper, L is a complete lattice with the least element 0 and the greatest element 1. For every family $\{b_i \mid i \in \Delta\}$, we can popularize some operations such as

$$\bigvee_{i\in\Delta} b_i = \sup\{b_i \mid i \in \Delta\}, \ \bigwedge_{i\in\Delta} b_i = \inf\{b_i \mid i \in \Delta\}.$$

A complete lattice L is called infinitely distributive if, for any $\alpha \in L$ and any $\Delta \subseteq L$,

$$\alpha \wedge (\bigvee_{\beta\in\Delta} \beta) = \bigvee_{\beta\in\Delta} (\alpha \wedge \beta) \ \text{ and } \ \alpha \vee (\bigwedge_{\beta\in\Delta} \beta) = \bigwedge_{\beta\in\Delta} (\alpha \vee \beta).$$

An L-subset of X is any function from X into L, which is introduced by Goguen [9] as a generalization of the notion of Zadeh's fuzzy subset. The class of L-subsets of X will be denoted by $F(X,L)$. In particular, if $L = [0,1]$, then it is appropriate to replace L-subset with fuzzy subset. In this case the set of all fuzzy subsets of X is denoted by $F(X)$. Let μ and ν be L-subsets of X. We say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted $\mu \leq \nu$. Then $\leq$ is a partial ordering on the set $F(X,L)$.

For each $\alpha \in L$, we define the level subset

$$\mu_\alpha \quad = \{x \in S \mid \alpha \leq \mu(x)\}.$$

Let μ_i $(i \in \Delta)$ be L-subsets of X. Define the intersection as follows:

$$(\bigcap_{i\in\Delta} \mu_i)(x) \quad = \bigwedge_{i\in\Delta} \mu_i(x),$$

for all $x \in X$. The characteristic function of a set $A \subseteq X$ is a function with value 1 if $y \in A$ and 0 if otherwise, and it is denoted by 1_A.

Throughout this paper, R stands for a commutative ring with unity, M is a module over R, and 0 denotes the zero element of M. $L(M)$ stands for all submodules of M, is a complete lattice with respect to set inclusion, called the submodules lattice of M. Note that $L(M)$ has initial element $\{0\}$ and final element M, and its binary operations $\wedge, \vee$ are defined by $I \wedge J = I \cap J$, $I \vee J = I + J$ for all $I, J \in L(M)$.

A *t-norm* on L is a commutative, associative mapping $T : L \times L \to L$ which is increasing in both arguments and for which $T(x, 1) = x$ for all $x \in L$. In the future text T will be a *t*-norm on L. The three basic t-norms on $[0, 1]$ the minimum T_M, the product T_P and the Lukasiewiez t-norm T_L are given by, respectively

$$T_M(x, y) = min\{x, y\};$$
$$T_P(x, y) = x \cdot y;$$
$$T_L(x, y) = max\{0, x + y - 1\}.$$

We write $T(x, y)$ and xTy interchangeably.
A t-norm T on a complete lattice L is called infinitely $\vee$-distributive if

$$\alpha T(\bigvee_{\beta\in\Delta} \beta) = \bigvee_{\beta\in\Delta} (\alpha T \beta),$$

for all $\alpha \in L$ and $\Delta \subseteq L$. If $T = \wedge$, then L is called infinitely $\vee$-distributive lattice.

Let μ and ν be L-subsets of module M. Define the L-subsets μ^ν and $\mu + \nu$ as follows:

$$\mu^\nu(x) = \begin{cases} \mu(0) \vee \nu(0), & x = 0, \\ \mu(x), & x \neq 0 \end{cases}$$

$$(\mu + \nu)(x) = \bigvee_{x=y+z} \mu(y) \wedge \nu(z),$$

Definition 2.1. Let μ and ν be L-subsets of a module M. Define the L-subset $\mu \oplus_T \nu$ as follows:

$$(\mu \oplus_T \nu)(x) = \mu(x) \vee \nu(x) \vee \bigvee_{x=y+z} \mu(y) T \nu(z),$$

for all $x \in M$.

