Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 2, (August 2015), pp. 299–308
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On zero-dimensional fuzzy spaces

Mohammad Abry, Jafar Zanjani

Received 9 January 2015; Revised 12 February 2015; Accepted 10 March 2015

Abstract. Using fuzzy topological spaces in the sense of Lowen and the concept of zero-dimensionality in the usual form, we show that every T_1 and normal fuzzy space is zero-dimensional. It is done by an analogous argument to Katětov and Tong characterization for normal spaces. A number of propositions and examples illustrate relative situations.

2010 AMS Classification: 54A40, 54F45, 54E35

Keywords: Fuzzy topology, Zero-dimensional space, Normal space, Pseudo-metrizable.

Corresponding Author: Mohammad Abry (`mabry@du.ac.ir`)

1. Introduction

Dimension theory has been one of the most challenging fuzzy topological issues since Chang [9] tried to define a topological structure over the system of fuzzy sets proposed by Zadeh in 1965 [26]. It has been studied by several authors and however the subject is still dealing with obstacles such as, lack of a satisfactory concept of a boundary as well as comprehensive definition of a dimension. Adnadjevic introduced the generalized fuzzy spaces, GF-spaces, and at the same time, defined two dimension functions $\mathrm{F-ind}$, $\mathrm{F-Ind}$ for them [4, 5]. The c-zero dimensional and strongly c-zero dimensional fuzzy topological spaces and the fuzzy covering dimension are defined by Ajmal and kohli [6]. Tarres and Cuchillo-Ibanez established that the c-zero dimensionality and the strongly c-zero dimensionality are not good extension (in the Lowen sense) and introduced a new definition of the boundary of a fuzzy set so that it would characterize clopen fuzzy sets as set with empty boundary [10, 11]. As an later strive Baiju and Sunil have extended the concept of covering dimension of general topological spaces to L-topological spaces [7].

The initial motivation to write down this paper is actually the Example 3.21 where it was proposed as an Erdős-like fuzzy space. However, an application of our results shows that the space is zero-dimensional and so has no deal with Erdős space. It is proved among other things that every T_1 and normal and so every pseudo-metrizable fuzzy space is zero-dimensional. On the other hand, regarding to the most of former works, passing from zero case would be the first basic step to develop the theory of fuzzy dimensions. The results of this paper may draw a much clear picture of the present situation to be able to figure out how far is it from a satisfactory theory of dimension. Some examples of zero- and non zero-dimensional spaces are also presented.

2. Preliminaries

This section consists of some definitions and known results that will be used in this paper. All undefined concepts and notations used here are standard by now and can be found in [11, 19, 24].

Definition 2.1. A fuzzy set in a nonempty set X is a function (membership function) from X into the closed unit interval $\mathbb{I} = [0, 1]$. A fuzzy set μ in X is called crisp if $\mu(X) \subset \{0, 1\}$. The family of all fuzzy sets on X is denoted by $\mathbb{I}^X$. For every fuzzy subset $\mu \in \mathbb{I}^X$, the support of μ is defined by $\mathrm{supp}(\mu) = \{\mathrm{x} \in \mathrm{X} : \mu(\mathrm{x}) > 0\} = \mu^{-1}(0, 1]$.

A fuzzy set μ is said to be contained in a fuzzy set η if $\mu(x) \leq \eta(x)$ for each x in X, denoted by $\mu \leq \eta$. The union and intersection of a family of fuzzy sets is defined by $\bigvee \mu_\alpha = \sup(\mu_\alpha)$ and $\bigwedge \mu_\alpha = \inf(\mu_\alpha)$, respectively.

Definition 2.2. For every $x \in X$ and every $\alpha \in (0, 1)$, the fuzzy set x_α with membership function

$$x_\alpha(y) = \begin{cases} \alpha & y = x, \\ 0 & y \neq x, \end{cases}$$

is called a fuzzy point. x_α is said to be contained in a fuzzy set μ, denoted by $x_\alpha \in \mu$, if $\alpha < \mu(x)$ [24].

