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Soft points and the structure of soft topological spaces

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Abstract. Soft set theory is a new mathematical tool to deal with uncertain problems. Although soft sets are so-called sets, we can not handle them like ordinary sets. The reason is that soft sets are defined by mappings and they lack " points ". In this paper, the concept of soft points is introduced and the relationship between soft points and soft sets is considered. We prove the fact that soft sets can be translated into soft point sets and then may conveniently deal with soft sets as same as ordinary sets. Moreover, we investigate some soft point sets, define neighborhoods of soft points and reveal the structure of soft topological spaces such as interior, closure, boundary and bases by using neighborhoods of soft points.

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1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We can not use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, theory of fuzzy sets [22], theory of interval mathematics, and theory of rough sets [17], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties [16]. For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [16] proposed a completely new approach, which is called theory of soft sets, for modeling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. [13, 15, 14] further studied soft set theory and used this theory to solve some decision making problems. Roy et al. [19] presented a fuzzy soft set theoretic approach towards decision making problems. Jiang et al. [11] generalized the adjustable approach to fuzzy soft sets based decision making. Jiang et al. [10] extended soft sets with description logics. Aktas and Cağman [2] defined soft groups. Feng et al. [6, 7] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [8] discussed the relationship between soft sets and topological spaces. Babitha et al. [4] proposed relations on soft sets. Shabir et al. [21] introduced soft topological spaces over the universe with a fixed set of parameters. Cağman et al. [5] defined soft topologies on soft sets. Sen [20] investigated algebraic structure of soft sets.

Although soft sets are so-called sets, we can not handle them like an ordinary set. The reason is that soft sets are defined by mappings and they lack " points ". Therefore, we need to introduce the concept of " points " in soft sets.

The purpose of this paper is to introduce the concept of soft points. We prove that soft sets can be translated into soft point sets and then may conveniently deal with soft sets as same as ordinary sets.

The rest of this paper is organized as follows. In Section 2, we give an overview of soft sets. In Section 3, we recall soft topological spaces. In Section 4, the concept of soft points is introduced, some soft point sets are studied and the relationship between soft points and soft sets is considered. We prove the fact that soft sets can be translated into soft point sets In Section 5, we illustrate that soft relations and soft operations can be conveniently handled. Neighborhoods of soft points are proposed and the structure of soft topological spaces such as interior, closure, boundary and bases is revealed by using them. Conclusions are in Section 6.

2. **Overview of soft sets**

In this section, we recall some basic concepts of soft sets.

Throughout this paper, U denotes an initial universe, E denotes the set of parameters, 2^U denotes the power set of U. We only consider the case where U and E are both nonempty finite sets.

Definition 2.1 ([16])**.** Let $A \subseteq E$. A pair (f, A) is called a soft set over U, if f is a mapping given by $f : A \to 2^U$. We denote (f, A) by f_A.

In other words, a soft set over U is the parameterized family of subsets of the universe U. For $\varepsilon \in A$, $f(\varepsilon)$ may be considered as the set of ε-approximate elements of f_A. Obviously, a soft set is not a set.

To illustrate this idea, let us consider the following Example 2.2.

Example 2.2. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of houses and let $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \subseteq E$ be a set of status of houses which stand for the parameters "beautiful", "modern", " cheap" and "in the green surroundings", respectively.

Now, we consider a soft set f_A, which describes the " attractiveness of the houses" that Mr.X is going to buy. In this case, to define the soft set f_A means to point out beautiful houses, modern houses and so on.

Consider the mapping f given by "houses(.)", where (.) is to be filled in by one of the parameters $\varepsilon_i \in A$. For instance, $f(\varepsilon_1)$ means "houses(beautiful)", and its

TABLE 1. Tabular representation of the soft set f_A

	h_1	h_2	h_3	h_4	h_5	h_6
ε_1	1	1	0	0	1	0
ε_2	0	0	0	0	0	0
ε_3	0	0	1	1	0	0
ε_4	0	0	1	1	0	1

functional value is the set consisting of all the beautiful houses in U. Let

$$f(\varepsilon_1) = \{h_1, h_2, h_5\}, \quad f(\varepsilon_2) = \varnothing, \ f(\varepsilon_3) = \{h_3, h_4\}, \ f(\varepsilon_4) = \{h_3, h_4, h_6\}.$$

Then a soft set f_A is described as the following Table 1. If $h_i \in f(\varepsilon_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where h_{ij} are the entries in the table (see Table 1).

Let $A \subseteq E$. Denote

$$S(U, A) = \{f_A : f_A \ is \ a \ soft \ set \ over \ U\}.$$

Definition 2.3. Let $A \subseteq E$ and let $f_A \in S(U, A)$. We define $(f_A)_E \in S(U, E)$ by

$$f_A(\varepsilon) = \begin{cases} f(\varepsilon) & \text{if } \ \varepsilon \in A, \\ \varnothing, & \text{if } \ \varepsilon \in E - A. \end{cases}$$

Obviously, $(f_E)_E = f_E$.

Definition 2.4 ([1])**.** Let $A, B \subseteq E$ and let $f_A \in S(U, A)$ and $g_B \in S(U, B)$.

(1) f_A is called a soft subset of g_B, if $A \subseteq B$ and for any $\varepsilon \in A$, $f(\varepsilon) \subseteq g(\varepsilon)$. We write $f_A \mathrel{\widetilde{\subset}} g_B$.

(2) f_A is called a soft super set of g_B, if $g_B \mathrel{\widetilde{\subset}} f_A$. We write $f_A \mathrel{\widetilde{\supset}} g_B$.

(3) f_A and g_B are called soft equal, if $A = B$ and $f(\varepsilon) = g(\varepsilon)$ for any $\varepsilon \in A$. We write $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \mathrel{\widetilde{\subset}} g_B$ and $f_A \mathrel{\widetilde{\supset}} g_B$.

Definition 2.5 ([13])**.** Let $A, B \subseteq E$ and let $f_A \in S(U, A)$ and $g_B \in S(U, B)$.

(1) $h_{A \cup B}$ is called the union of f_A and g_B, if

$$h(\varepsilon) = \begin{cases} f(\varepsilon) & \text{if } \ \varepsilon \in A - B, \\ g(\varepsilon) & \text{if } \ \varepsilon \in B - A, \\ f(\varepsilon) \cup g(\varepsilon) & \text{if } \ \varepsilon \in A \cap B. \end{cases}$$

We write $f_A \mathbin{\widetilde{\cup}} g_B = h_{A \cup B}$.

(2) $h_{A \cap B}$ is called the intersection of f_A and g_B, if $h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon)$ for any $\varepsilon \in A \cap B$. We write $f_A \mathbin{\widetilde{\cap}} g_B = h_{A \cap B}$.