Definition 2.2. Let μ be an L-subset of a module M. Then μ is called a TL-submodule of M if

$$\mu(x - y) \geq \mu(x) T \mu(y) \text{ and } \mu(rx) \geq \mu(x),$$

for all $x, y \in M$ and for all $r \in R$. The family of all TL-submodules is denoted by $FL(M, T, L)$. In particular, when $T = \wedge$ ($L = [0, 1]$ resp.), a TL-submodule of M is referred to as an L-submodule (fuzzy submodule resp.) of M. The family of all L-submodules and the family of all fuzzy submodules is denoted by $FL(M, L)$ and $FL(M)$ repectively.

Lemma 2.3. Let $\mu \in FL(M, T, L)$.

(i) If μ_α is a classical submodule of M for any $\alpha \in L$, then μ is a TL-submodule of M,
(ii) If μ is an L-submodule of M, then $\mu_\alpha(\neq \varnothing)$ is a submodule of M.

The following example shows that the converse of Lemma 2.3(i) may not be true in general for an arbitrary t-norm.
Example 2.4. We define a fuzzy subset of $\mathbb{Z}$ as follows:

$$\mu(x) = \begin{cases} 0.6, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.5, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.4, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.3, & \text{if} \quad \text{otherwise.} \end{cases}$$

It is easy to see that $\mu \in FL(\mathbb{Z}, T_P)$. Because of $\mu_{0.4} = 2\mathbb{Z} \cup 3\mathbb{Z}$, $\mu_{0.4}$ is not a submodule of $\mathbb{Z}$.
Theorem 2.5. Let μ_i $(i \in \Delta)$ be TL-submodules of a module M. Then $\bigcap_{i\in\Delta} \mu_i$ is a TL-submodule of M.

Corollary 2.6. $FL(M, T, L)$ is a complete lattice under the ordering of L-set inclusion such that $\bigwedge_{i\in\Delta} \mu_i = \bigcap_{i\in\Delta} \mu_i$ for all $\mu_i \in FL(M, T, L)$ $(i \in \Delta)$.

Lemma 2.7. Let A, B be subsets of M. Then

(i) A is a submodule of M if and only if 1_A is an L-submodule of M,
(ii) If A, B are submodules of M, then $1_A \vee 1_B = 1_{A+B}$ and $1_A \wedge 1_B = 1_{A\cap B}$,
(iii) $\{1_A \mid A$ is an submodule of $M\}$ is a sublattice of $FL(M, L)$.

3. The modularity of $FL(M, T, L)$

In this section, we will investigate modularity of the lattice of TL-submodules of a module M.
Lemma 3.1. Let T be an infinitely $\vee$-distributive t-norm and $\mu, \nu \in FL(M, T, L)$. Then $\mu \oplus_T \nu \in FL(M, T, L)$.

Proof. Let $x, y \in M$. Then

$$\begin{aligned}
&\mu \oplus_T \nu(x) T \mu \oplus_T \nu(y) \\
&= [\mu(x) \vee \nu(x) \vee (\bigvee_{x=a+b} \mu(a)T\nu(b))]T[\mu(y) \vee \nu(y) \vee (\bigvee_{y=c+d} \mu(c)T\nu(d))] \\
&= [(\mu(x) \vee \nu(x))T(\mu(y) \vee \nu(y))] \vee [(\mu(x) \vee \nu(x))T(\bigvee_{y=c+d} \mu(c)T\nu(d))] \vee [(\mu(y) \vee \nu(y)) \\
&\quad T(\bigvee_{x=a+b} \mu(a)T\nu(b))] \vee [(\bigvee_{x=a+b} \mu(a)T\nu(b))T(\bigvee_{y=c+d} \mu(c)T\nu(d))] \\
&= (\mu(x)T\mu(y)) \vee (\mu(x)T\nu(y)) \vee (\nu(x)T\mu(y)) \vee (\nu(x)T\nu(y)) \vee (\bigvee_{y=c+d} \mu(x)T\mu(c)T\nu(d)) \\
&\quad \vee (\bigvee_{y=c+d} \nu(x)T\mu(c)T\nu(d)) \vee (\bigvee_{x=a+b} \mu(a)T\nu(b)T\mu(y)) \vee (\bigvee_{x=a+b} \mu(a)T\nu(b)T\nu(y)) \\
&\quad \vee (\bigvee_{\substack{x=a+b \\ y=c+d}} \mu(a)T\nu(b)T\mu(c)T\nu(d)) \\
&\leq \mu(x+y) \vee \nu(x+y) \vee (\mu(x)T\nu(y)) \vee (\mu(y)T\nu(x)) \vee (\bigvee_{y=c+d} \mu(x+c)T\nu(d)) \\
&\quad \vee (\bigvee_{y=c+d} \mu(c)T\nu(d+x)) \vee (\bigvee_{x=a+b} \mu(a+y)T\nu(b)) \\
&\quad \vee (\bigvee_{x=a+b} \mu(a)T\nu(b+y)) \vee (\bigvee_{\substack{x=a+b \\ y=c+d}} \mu(a+c)T\nu(b+d)) \\
&\leq \mu(x+y) \vee \nu(x+y) \vee (\bigvee_{x+y=u+v} \mu(u)T\nu(v)) \\
&= \mu \oplus_T \nu(x+y).
\end{aligned}$$