We omit the case $\alpha = \mu(x)$ in the above definition for some technical reasons. For instance, every second countable fail to be first countable if we allow the equality. Anyway, any fuzzy set is the union of all points which are contained in it and that for every two fuzzy sets μ and ν we have $\mu \leq \nu$ if and only if $x_\alpha \in \mu$ implies $x_\alpha \in \nu$, for every fuzzy point x_α. Any point x_1 is called a crisp point. We denote a constant fuzzy set whose unique value is $c \in [0, 1]$ by c_X.

Definition 2.3 ([19])**.** A fuzzy topology is a family δ of fuzzy sets in X which satisfies the following conditions:

(i) $\forall c \in \mathbb{I}$, $c_X \in \delta$,
(ii) $\forall \mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$,
(iii) $\forall (\mu_j)_{j \in J} \subset \delta \Rightarrow \bigvee_{j \in J} \mu_j \in \delta$.

δ is called a fuzzy topology for X, and the pair (X, δ) is called a fuzzy topological space. Open fuzzy sets, closed fuzzy sets and fuzzy clopens are defined as usual. In Chang's definition of fuzzy topology, which we will refer to as quasi fuzzy topology, condition (i) should be replaced by $(i)'$ $0, 1 \in \delta$. A base or subbase for a fuzzy space have the same meaning in the classic sense.

Definition 2.4 ([22]). Let $P = \{x_\alpha : x \in X, \alpha \in (0, 1]\}$ and $P_* = P \cup \{x_0 : x \in X\}$. A fuzzy metric on X is a mapping $d : P_* \times P_* \to [0, \infty)$ which satisfies, for all $x_\alpha, y_\beta, z_\gamma \in P_*$, the following conditions,

(1) If $y_\beta \subset x_\alpha$ (*i.e.*, $y = x$, $\beta \leq \alpha$), then $d(x_\alpha, y_\beta) = 0$,
(2) $d(x_\alpha, z_\gamma) \leq d(x_\alpha, y_\beta) + d(y_\beta, z_\gamma)$,
(3) $d(x_\alpha, y_\beta) = d(y_{1-\beta}, x_{1-\alpha})$,
(4) If $y_\beta \not\subset x_\alpha$, then $d(x_\alpha, y_\beta) > 0$.

For instance, it is clear that the mapping $d(x_\alpha, y_\beta) = \max\{d_1(x, y), \beta - \alpha\}$, where d_1 is a metric on X, is a fuzzy metric. In the above definition , if (4) (or (3),(4)) is omitted, then d is called a fuzzy pseudo-metric (or fuzzy quasi-metric).

Let d be a fuzzy quasi-metric for X, then for any $x_\alpha \in P_*$ and $\varepsilon > 0$, $B_\varepsilon(x_\alpha) = \bigvee\{y_\beta : d(x_\alpha, y_\beta) < \varepsilon\}$ is a fuzzy set which is called an ε-open ball of x_α. The family of all fuzzy open balls $B = \{B_\varepsilon(x_\alpha) : x_\alpha \in P,\ \varepsilon > 0\}$, corresponding to fuzzy (quasi-, pseudo-) metric d, forms a base for some fuzzy topology δ_d on X.

Recall that a fuzzy topological space (X, δ) is said to be T_1 if every fuzzy point is a closed fuzzy set, and it is called regular (normal) when for every fuzzy point x_α (for every closed fuzzy set ν) in X and for every open fuzzy set μ such that $x_\alpha \in \mu$ ($\nu \leq \mu$), there exists an open fuzzy set η such that $x_\alpha \in \eta \leq \mathrm{cl}(\eta) \leq \mu$ ($\nu \leq \eta \leq \mathrm{cl}(\eta) \leq \mu$). The following Proposition is standard.