Remark 2.6. Let $A, B, C \subseteq E$ and let $f_A \in S(U, A)$, $g_B \in S(U, B)$ and $h_C \in S(U, C)$. Then

(1) $f_A \mathbin{\widetilde{\cap}} g_B \mathrel{\widetilde{\subset}} f_A$ (*or* g_B) $\mathrel{\widetilde{\subset}} f_A \mathbin{\widetilde{\cup}} g_B$.

(2) If $h_C \mathrel{\widetilde{\subset}} f_A$ and $h_C \mathrel{\widetilde{\subset}} g_B$, then $h_C \mathrel{\widetilde{\subset}} f_A \mathbin{\widetilde{\cap}} g_B$.

(3) If $h_C \mathrel{\widetilde{\supset}} f_A$ and $h_C \mathrel{\widetilde{\supset}} g_B$, then $h_C \mathrel{\widetilde{\supset}} f_A \mathbin{\widetilde{\cup}} g_B$.

Definition 2.7 ([21])**.** Let $A \subseteq E$ and let $f_A, g_A, h_A \in S(U, A)$. h_A is called the difference of f_A and g_A, if $h(\varepsilon) = f(\varepsilon) - g(\varepsilon)$ for any $\varepsilon \in A$. We write $h_A = f_A - g_A$.

Definition 2.8 ([13])**.** Let $A \subseteq E$ and let $f_A, g_A \in S(U, A)$. g_A is called the relative complement of f_A, if $g(\varepsilon) = U - f(\varepsilon)$ for any $\varepsilon \in A$. We write $g_A = f'_A$ *or* $(f_A)'$.

Proposition 2.9 ([3])**.** *Let $A \subseteq E$ and let $f_A, g_A \in S(U, A)$. Then*
(1) $(f_A \mathbin{\widetilde{\cup}} g_A)' = f'_A \mathbin{\widetilde{\cap}} g'_A$.
(2) $(f_A \mathbin{\widetilde{\cap}} g_A)' = f'_A \mathbin{\widetilde{\cup}} g'_A$.

Remark 2.10. Let $A \subseteq E$ and let $f_A, g_A \in S(U, A)$. Then
(1) $(f'_A)' = f_A$.
(2) $f_A \mathbin{\widetilde{\subset}} g_A \iff (f_A)' \mathbin{\widetilde{\supset}} (g_A)'$.

Definition 2.11 ([21])**.** Let $A \subseteq E$ and let $X \subseteq U$. The soft set X_A over U is defined by $X(\varepsilon) = X$ for any $\varepsilon \in A$.

In this paper, U_E and $\varnothing_E$ are also denoted by $\widetilde{U}$ and $\widetilde{\varnothing}$, respectively.

Remark 2.12. Let $f_A, g_A \in S(U, A)$. Then
(1) $U_A - f_A = f'_A$,
(2) $f_A \mathbin{\widetilde{\cap}} g_A = \varnothing_A \iff f_A \mathbin{\widetilde{\subset}} g'_A$,
(3) $f_A - g_A = f_A \mathbin{\widetilde{\cap}} g'_A$.

Proposition 2.13. *Let $A, B \subseteq E$ and let $f_A \in S(U, A)$ and $g_B \in S(U, B)$. Then*
(1) $(f_A \mathbin{\widetilde{\cup}} g_B)_E = (f_A)_E \mathbin{\widetilde{\cup}} (g_B)_E$.
(2) $(f_A \mathbin{\widetilde{\cap}} g_B)_E = (f_A)_E \mathbin{\widetilde{\cap}} (g_B)_E$.
(3) $(f_A - g_A)_E = (f_A)_E - (g_A)_E$.

Proof. (1) Denote $h_C = f_A \mathbin{\widetilde{\cup}} g_B$ with $C = A \cup B$. We will show that

$$h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon) \text{ for any } \varepsilon \in E.$$

Case 1 If $\varepsilon \in A - B$, then $\varepsilon \in C$ and $\varepsilon \in E - B$. Since $h_C(\varepsilon) = h(\varepsilon) = f(\varepsilon)$, $f_A(\varepsilon) = f(\varepsilon)$ and $g_B(\varepsilon) = \varnothing$, we have $h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon)$.
Case 2 If $e \in B - A$, then $\varepsilon \in C$ and $\varepsilon \in E - A$. Since $h_C(\varepsilon) = h(\varepsilon) = g(\varepsilon)$, $f_A(\varepsilon) = \varnothing$ and $g_B(\varepsilon) = g(\varepsilon)$, we have $h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon)$.
Case 3 If $\varepsilon \in A \cap B$, then $\varepsilon \in C$. Since $h_C(\varepsilon) = h(\varepsilon) = f(\varepsilon) \cup g(\varepsilon)$, $f_A(\varepsilon) = f(\varepsilon)$ and $g_B(\varepsilon) = g(\varepsilon)$, we have $h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon)$.
Case 4 If $\varepsilon \in E - A \cup B$, then $\varepsilon \in (E - A) \cap (E - B) \cap (E - C)$. Since $h_C(\varepsilon) = \varnothing$, $f_A(\varepsilon) = \varnothing$ and $g_B(\varepsilon) = \varnothing$, we have $h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon)$.
Thus,

$$h_C(\varepsilon) = f_A(\varepsilon) \cup g_B(\varepsilon) \text{ for any } \varepsilon \in E.$$

This implies that $(h_C)_E = (f_A)_E \mathbin{\widetilde{\cup}} (g_B)_E$. Hence $(f_A \mathbin{\widetilde{\cup}} g_B)_E = (f_A)_E \mathbin{\widetilde{\cup}} (g_B)_E$.
(2) The proof is similar to (1).
(3) The proof is similar to (1). □

3. Soft topological spaces

In this section, we recall some concepts of soft topological spaces.

Definition 3.1 ([21]). $\tau \subseteq S(U,E)$ is called a soft topology over U, if (i) $\widetilde{\varnothing}, \widetilde{U} \in \tau$; (ii) the union of any number of soft sets in τ belongs to τ; (iii) the intersection of any two soft sets in τ belongs to τ.

The triplet (U,τ,E) is called a soft topological space over U. Every element of τ is called soft open set in U and its relative complement is called soft closed set in U.

Definition 3.2 ([21]). Let (U,τ,E) be a soft topological space over U. For any $f_E \in S(U,E)$, the soft closure of f_E is defined by

$$cl(f_E) = \widetilde{\cap}\ \{g_E : f_E \mathrel{\widetilde{\subset}} g_E \text{ and } g_E \text{ is a soft closed set in } U\}.$$

Proposition 3.3 ([21]). *Let (U,τ,E) be a soft topological space over U and let $f_E \in S(U,E)$. Then*

(1) $cl(\widetilde{\varnothing}) = \widetilde{\varnothing}$ *and* $cl(\widetilde{U}) = \widetilde{U}$.
(2) $f_E \mathrel{\widetilde{\subset}} cl(f_E)$.
(3) f_E *is a soft closed set in* $U \iff f_E = cl(f_E)$.
(4) $cl(cl(f_E)) = cl(f_E)$.