Hence $\mu \oplus_T \nu(x) T \mu \oplus_T \nu(y) \leq \mu \oplus_T \nu(x+y)$. Next consider

$$\begin{aligned}
&\mu \oplus_T \nu(-x) \\
&\quad = \mu(-x) \vee \nu(-x) \vee (\bigvee_{-x=a+b} \mu(a)T\nu(b)) \\
&\quad = \mu(-x) \vee \nu(-x) \vee (\bigvee_{x=(-a)+(-b)} \mu(-(-a))T\nu(-(-b)) \\
&\quad \leq \mu(x) \vee \nu(x) \vee (\bigvee_{x=u+v} \mu(u)T\nu(v))] \\
&\quad = \mu \oplus_T \nu(x).
\end{aligned}$$

Hence $\mu \oplus_T \nu(-x) \leq \mu \oplus_T \nu(x)$. Let $r \in R$. Then

$$\begin{aligned}
\mu \oplus_T \nu(x) \quad &= \mu(x) \vee \nu(x) \vee (\bigvee_{x=a+b} \mu(a)T\nu(b)) \\
&\leq \mu(rx) \vee \nu(rx) \vee (\bigvee_{x=a+b} \mu(ra)T\nu(rb)) \\
&\leq \mu(rx) \vee \nu(rx) \vee (\bigvee_{rx=u+v} \mu(u)T\nu(v))] \\
&= \mu \oplus_T \nu(rx)
\end{aligned}$$

Thus $\mu \oplus_T \nu \in FL(M,T,L)$.□

Lemma 3.2. Let T be an infinitely $\vee$-distributive t-norm and $\mu,\nu \in FL(M,T,L)$. Then $\mu \vee \nu = \mu \oplus_T \nu$.
Proof. From the Lemma 3.1, $\mu \oplus_T \nu \in FL(M,T,L)$. It is clear that $\mu \leq \mu \oplus_T \nu$ and $\nu \leq \mu \oplus_T \nu$. Let $\theta \in FL(M,T,L)$ such that $\mu \leq \theta$ and $\nu \leq \theta$. Then

$$\mu(a)T\nu(b) \leq \theta(a)T\theta(b) \leq \theta(a+b) = \theta(x)$$

for all $x = a+b$. By the definition of $\mu \oplus_T \nu$, it folows that $\mu \oplus_T \nu \leq \theta$. Hence $\mu \vee \nu = \mu \oplus_T \nu$. □

We will point out that the $FL(M,T,L)$ may not be a modular lattice by giving an example for an arbitrary t-norm.
Example 3.3. We define fuzzy subsets of $\mathbb{Z}$ as follows:

$$\mu_1(x) = \begin{cases} 0.8, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.7, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.42, & \text{if} \quad \text{otherwise}. \end{cases} \qquad \mu_2(x) = \begin{cases} 0.8, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.4, & \text{if} \quad \text{otherwise}. \end{cases}$$

$$\mu_3(x) = \begin{cases} 0.8, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.5, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.7, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.35, & \text{if} \quad \text{otherwise}. \end{cases} \qquad \mu_4(x) = \begin{cases} 0.8, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.36, & \text{if} \quad \text{otherwise}. \end{cases}$$

$$\mu_5(x) = \begin{cases} 0.8, & \text{if} \quad x \in 6\mathbb{Z}, \\ 0.5, & \text{if} \quad x \in 2\mathbb{Z} \quad x \notin 3\mathbb{Z}, \\ 0.6, & \text{if} \quad x \in 3\mathbb{Z} \quad x \notin 2\mathbb{Z}, \\ 0.35, & \text{if} \quad \text{otherwise}. \end{cases}$$

It is easy to see that $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \in FL(\mathbb{Z}, T_P)$ such that whose Hasse diagram is presented in Figure 1.