Proposition 2.5. *A fuzzy topological space X is normal if and only if for any two closed fuzzy sets μ and ν with the property that $\mu(x) + \nu(x) \leq 1$ for every $x \in X$ and that $\mu(x) + \nu(x) = 1$ implies $\mu(x) = 1$ or $\nu(x) = 1$, there exist open fuzzy sets η_1 and η_2 such that $\mu \leq \eta_1$, $\nu \leq \eta_2$ and $\eta_1(x) + \eta_2(x) \leq 1$ for every $x \in X$.*

Theorem 2.6 ([22]). *If fuzzy topological space (X, δ) is fuzzy regular and has a σ-fuzzy locally finite base, then it is fuzzy pseudo-metrizable.*

Definition 2.7 ([14]). To every regular space X one assigns the small inductive dimension of X, denoted by ind X, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition of the dimension function ind consists in the following conditions:

(i) ind X $= -1$ if and only if $X = \varnothing$,
(ii) ind X $\leq$ n where n = 0, 1, ... , if for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and ind FrU $\leq$ n $-$ 1,
(iii) ind X = n if ind X $\leq$ n and ind X $>$ n $-$ 1, i.e., the inequality ind X $\leq$ n $-$ 1 does not hold,
(iv) ind X $= \infty$ if ind X $>$ n for n = -1, 0, 1, ...

Example 2.8. The space $\mathbb{Q}^c \subset \mathbb{R}$ of irrational numbers is zero-dimensional because it has a countable base consisting of open-and-closed sets, viz., the sets of the form $\mathbb{Q}^c \cap (a, b)$, where a and b are rational numbers.

3. Zero-dimensional spaces

The concept of the boundary is essential in the definition of inductive dimensions of topological spaces according to the Definition 2.7, see also [14]. By a similar manner Adnadjevic defined the inductive dimensions for generalized fuzzy spaces [4]. Cuchillo and Tarres [11] proposed a new definition of a fuzzy boundary as follows,

Definition 3.1 ([11])**.** Let μ be a fuzzy set in a fuzzy topological space X. The fuzzy boundary of μ, denoted by $\mathrm{Fr}(\mu)$, is defined as the infimum of all closed fuzzy sets σ in X with the property $\sigma(x) \geq \overline{\mu}(x)$ for all $x \in X$ for which $\overline{\mu}(x) - \mu^{\circ}(x) > 0$.

It is ready to see that μ is clopen fuzzy set if and only if $\mathrm{Fr}(\mu) = 0_{\mathrm{X}}$. So regarding this boundary if the definition of the Adnadjevic's dimension function is particularized, in the case of zero-dimensionality, we get the following well known definition.

Definition 3.2 ([11])**.** A fuzzy topological space X is zero-dimensional, ($\mathrm{ind(X)} = 0$) if for each fuzzy point x_α in X and every open fuzzy set μ containing x_α, there exists a clopen fuzzy set σ (equivalentely, a fuzzy set σ in X with $\mathrm{Fr}(\sigma) = 0_{\mathrm{X}}$) such that $x_\alpha \in \sigma \leq \mu$.

Throughout this paper we use the above definition.

Example 3.3. Let δ be the fuzzy topology on $X = [0, 1]$ with subbase $\{c_X : c \in \mathbb{I}\} \cup \{\mu\}$, where

$$\mu(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{3} & \frac{1}{2} < x \leq 1, \end{cases}$$

Clearly any non-constant open fuzzy sets has the form

$$\nu(x) = \begin{cases} a & 0 \leq x \leq \frac{1}{2}, \\ b & \frac{1}{2} < x \leq 1, \end{cases} \quad ; \quad 0 \leq a \leq b \leq \frac{1}{3}$$

There exists no clopen fuzzy set σ such that $\sigma \leq \mu$ because the constant fuzzy sets are the only clopen fuzzy sets. Thus $\mathrm{ind(X)} \neq 0$. Note that X does not satisfy even the rare separation axiom T_1.

Example 3.4. Let $X = \{0, 1\}$ with the fuzzy topology generated by the subbasis: $\{\gamma, c_X : c \in \mathbb{I}\}$, where $\gamma : X \to \mathbb{I}$ is the fuzzy set defined by $\gamma(x) = 1 - x$. It is easy to see that the non-constant open fuzzy sets are

$$\delta(x) = \begin{cases} a & x = 0, \\ 0 & x = 1, \end{cases} \quad ; \quad 0 \leq a \leq 1$$

The fuzzy point $0_{\frac{1}{2}}$ is contained in the open fuzzy set γ and there exists no clopen fuzzy set ν with the property $0_{\frac{1}{2}} \in \nu \leq \gamma$. Thus $\mathrm{ind(X)} \neq 0$. Note that $\mathrm{ind(X)} = 0$ if $\{c_X : c \in \mathbb{I}\}$ be the fuzzy topology on X.