Definition 3.4 ([9]). Let (U,τ,E) be a soft topological space over U. For any $f_E \in S(U,E)$, the soft interior of f_E is defined by

$$int(f_E) = \widetilde{\cup}\ \{g_E : g_E \mathrel{\widetilde{\subset}} f_E \text{ and } g_E \in \tau\}.$$

Proposition 3.5 ([9]). *Let (U,τ,E) be a soft topological space over U. Then for any $f_E \in S(U,E)$,*

$$int(f_E) = \widetilde{U} - cl(\widetilde{U} - f_E).$$

Proposition 3.6 ([9]). *Let (U,τ,E) be a soft topological space over U and let $f_E \in S(U,E)$. Then*

(1) $int(\widetilde{\varnothing}) = \widetilde{\varnothing}$ *and* $int(\widetilde{U}) = \widetilde{U}$.
(2) $int(f_E) \mathrel{\widetilde{\subset}} f_E$.
(3) $f_E \in \tau \iff f_E = int(f_E)$.
(4) $int(int(f_E)) = int(f_E)$.

Definition 3.7 ([9]). Let (U,τ,E) be a soft topological space over U. For any $f_E \in S(U,E)$, the soft boundary of f_E is defined by

$$\partial(f_E) = cl(f_E) \mathbin{\widetilde{\cap}} cl(f'_E).$$

4. **Soft points**

In this section, we will introduce the concept of soft points and consider the relationship between soft points and soft sets.

4.1. **The concept of soft points.**

Definition 4.1. Let $f_E \in S(U,E)$. f_E is called a soft point over U, if there exist $e \in E$ and $x \in U$ such that

$$f(\varepsilon) = \begin{cases} \{x\}, & \text{if } \varepsilon = e, \\ \varnothing, & \text{if } \varepsilon \in E - \{e\}. \end{cases}$$

We denote f_E by $(x_e)_E$. In this case, x is called the support point of $(x_e)_E$, $\{x\}$ is called the support set of $(x_e)_E$ and e is called the expressive parameter of $(x_e)_E$.

The family of all soft points over U is denoted by $P(U, E)$.

Let $A \subseteq E$ and let $f_A \in S(U, A)$. We denote

$$P(U, A) = \{(x_e)_E \in P(U, E) : e \in A\},$$

$$\mathcal{F}(A) = \{(x_e)_E : x \in f(e) \ and \ e \in A\}.$$

Remark 4.2. (1) $(x_e)_E \in \mathcal{F}(A) \Longleftrightarrow x \in f(e) \ and \ e \in A$.

(2) $\mid \mathcal{F}(A) \mid = \sum\limits_{e \in A} \mid f(e) \mid$.

Example 4.3. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $E = \{e_1, e_2, e_3, e_4\}$. We define $f(e_1) = \{x_1, x_4\}$, $f(e_2) = U$, $f(e_3) = \{x_5\}$, $f(e_4) = \varnothing$.

Then $\mathcal{F}(E) = \{((x_1)_{e_1})_E, ((x_4)_{e_1})_E, ((x_1)_{e_2})_E, ((x_2)_{e_2})_E, ((x_3)_{e_2})_E, ((x_4)_{e_2})_E, ((x_5)_{e_2})_E, ((x_5)_{e_3})_E\}$ and

$$P(U, E) = \{((x_i)_{e_j})_E : 1 \leq i \leq 5, 1 \leq j \leq 4\}.$$

Proposition 4.4. *Let $A, B \subseteq E$. Then*

(1) $A \subseteq B \Longrightarrow P(U, A) \subseteq P(U, B)$.

(2) $P(U, E) = \bigcup\{P(U, A) : A \subseteq E\}$.

Proof. These are obvious. □

Proposition 4.5. *Let $f_A, g_A, h_A \in S(U, A)$.*

(1) *If $g_A \ \widetilde{\subset} \ f_A$, then $\mathcal{G}(A) \subseteq \mathcal{F}(A)$.*

(2) *If $f_A = g_A \ \widetilde{\cap} \ h_A$, then $\mathcal{F}(A) = \mathcal{G}(A) \cap \mathcal{H}(A)$.*

(3) *If $f_A = g_A \ \widetilde{\cup} \ h_A$, then $\mathcal{F}(A) = \mathcal{G}(A) \cup \mathcal{H}(A)$.*

(4) *If $f_A = g_A - h_A$, then $\mathcal{F}(A) = \mathcal{G}(A) - \mathcal{H}(A)$.*

Proof. (1) This is obvious.

(2) Let $(x_e)_E \in \mathcal{F}(A)$. Then $x \in f(e)$. Since $f_A = g_A \ \widetilde{\cap} \ h_A$, we have $x \in g(e)$ and $x \in h(e)$. Thus $(x_e)_E \in \mathcal{G}(A)$ and $(x_e)_E \in \mathcal{F}(A)$. Hence $(x_e)_E \in \mathcal{G}(A) \cap \mathcal{H}(A)$. Conversely. The proof is similar.

(3) The proof is similar to (2).

(4) The proof is similar to (2). □

Proposition 4.6. (1) *If $f_A = U_A$, then $P(U, A) = \mathcal{F}(A)$.*

(2) $P(U, A) = \cup \ \{\mathcal{F}(A) : f_A \in S(U, A)\}$.

Proof. (1) This is obvious.

(2) Let $f_A \in S(U, A)$. Since $f_A \ \widetilde{\subset} \ U_A$, by Proposition 4.5 and (1) we have $\mathcal{F}(A) \subseteq P(U, A)$. Thus $P(U, A) \supseteq \cup\{\mathcal{F}(A) : f_A \in S(U, A)\}$.

Conversely, since $U_A \in S(U, A)$, by (1) we have $P(U, A) \subseteq \cup\{\mathcal{F}(A) : f_A \in S(U, A)\}$.

Hence $P(U, A) = \cup\{\mathcal{F}(A) : f_A \in S(U, A)\}$. □

4.2. **Soft points and soft sets.**

Definition 4.7. Let $A \subseteq E$ and let $f_A \in S(U, A)$ and $(x_e)_E \in P(U, E)$. We define $(x_e)_E \ \widetilde{\in} \ f_A$ by

$$(x_e)_E \ \widetilde{\subset} \ (f_A)_E.$$

Note that $(x_e)_E \ \widetilde{\notin} \ f_A$, if $(x_e)_E \ \widetilde{\not\subset} \ (f_A)_E$.

Remark 4.8. (1) $(x_e)_E = (x'_{e'})_E \iff x = x'$ and $e = e'$.
(2) $(x_e)_E \widetilde{\in} f_A \iff x \in f(e)$ and $e \in A \iff (x_e)_E \in \mathcal{F}(A)$.
(3) $(x_e)_E \widetilde{\in} f_A$ and $f_A \widetilde{\subset} g_B \implies (x_e)_E \widetilde{\in} g_B$.