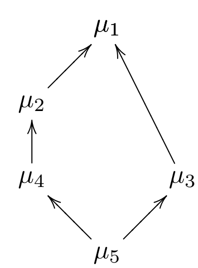

FIGURE 1.

Therefore, $FL(\mathbb{Z}, T_P)$ is not a modular lattice.

Consider the lattice of L-ideals of a ring: the join is given by $\mu^{\nu} + \nu^{\mu}$, where μ, ν are L-ideals [13]. The join is obtained similarly in the lattice of L-submodules of a module. However, in the lattice of TL-submodules of a module, $\mu^{\nu} + \nu^{\mu}$ may not be join of μ and ν. In Example 3.3, $(\mu_2 \vee_{T_P} \mu_3)(1) = 0,42$, whereas $(\mu_2^{\mu_3} + \mu_3^{\mu_2})(1) = 0,6$.

Theorem 3.4. If L is an infinitely $\vee$-distributive lattice, then $FL(M, L)$ is a modular lattice.
Proof. Let $\mu, \nu, \theta \in FL(M, L)$ such that $\mu \leq \nu$. We have

$$\begin{aligned}
&\nu \wedge (\mu \vee \theta)(x) \\
&\quad = \nu(x) \wedge (\mu \oplus_T \nu)(x) \\
&\quad = \nu(x) \wedge [\mu(x) \vee \theta(x) \vee (\bigvee_{x=a+b} \mu(a) \wedge \theta(b))] \\
&\quad = (\nu(x) \wedge \mu(x)) \vee (\nu(x) \wedge \theta(x)) \vee (\bigvee_{x=a+b} \nu(x) \wedge \mu(a) \wedge \theta(b)) \\
&\quad = \mu(x) \vee (\nu(x) \wedge \theta(x)) \vee (\bigvee_{x=a+b} \nu(x) \wedge \nu(a) \wedge \mu(a) \wedge \theta(b)) \\
&\quad \leq \mu(x) \vee (\nu(x) \wedge \theta(x)) \vee (\bigvee_{x=a+b} \nu(b) \wedge \mu(a) \wedge \theta(b)) \\
&\quad = \mu \oplus_T (\nu \wedge \theta) \\
&\quad = \mu \vee (\nu \wedge \theta)
\end{aligned}$$

Hence $\nu \wedge (\mu \vee \theta) \leq \mu \vee (\nu \wedge \theta)$. Since the modularity inequality is valid for every lattice, we have $\mu \vee (\nu \wedge \theta) \leq \nu \wedge (\mu \vee \theta)$. Therefore, $\mu \vee (\nu \wedge \theta) = \nu \wedge (\mu \vee \theta)$. This establishes that $FL(M, L)$ is a modular lattice.□

By the Theorem 3.4, we immediately get the next corollary.
Corollary 3.5. $FL(M)$ is a modular lattice.
Remark 3.6. Let T_1, T_2 be t-norms such that $T_1 < T_2$. Although $FL(M, T_2, L) \subseteq FL(M, T_1, L)$, $FL(M, T_2, L)$ is not a sublattice of $FL(M, T_1, L)$, in general.This situation can be seen in the following example:
Example 3.7. μ_2 and μ_3 (see Example 3.3.) are both T_P-submodules and T_L-submodules of $\mathbb{Z}$. Also, we know that $T_L < T_P$.

$$\begin{aligned}
\mu_2 \vee_{T_L} \mu_3(1) &= \mu_2 \oplus_{T_L} \mu_3(1) \\
&= \mu_2(1) \vee \mu_3(1) \vee \bigvee_{1=a+b} \mu_2(a) T_L \mu_3(a) \\
&= 0.4 \vee 0.35 \vee 0.3 \vee 0.1 = 0.4
\end{aligned}$$

Since $\mu_2 \vee_{T_P} \mu_3(1) = 0.42$, $FL(\mathbb{Z}, T_P)$ is not a sublattice of $FL(\mathbb{Z}, T_L)$.

4. Conclusion

In this study we proved modularity of the lattice of L-submodules of a module. There arise an open question.
Open Problem A necessary and sufficient condition in order that the lattice of TL-submodules of a module is distributive (or modular).

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