Remark 3.5. Let (X, δ) be a fuzzy topological space and $Y \subset X$, then the family $\delta_1 = \{\mu|_Y : \mu \in \delta\}$ is a fuzzy topology for Y and (Y, δ_1) is called a subspace of (X, δ). If $\mathrm{ind(X)} = 0$, then $\mathrm{ind(Y)} = 0$. Note that restriction of every clopen fuzzy subset of X on subspace Y is a clopen fuzzy set in Y.

Following [19], we denote by $(X, \omega(\tau))$ the fuzzy topological space generated by a topological space (X, τ) consisting of all lower semi continuous functions $f : X \to \mathbb{I}$. The space $(X, \omega(\tau))$ is called the induced fuzzy topological space of (X, τ). Any fuzzy topology which is equal to $\omega(\tau)$ for some topology τ is called a topologically generated fuzzy topology. Clearly, a fuzzy set in $\omega(\tau)$ is clopen if and only if it is continuous with respect to τ. An extension of a topological property for fuzzy spaces is called a good extension when the underlying topology τ has this property if and only if the induced fuzzy topology $\omega(\tau)$ has it [20].

Proposition 3.6. *Zero-dimensionality in quasi fuzzy topological spaces is a good extension. Indeed* $\mathrm{ind}(\mathrm{X}, \tau) = 0$ *if and only if* $\mathrm{ind}(\mathrm{X}, \omega(\tau)) = 0$.

Proof. This is straightforward. □

Recall that for a topological space (X, τ), the quasi fuzzy topology generated by subbase $\{\chi_U : U \in \tau\}$ is called the characteristic quasi fuzzy topology on X.

Proposition 3.7. *Let X be the characteristic quasi fuzzy topological space of any non zero-dimensional space. Then* $\mathrm{ind}(\mathrm{X}) \neq 0$.

Proof. This is straightforward. □

The following proposition provides a base for induced topology consisting of finite intersections of constants and the charactristic functions over the open fuzzy sets.

Proposition 3.8. *Let (X, τ) be a topological space. If δ is the fuzzy topology generated by the subbase $\{\chi_U : U \in \tau\} \cup \{c_X : 0 \leq c \leq 1\}$ on X then δ contains all lower semi-continuous mappings, that is, $\delta = \omega(\tau)$.*

Proof. Let $f : X \to \mathbb{I}$ be a lower semi-continuous function. For every x in X and every $\varepsilon > 0$ choose $0 < c < 1$ with $f(x) - \varepsilon < c < f(x)$ and let $U = \{z \in X : f(z) > c\}$. Thus U is an open neighborhood of x and for every $z \in U$ we have $0 < f(z) - c \cdot \chi_U(z) < \varepsilon$ where $c \cdot \chi_U$ agrees with the open fuzzy set $\chi_U \wedge c_X$. Now it is easy to verify that

$$f = \sup\{c \cdot \chi_U : x \in X \text{ and } \varepsilon > 0\}.$$

□

Using characteristic functions over open sets of τ easily implies that if $\omega(\tau)$ is T_1, reagular or normal then τ is too. Next proposition presents a generalization of this fact for the separation properties T_1 and regularity.

Proposition 3.9. *Let δ be a fuzzy topology on a set X. Then the family $\{U_{\mu,t} : \mu \in \delta$ and $0 < t < 1\}$ where $U_{\mu,t} = \{x \in X : \mu(x) > t\}$ generates a topology $\tau = i(\delta)$ on X such that $\omega(\tau)$ is finer than δ. Indeed, τ is the coarsest topolology on X that makes all members of δ lower semi-continuous. Moreover, if δ is T_1 or regular then τ is too.*