Theorem 4.9. *Let $f_A \in S(U, A)$ and let $(x_e)_E \in P(U, A)$. Then*

$$(x_e)_E \widetilde{\in} f_A \ \text{ if and only if } \ (x_e)_E \widetilde{\notin} f'_A.$$

Proof. Let $(x_e)_E \widetilde{\in} f_A$. Then $x \in f(e)$ and $e \in A$. This implies that $x \notin U - f(e) = f'(e)$. By Remark 4.8, $(x_e)_E \widetilde{\notin} f'_A$.

Conversely. Let $(x_e)_E \widetilde{\notin} f'_A$. Since $(x_e)_E \in P(U, A)$, $e \in A$. By Remark 4.8, $x \notin f'(e)$. Thus $x \in f(e)$. This show that $(x_e)_E \widetilde{\in} f_A$. □

Theorem 4.10. *Let $f_A \in S(U, A)$. Then*

$$f_A = \widetilde{\cup}\, \mathcal{F}(A).$$

Proof. Denote $h_A = \widetilde{\cup}\, \mathcal{F}(A)$. Then $h_A = \widetilde{\cup}\, \{(x_e)_E : x \in f(e) \text{ and } e \in A\}$. Thus

$$h_A = \widetilde{\bigcup_{e\in A}} \ \widetilde{\bigcup_{x\in f(e)}} (x_e)_E.$$

For any $\varepsilon \in A$, we have

$$\begin{aligned} h(\varepsilon) &= \bigcup_{e\in A} \bigcup_{x\in f(e)} x_e(\varepsilon) = (\bigcup_{x\in f(\varepsilon)} x_e(e)) \bigcup (\bigcup_{\varepsilon\in A-\{e\}} \bigcup_{x\in f(e)} x_e(\varepsilon)) \\ &= (\bigcup_{x\in f(\varepsilon)} \{x\}) \bigcup \varnothing = f(\varepsilon). \end{aligned}$$

This show that $h_A = f_A$. Hence $f_A = \widetilde{\cup}\, \mathcal{F}(A)$. □

Remark 4.11. Theorem 4.10 reveal the fact that soft sets can be translated into soft point sets.

5. Some related results with soft points

In this section, we obtain some related results with soft points.

5.1. Soft relations and soft operations.

Theorem 5.1. *Let $A, B \subseteq E$ and let $f_A \in S(U, A)$ and $g_B \in S(U, B)$. Then*
(1) $f_A \widetilde{\subset} g_B \iff A \subseteq B$ *and* $\mathcal{F}(A) \subseteq \mathcal{G}(B)$.
(2) $f_A = g_B \iff A = B$ *and* $\mathcal{F}(A) = \mathcal{G}(B)$.

Proof. (1) Necessity. Let $(x_e)_E \in \mathcal{F}(A)$. By Proposition 4.6, $(x_e)_E \in P(U, A)$. Since $A \subseteq B$, by Proposition 4.4, we have $P(U, A) \subseteq P(U, B)$. Thus $(x_e)_E \in P(U, B)$ and so $(x_e)_E \in \mathcal{G}(B)$. Hence $\mathcal{F}(A) \subseteq \mathcal{G}(B)$.

Sufficiency. Let $A \subseteq B$. For any $e \in A$ and $x \in f(e)$, we have $(x_e)_E \in \mathcal{F}(A)$. Since $\mathcal{F}(A) \subseteq \mathcal{G}(B)$, $(x_e)_E \in \mathcal{G}(B)$. Then $x \in g(e)$ and $e \in B$. Thus $f(e) \subseteq g(e)$ and $f_A \widetilde{\subset} g_B$.

(2) This holds by (1). □

Theorem 5.2. *Let $A, B, C \subseteq E$ and let $f_A \in S(U, A)$, $g_B \in S(U, B)$ and $h_C \in S(U, C)$.*

(1) *If $C = A \cup B$, then $h_C = f_A \,\widetilde{\cup}\, g_B \Longleftrightarrow \mathcal{H}(C) = \mathcal{F}(A) \cup \mathcal{G}(B)$.*
(2) *If $C = A \cap B$, then $h_C = f_A \,\widetilde{\cap}\, g_B \Longleftrightarrow \mathcal{H}(C) = \mathcal{F}(A) \cap \mathcal{G}(B)$.*
(3) *$f_A = g_A - h_A \iff \mathcal{F}(A) = \mathcal{G}(A) - \mathcal{H}(A)$.*

Proof. (1) Necessity. Let $(x_e)_E \in \mathcal{F}(A) \cup \mathcal{G}(B)$. Then $x \in f(e)$ and $e \in A$, or $x \in g(e)$ and $e \in B$.

Case 1 If $e \in A - B$, then $x \in f(e) = h(e)$ and $e \in C$. Thus $(x_e)_E \in \mathcal{H}(C)$.

Case 2 If $e \in B - A$, then $x \in g(e) = h(e)$ and $e \in C$. Thus $(x_e)_E \in \mathcal{H}(C)$.

Case 3 If $e \in A \cap B$, then $x \in f(e) \cup g(e) = h(e)$ and $e \in C$. Thus $(x_e)_E \in \mathcal{H}(C)$.

Hence $\mathcal{F}(A) \cup \mathcal{G}(B) \subseteq \mathcal{H}(C)$.

Conversely. Let $(x_e)_E \in \mathcal{H}(C)$. Then $x \in h(e)$ and $e \in C$.

Case 1: If $e \in A - B$, then $x \in h(e) = f(e)$ and $e \in A$. Thus $(x_e)_E \in \mathcal{F}(A)$ and so $(x_e)_E \in \mathcal{F}(A) \cup \mathcal{G}(B)$.

Case 2: If $e \in B - A$, then $x \in h(e) = g(e)$ and $e \in B$. Thus $(x_e)_E \in \mathcal{G}(B)$ and so $(x_e)_E \in \mathcal{G}(B) \cup \mathcal{F}(A)$.

Case 3: If $e \in A \cap B$, then $x \in h(e) = f(e) \cup g(e)$. So $x \in f(e)$ and $e \in A$, or $x \in g(e)$ and $e \in B$. Thus $(x_e)_E \in \mathcal{F}(A) \cup \mathcal{G}(B)$.

Hence $\mathcal{H}(C) \subseteq \mathcal{F}(A) \cup \mathcal{G}(B)$.

Therefore, $\mathcal{F}(A) \cup \mathcal{G}(B) = \mathcal{H}(C)$.

Sufficiency. Denote $l_C = f_A \,\widetilde{\cup}\, g_B$. By the proof of Necessity, $\mathcal{L}(C) = \mathcal{F}(A) \cup \mathcal{G}(B)$. Since $\mathcal{F}(A) \cup \mathcal{G}(B) = \mathcal{H}(C)$, $\mathcal{L}(C) = \mathcal{H}(C)$. By Theorem 5.1, $h_C = l_C$. Thus $f_A \,\widetilde{\cup}\, g_B = h_{A \cup B}$.

(2) The proof is similar to (1).

(3) This holds by Proposition 4.5. □

5.2. **Neighborhoods of soft points.**

Definition 5.3. Let (U, τ, E) be a soft topological space over U. Let $f_E \in S(U, E)$ and $(x_e)_E \in P(U, E)$. Then

(1) f_E is called a neighborhood of $(x_e)_E$, if there exists $g_E \in \tau$ such that $(x_e)_E \,\widetilde{\in}\, g_E \,\widetilde{\subset}\, f_E$.

(2) f_E is called an open neighborhood of $(x_e)_E$, if $f_E \in \tau$ and f_E is a neighborhood of $(x_e)_E$.

The family of all neighborhoods of $(x_e)_E$ is denoted by $\mathbb{N}_{(x_e)_E}$. Denote

$$\tau_{(x_e)_E} = \{f_E \in \tau : (x_e)_E \,\widetilde{\in}\, f_E\}.$$

Then $\tau_{(x_e)_E}$ means the family of all open neighborhoods of $(x_e)_E$.

Example 5.4. Let $U = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$. We define

$g_1(e_1) = \{x_2\},\ g_1(e_2) = \{x_3\};$
$g_2(e_1) = \{x_2\},\ g_2(e_2) = \{x_1, x_3\};$
$g_3(e_1) = \{x_2, x_3\},\ g_3(e_2) = \{x_3\};$
$g_4(e_1) = \{x_2, x_3\},\ g_4(e_2) = \{x_1, x_3\};$
$g_5(e_1) = \{x_1, x_2\},\ g_5(e_2) = \{x_2, x_3\};$
$g_6(e_1) = \{x_1, x_2\},\ g_6(e_2) = U;$
$g_7(e_1) = U,\ g_7(e_2) = \{x_2, x_3\};$

$f(e_1) = \{x_1, x_2\},\ f(e_2) = \{x_1, x_3\}$.

Put $\tau = \{\widetilde{\varnothing}, \widetilde{U}, (g_1)_E, (g_2)_E, \cdots, (g_7)_E\}$. Then (U, τ, E) is a soft topological space over U.

Obviously, $(g_2)_E \widetilde{\subset} f_E \notin \tau$. Since $x_1 \in g_2(e_2)$, $((x_1)_{e_2})_E \widetilde{\in} (g_2)_E$. Then $((x_1)_{e_2})_E \widetilde{\in} (g_2)_E \widetilde{\subset} f_E$. So f_E is a neighborhood of $((x_1)_{e_2})_E$. But f_E is not an open neighborhood of $((x_1)_{e_2})_E$.

Since $x_1 \in g_4(e_2)$ $x_1 \in g_6(e_2)$, $((x_1)_{e_2})_E \widetilde{\in} (g_4)_E \in \tau$, $((x_1)_{e_2})_E \widetilde{\in} (g_6)_E \in \tau$. Thus $\tau_{((x_1)_{e_2})_E} = \{(g_2)_E, (g_4)_E, (g_6)_E, \widetilde{U}\}$.

Proposition 5.5. *Let (U, τ, E) be a soft topological space over U and let $\mathbb{N}_{(x_e)_E}$ is a neighborhood system of $(x_e)_E$. Then*

(1) *If $f_E \in \mathbb{N}_{(x_e)_E}$, then $(x_e)_E \widetilde{\in} f_E$;*

(2) *If f_E, $g_E \in \mathbb{N}_{(x_e)_E}$, then $f_E \widetilde{\cap} g_E \in \mathbb{N}_{(x_e)_E}$;*

(3) *If $g_E \widetilde{\subset} f_E$ and $g_E \in \mathbb{N}_{(x_e)_E}$, then $f_E \in \mathbb{N}_{(x_e)_E}$;*

Proof. (1) Let $f_E \in \mathbb{N}_{(x_e)_E}$. Then there exists $g_E \in \tau$ such that $(x_e)_E \widetilde{\in} g_E \widetilde{\subset} f_E$. Thus $x \in g(e) \subseteq f(e)$. By Remark 4.8, $(x_e)_E \widetilde{\in} f_E$.

(2) Let f_E, $g_E \in \mathbb{N}_{(x_e)_E}$. Then there exist h_E, $k_E \in \tau$ such that $(x_e)_E \widetilde{\in} h_E \widetilde{\subset} f_E$ and $(x_e)_E \widetilde{\in} k_E \widetilde{\subset} g_E$. So $x \in h(e) \cap k(e)$ and $h_E \widetilde{\cap} k_E \widetilde{\subset} f_E \widetilde{\cap} g_E$. By Remark 4.8, $(x_e)_E \widetilde{\in} h_E \widetilde{\cap} k_E$. Now $h_E \widetilde{\cap} k_E \in \tau$. Thus $f_E \widetilde{\cap} g_E \in \mathbb{N}_{(x_e)_E}$.

(3) Let $g_E \in \mathbb{N}_{(x_e)_E}$. Then there exists $h_E \in \tau$ such that $(x_e)_E \widetilde{\in} h_E \widetilde{\subset} g_E$. Since $g_E \widetilde{\subset} f_E$, $(x_e)_E \in h_E \widetilde{\subset} f_E$. Thus $f_E \in \mathbb{N}_{(x_e)_E}$. □

Theorem 5.6. *Let (U, τ, E) be a soft topological space over U and let $f_E \in S(U, E)$. Then the following are equivalent.*

(1) *$f_E \in \tau$;*

(2) *$f_E \in \tau_{(x_e)_E}$ for any $(x_e)_E \widetilde{\in} f_E$;*

(3) *$f_E \in \mathbb{N}_{(x_e)_E}$ for any $(x_e)_E \widetilde{\in} f_E$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Let $f_E \in \mathbb{N}_{(x_e)_E}$ for any $(x_e)_E \widetilde{\in} f_E$. Then there exists $g(x_e)_E \in \tau$ such that $(x_e)_E \widetilde{\in} g(x_e)_E \widetilde{\subset} f_E$. So $(x_e)_E \widetilde{\subset} g(x_e)_E \widetilde{\subset} f_E$. This implies that

$$\widetilde{\cup}\ \{(x_e)_E : x \in f(e) \text{ and } e \in E\} \widetilde{\subset} \widetilde{\cup}\ \{g(x_e)_E : x \in f(e) \text{ and } e \in E\} \widetilde{\subset} f_E$$

By Theorem 4.10, $f_E = \widetilde{\cup}\ \{(x_e)_E : x \in f(e) \text{ and } e \in E\}$. Then $f_E = \widetilde{\cup}\ \{g(x_e)_E : x \in f(e) \text{ and } e \in E\}$. Hence $f_E \in \tau$. □

Corollary 5.7. *Let (U, τ, E) be a soft topological space over U. Then*

$$\tau = \{f_E : f_E \in \mathbb{N}_{(x_e)_E} \text{ for any } (x_e)_E \widetilde{\in} f_E\}.$$

Proof. This holds by Theorem 5.6. □

5.3. **The structure on soft topological spaces.**

Definition 5.8. Let (U, τ, E) be a soft topological space over U. Let $f_E \in S(U, E)$ and $(x_e)_E \in P(U, E)$.

(1) $(x_e)_E$ is called an interior point of f_E, if $g_E \widetilde{\subset} f_E$ for some $g_E \in \mathbb{N}_{(x_e)_E}$.

(2) $(x_e)_E$ is called a adherent point of f_E, if $g_E \widetilde{\cap} f_E \neq \widetilde{\varnothing}$ for any $g_E \in \mathbb{N}_{(x_e)_E}$.

(3) $(x_e)_E$ is called a boundary point of f_E, if $g_E \widetilde{\cap} f_E \neq \widetilde{\varnothing}$ and $g_E \widetilde{\cap} f'_E \neq \widetilde{\varnothing}$ for any $g_E \in \mathbb{N}_{(x_e)_E}$.

Remark 5.9. $(x_e)_E$ is a boundary point of $f_E \Longleftrightarrow (x_e)_E$ is a adherent point of f_E and $(x_e)_E$ is a adherent point of f'_E.

Proposition 5.10. *Let (U, τ, E) be a soft topological space over U. Let $f_E \in S(U, E)$ and $(x_e)_E \in P(U, E)$.*

(1) *$(x_e)_E$ is an interior point of f_E, if $g_E \tilde{\subset} f_E$ for some $g_E \in \tau_{(x_e)_E}$.*

(2) *$(x_e)_E$ is a adherent point of f_E, if $g_E \tilde{\cap} f_E \neq \tilde{\varnothing}$ for any $g_E \in \tau_{(x_e)_E}$.*

(3) *$(x_e)_E$ is a boundary point of f_E, if $g_E \tilde{\cap} f_E \neq \tilde{\varnothing}$ and $g_E \tilde{\cap} f'_E \neq \tilde{\varnothing}$ for any $g_E \in \tau_{(x_e)_E}$.*

Proof. The proof is straightforward. □

Theorem 5.11. *Let (U, τ, E) be a soft topological space over U. Then for any $f_E \in S(U, E)$*

$$int(f_E) = \tilde{\cup} \, \{(x_e)_E : (x_e)_E \textit{ is an interior point of } f_E\}.$$

Proof. Denote $h_E = int(f_E)$. We will show that

$$\mathcal{H}(E) = \{(x_e)_E : (x_e)_E \text{ is an interior point of } f_E\}.$$

Let $(x_e)_E \in \mathcal{H}(E)$. Then $x \in h(e)$. Since $h(e) = \cup \, \{g(e) : g_E \tilde{\subset} f_E \textit{ and } g_E \in \tau\}$, there exists $g_E \in \tau$ such that $g_E \tilde{\subset} f_E$ and $x \in g(e)$. $x \in g(e)$ implies that $(x_e)_E \tilde{\in} g_E$. Then $f_E \in \mathbb{N}_{(x_e)_E}$. Thus $(x_e)_E$ is an interior point of f_E. This show that $\mathcal{H}(E) \subseteq \{(x_e)_E : (x_e)_E$ is an interior point of $f_E\}$.

Conversely. Let $(x_e)_E$ is an interior point of f_E. By Proposition 5.10, $g_E \tilde{\subset} f_E$ for some $g_E \in \tau_{(x_e)_E}$. $(x_e)_E \tilde{\in} g_E$ implies that $x \in g(e)$. Then $(x_e)_E \in \mathcal{G}(E)$. Note that $g_E \in \tau$ and $g_E \tilde{\subset} f_E$. By Remark 2.6, $g_E \tilde{\subset} int(f_E) = h_E$. By Proposition 4.5, $\mathcal{G}(E) \subseteq \mathcal{H}(E)$. Hence $(x_e)_E \in \mathcal{H}(E)$. This show that $\mathcal{H}(E) \supseteq \{(x_e)_E : (x_e)_E$ is an interior point of $f_E\}$.

Hence $\mathcal{H}(E) = \{(x_e)_E : (x_e)_E$ is an interior point of $f_E\}$.

By Theorem 4.10, $h_E = \tilde{\cup} \, \mathcal{H}(E)$.

Therefore, $int(f_E) = \tilde{\cup} \, \{(x_e)_E : (x_e)_E$ is an interior point of $f_E\}$. □

Theorem 5.12. *Let (U, τ, E) be a soft topological space over U. Then for any $f_E \in S(U, E)$*

$$cl(f_E) = \tilde{\cup} \, \{(x_e)_E : (x_e)_E \textit{ is a adherent point of } f_E\}.$$

Proof. Denote $h_E = cl(f_E)$. We will show that

$$\mathcal{H}(E) = \{(x_e)_E : (x_e)_E \text{ is a adherent point of } f_E\}.$$

Let $(x_e)_E \in \mathcal{H}(E)$. Suppose that $f_E \tilde{\cap} g_E = \tilde{\varnothing}$ for some $g_E \in \tau_{(x_e)_E}$. Since $f_E \tilde{\cap} g_E = \tilde{\varnothing}$, by Remark 2.12, we have $f_E \tilde{\subset} g'_E$. Note that $h_E = \tilde{\cap} \, \{l_E : f_E \tilde{\subset} l_E \textit{ and } l'_E \in \tau\}$. By Remark 2.6, $h_E \tilde{\subset} g'_E$. Since $(x_e)_E \in \mathcal{H}(E)$, $x \in h(e)$. Then $x \in g'(e)$. This implies that $x \notin g(e)$. Since $g_E \in \tau_{(x_e)_E}$, $(x_e)_E \tilde{\in} g_E$. Thus $x \in g(e)$, a contradiction. Hence $f_E \tilde{\cap} g_E \neq \tilde{\varnothing}$ for any $g_E \in \tau_{(x_e)_E}$. By Proposition 5.10, $(x_e)_E$ is a adherent point of f_E. This show that $\mathcal{H}(E) \subseteq \{(x_e)_E : (x_e)_E$ is a adherent point of $f_E\}$.

Conversely. Let $(x_e)_E$ is a adherent point of f_E. Suppose that $(x_e)_E \notin \mathcal{H}(E)$. Then $x \notin h(e)$. By Remark 4.8, $(x_e)_E \tilde{\notin} h_E$. By Theorem 4.9, $(x_e)_E \tilde{\in} h'_E$. By

Proposition 3.3, $h'_E \in \tau$ and $f_E \widetilde{\subset} h_E = (h'_E)'$. By Remark 2.12, $f_E \widetilde{\cap} h'_E = \widetilde{\varnothing}$. Since $(x_e)_E$ is a adherent point of f_E, by Proposition 5.10, we have $f_E \widetilde{\cap} g_E \neq \widetilde{\varnothing}$ for any $g_E \in \tau_{(x_e)_E}$. Now $h'_E \in \tau_{(x_e)_E}$. This implies that $f_E \widetilde{\cap} h'_E \neq \widetilde{\varnothing}$, a contradiction. This show that $\mathcal{H}(E) \supseteq \{(x_e)_E : (x_e)_E$ is a adherent point of $f_E\}$.

Hence $\mathcal{H}(E) = \{(x_e)_E : (x_e)_E$ is a adherent point of $f_E\}$.

By Theorem 4.10, $h_E = \widetilde{\cup}\, \mathcal{H}(E)$.

Therefore, $cl(f_E) = \widetilde{\cup}\, \{(x_e)_E : (x_e)_E$ is a adherent point of $f_E\}$. □

Theorem 5.13. *Let (U, τ, E) be a soft topological space over U. Then for any $f_E \in S(U, E)$*

$$\partial(f_E) = \widetilde{\cup}\, \{(x_e)_E : (x_e)_E \textit{ is a boundary point of } f_E\}.$$

Proof. Denote $h_E = \partial(f_E)$, $g_E = cl(f_E)$ and $l_E = cl(f'_E)$. Since $h_E = g_E \widetilde{\cap} l_E$, by Proposition 4.5, we have $\mathcal{H}(E) = \mathcal{G}(E) \cap \mathcal{L}(E)$. By the proof of Theorem 5.12, $\mathcal{G}(E) = \{(x_e)_E : (x_e)_E$ is a adherent point of $f_E\}$ and $\mathcal{L}(E) = \{(x_e)_E : (x_e)_E$ is a adherent point of $f'_E\}$. Thus

$$\mathcal{H}(E) = \{(x_e)_E : (x_e)_E \text{ is both a adherent point of } f_E \text{ and } f'_E\}.$$

Hence $\mathcal{H}(E) = \{(x_e)_E : (x_e)_E$ is a boundary point of $f_E\}$.

By Theorem 4.10, $h_E = \widetilde{\cup}\, \mathcal{H}(E)$.

Therefore, $\partial(f_E) = \widetilde{\cup}\, \{(x_e)_E : (x_e)_E$ is a boundary point of $f_E\}$. □

Definition 5.14. Let (U, τ, E) be a soft topological space over U and let $\mathcal{B} \subseteq \tau$. $\mathcal{B}$ is called a base for (U, τ, E), if for any $f_E \in \tau$, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$f_E = \widetilde{\cup}\, \{b_E : b_E \in \mathcal{B}'\}.$$

Example 5.15. Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. We define

$f_1(e_1) = \varnothing,\ f_1(e_2) = \{x_1\};\ \ f_2(e_1) = \varnothing,\ f_2(e_2) = \{x_2\};$
$f_3(e_1) = \varnothing,\ f_3(e_2) = \{x_1, x_2\};\ \ f_4(e_1) = \{x_1\},\ f_4(e_2) = \varnothing;$
$f_5(e_1) = \{x_2\},\ f_5(e_2) = \varnothing;\ \ f_6(e_1) = \{x_1, x_2\},\ f_6(e_2) = \varnothing;$
$f_7(e_1) = \{x_1\},\ f_7(e_2) = \{x_2\};\ \ f_8(e_1) = \{x_2\},\ f_8(e_2) = \{x_1\};$
$f_9(e_1) = \{x_2\},\ f_9(e_2) = \{x_2\};\ \ f_{10}(e_1) = \{x_2\},\ f_{10}(e_2) = \{x_1, x_2\};$
$f_{11}(e_1) = \{x_1, x_2\},\ f_{11}(e_2) = \{x_2\};\ \ f_{12}(e_1) = \{x_1, x_3\},\ f_{12}(e_2) = \{x_1, x_3\};$
$f_{13}(e_1) = U,\ f_{13}(e_2) = \{x_1, x_3\};\ \ f_{14}(e_1) = \{x_1, x_3\},\ f_{14}(e_2) = U;$
$f_{15}(e_1) = \{x_1, x_2\},\ f_{15}(e_2) = \{x_1\};\ \ f_{16}(e_1) = \{x_1\},\ f_{16}(e_2) = \{x_1\};$
$f_{17}(e_1) = \{x_1\},\ f_{17}(e_2) = \{x_1, x_2\};\ \ f_{18}(e_1) = \{x_1, x_2\},\ f_{18}(e_2) = \{x_1, x_2\}.$

Put $\tau = \{\widetilde{\varnothing}, \widetilde{U}, (f_1)_E, (f_2)_E, \cdots, (f_{18})_E\}$. Then (U, τ, E) is a soft topological space over U.

Put $\mathcal{B} = \{\widetilde{\varnothing}, (f_1)_E, (f_2)_E, (f_4)_E, (f_5)_E, (f_9)_E, (f_{14})_E, (f_{15})_E, (f_{17})_E\}$. Then $\mathcal{B}$ is a base for (U, τ, E).

Theorem 5.16. *Let (U, τ, E) be a soft topological space over U and let $f_E \in S(U, E)$ and let $\mathcal{B}$ be a base for (U, τ, E). Then*

$$f_E \in \tau \iff \forall\ (x_e)_E \in \mathcal{F}(E)\ \exists\ b_E \in \mathcal{B} \textit{ such that } (x_e)_E \widetilde{\in} b_E \widetilde{\subset} f_E.$$

Proof. Necessity. Suppose that $f_E \in \tau$. Let $(x_e)_E \in \mathcal{F}(E)$. Since $\mathcal{B}$ is base, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $f_E = \widetilde{\cup}\, \{b_E : b_E \in \mathcal{B}'\}$. By Remark 4.8, $x \in f(e) = \cup\, \{b(e) :$

$b_E \in \mathcal{B}'$ *for any* $e \in E\}$. Then there exists $b_E \in \mathcal{B}'$ such that $x \in b(e)$. By Remark 4.8, $(x_e)_E \,\widetilde{\in}\, b_E$. Note that $b_E \,\widetilde{\subset}\, f_E$. Hence $(x_e)_E \,\widetilde{\in}\, b_E \,\widetilde{\subset}\, f_E$.

Sufficiency. Let $x \in f(e)$ and $e \in E$. Then $(x_e)_E \in \mathcal{F}(E)$. So there exists $b(e,x)_E \in \mathcal{B}$ such that $(x_e)_E \,\widetilde{\in}\, b(e,x)_E \,\widetilde{\subset}\, f_E$. So $(x_e)_E \,\widetilde{\subset}\, b(e,x)_E \,\widetilde{\subset}\, f_E$. This implies that

$$\widetilde{\cup}\,\{(x_e)_E : x \in f(e) \textit{ and } e \in E\} \,\widetilde{\subset}\, \widetilde{\cup}\,\{b(e,x)_E : x \in f(e) \textit{ and } e \in E\} \,\widetilde{\subset}\, f_E$$

By Theorem 4.10, $f_E = \widetilde{\cup}\,\{(x_e)_E : x \in f(e) \textit{ and } e \in E\}$. Then $f_E = \widetilde{\cup}\,\{b(e,x)_E : x \in f(e) \textit{ and } e \in E\}$. Note that $\mathcal{B} \in \tau$. Thus $f_E \in \tau$. □

Corollary 5.17. *Let (U, τ, E) be a soft topological space over U and let $\mathcal{B}$ be a base for (U, τ, E). Then*

$$\tau = \{f_E \in S(U,E) : \forall\ (x_e)_E \,\widetilde{\in}\, f_E,\ \exists\ b_E \in \mathcal{B} \textit{ such that } (x_e)_E \,\widetilde{\in}\, b_E \,\widetilde{\subset}\, f_E\ \}.$$

Proof. This holds by Theorem 5.16. □

Theorem 5.18. *Let (U, τ, E) be a soft topological space over U and let $\mathcal{B} \subseteq \tau$. Then the following are equivalent.*

(1) *$\mathcal{B}$ is base;*

(2) *For any $(x_e)_E \in P(U,E)$ and $f_E \in \tau_{(x_e)_E}$, there exists $b_E \in \mathcal{B}$ such that $(x_e)_E \,\widetilde{\in}\, b_E \,\widetilde{\subset}\, f_E$;*

(3) *For any $(x_e)_E \in P(U,E)$ and $f_E \in \mathbb{N}_{(x_e)_E}$, there exists $b_E \in \mathcal{B}$ such that $(x_e)_E \,\widetilde{\in}\, b_E \,\widetilde{\subset}\, f_E$.*

Proof. (1) ⇒ (2). Let $(x_e)_E \in P(U,E)$ and let $f_E \in \tau_{(x_e)_E}$. Then there exists $g_E \in \tau$ such that $(x_e)_E \,\widetilde{\in}\, g_E \,\widetilde{\subset}\, f_E$. By (1) there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$g_E = \widetilde{\cup}\,\{b_E : b_E \in \mathcal{B}'\}.$$

Thus $(x_e)_E \,\widetilde{\in}\, g_E$. By Remark 4.8, $x \in g(e) = \cup\,\{b(e) : b_E \in \mathcal{B}'$ *for any* $e \in E\}$. Then there exists $b_E \in \mathcal{B}'$ such that $x \in b(e)$. By Remark 4.8, $(x_e)_E \,\widetilde{\in}\, b_E$. Hence $b_E \,\widetilde{\subset}\, g_E \,\widetilde{\subset}\, f_E$ and $b_E \,\widetilde{\subset}\, f_E$.

(2) ⇒ (3). This is obvious.

(3) ⇒ (1). Let $f_E \in \tau$. Let $x \in f(e)$ and $e \in E$. By Remark 4.8, $(x_e)_E \,\widetilde{\in}\, f_E$. Thus $f_E \in \mathbb{N}_{(x_e)_E}$. By (2) there exists $b(e,x)_E \in \mathcal{B}$ such that $(x_e)_E \,\widetilde{\in}\, b(e,x)_E \,\widetilde{\subset}\, f_E$. So $(x_e)_E \,\widetilde{\subset}\, b(e,x)_E \,\widetilde{\subset}\, f_E$. This implies that

$$\widetilde{\cup}\,\{(x_e)_E : x \in f(e) \textit{ and } e \in E\} \,\widetilde{\subset}\, \widetilde{\cup}\,\{b(e,x)_E : x \in f(e) \textit{ and } e \in E\} \,\widetilde{\subset}\, f_E$$

By Theorem 4.10, $f_E = \widetilde{\cup}\,\{(x_e)_E : x \in f(e) \textit{ and } e \in E\}$. Then $f_E = \widetilde{\cup}\,\{b(e,x)_E : x \in f(e) \textit{ and } e \in E\}$. Put $\mathcal{B}' = \{b(e,x)_E : x \in f(e) \textit{ and } e \in E\}$. Then $\mathcal{B}$ is base. □

Let (U, τ, E) be a soft topological space over U and let $X \subseteq U$. Denote $\tau_X = \{g_E \,\widetilde{\cap}\, X_E : g_E \in \tau\}$.

It is easy to prove that (X, τ_X, E) is a soft topological space over X. (X, τ_X, E) is called a subspace of (U, τ, E).

Theorem 5.19 ([21]). *Let (U, τ, E) be a soft topological space over U, let (X, τ_X, E) be its subspace and let $g_E \,\widetilde{\subset}\, X_E$. Then*

(1) *g_E is soft open in X if and only if $g_E = f_E \,\widetilde{\cap}\, X_E$ for some $f_E \in \tau$.*

(2) *g_E is soft closed in X if and only if $g_E = f_E \,\widetilde{\cap}\, X_E$ for some soft closed set f_E in U.*

Theorem 5.20. *Let (U, τ, E) be a soft topological space over U, let $\mathcal{B}$ be a base for (U, τ, E) and let $X \subseteq U$. Then $\mathcal{B}_X = \{f_E \widetilde{\cap} X_E : f_E \in \mathcal{B}\}$ is a base for (X, τ_X, E).*

Proof. Let $g_E \in \tau_X$. By Theorem 5.19, there exists $f_E \in \tau$ such that $g_E = f_E \widetilde{\cap} X_E$. Let $x \in f(e)$ and $e \in E$. By Remark 4.8, $(x_e)_E \widetilde{\in} f_E$. Thus $f_E \in \mathbb{N}_{(x_e)_E}$. By Theorem 5.18, there exists $b(e,x)_E \in \mathcal{B}$ such that $(x_e)_E \widetilde{\in} b(e,x)_E \widetilde{\subset} f_E$. This implies that $(x_e)_E \widetilde{\in} b(e,x)_E \widetilde{\cap} X_E \widetilde{\subset} f_E \widetilde{\cap} X_E = g_E$. Note that $b(e,x)_E \widetilde{\cap} X_E \in \mathcal{B}_X$. By Theorem 5.18, $\mathcal{B}_X$ is a base for (X, τ_X, E). $\square$

6. **Conclusions**

In this paper, we defined soft points, researched some soft point sets and presented neighborhoods of soft points. It was worthwhile to mention that soft sets can be translated into soft point sets and then we may conveniently deal with soft relations, soft operations and soft topological spaces. In future work, we will study continuous mappings between soft topological spaces by using soft points and their neighborhoods.

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