Proof. Note that any $\mu \in \delta$ is clearly a lower semi-continuous function with respect to τ and hence $\delta \subset \omega(\tau)$. Suppose that δ is T_1. Let x, y be two distinct points in X and let $0 < \alpha < 1$ be an arbitrary number. There exists $\mu \in \delta$ that contains x_α but misses y_α. So if $\alpha < c < \mu(x)$ then $U_{\mu,c}$ is an open neighborhood of x which does not contain y. Finally suppose that δ is regular and let x be an arbitrary point in X that is contained in open set $U_{\mu,t}$. Choose a number α so that $t < \alpha < \mu(x)$. There exists an open fuzzy set ν with $x_\alpha \in \nu \leq \bar{\nu} \leq \mu$. Since $\alpha < \nu(x)$ we may choose a number

s with $\alpha < s < \nu(x)$. Now $U_{\nu,s}$ is an open neighborhood of x that is contained in the closed set $E = \{z \in X : \bar{\nu}(z) \geq s\}$ and that $E \subset \{z \in X : \mu(z) > t\}$. It implies that $x \in U_{\nu,s} \subset \overline{U_{\nu,s}} \subset U_{\mu,t}$. Thus τ is regular. □

The converse is also true and for the separation properties T_1 and regularity is readily obtained by applying characteristic functions. The property of being normal is a consequence of the more strong result due to Katětov [17] and Tong [25]. We bring a proof of it for the sake of completeness.

Proposition 3.10. *Let X be a normal space. If f and g are lower and upper semi-continuous functions from X to $\mathbb{I}$ respectively, with $g \leq f$ then there exists a continuous mapping $h : X \to \mathbb{I}$ such that $g \leq h \leq f$.*

Proof. For each pair (m, n) of natural numbers with $m \leq n$ we define disjoint closed subsets $E_{m,n} = \{x : f(x) \leq m/n\}$ and $F_{m,n} = \{x : g(x) \geq m/n + 1/2n\}$ of X. Let $f_{m,n}$ be a continuous function that is equal to $m/n + 1/2n$ on $E_{m,n}$ and to 1 on $F_{m,n}$. Put $f_i = \min_{j<i} f_{m,n}$. It is readily verified that for every $x \in X$ we have $g(x) \leq f_i(x)$ and that $f_n < f(x)$ or $|f_n(x) - f(x)| < 3/2n$. It follows that if $f_0 = \inf f_n$ then $g(x) \leq f_0(x) \leq f(x)$ for every $x \in X$. Applying this result to functions $-f$ and $-f_0$ we obtain $g_0 = \sup g_n$ such that $f_0(x) \leq g_0(x) \leq f(x)$ for every $x \in X$. With no loss of generality we may assume that $f_{n+1} \leq f_n$ and $g_{n+1} \geq g_n$ for every $n \in \mathbb{N}$ and every $x \in X$. Now if we let $k_i = \max_{j \leq i}[\min\{f_j, g_j\}]$ and $l_i = \max\{k_{i-1}, f_i\}$ then $\sup k_i = \inf l_i$ and so if we define $h(x)$ as this common value then h is as required. □

Note that the above result asserts that for any normal space (X, τ) the induced fuzzy space $(X, \omega(\tau))$ is zero-dimensional.

Remark 3.11. It is proved that a T_1 space X is Tychonoff if and only if for every lower semi-continuous function $h : X \to \mathbb{I}$ there exists a collection of real continuous mappings $\{f_s\}$ such that $h = \sup f_s$ [8, Proposition 5, Page 146]. For metric spaces h can be expressed as a limit of an increasing sequence of continuous functions; see [18, Theorem 23.19]. In other words, when (X, τ) is Tychonoff the collection of all real continuous functions in $(X, \omega(\tau))$ forms a base for the induced fuzzy topology. This yields the following result.

Proposition 3.12. *(X, τ) is Tychonoff if and only if $(X, \omega(\tau))$ is zero-dimensional.*

The normality of a fuzzy topology δ does not imply the normality of $i(\delta)$ in general.

Example 3.13. Let $X = \mathbb{I}$, $A = [0, 1/2]$, $\nu = \frac{1}{3}\chi_A$ and

$$\mu(x) = \begin{cases} \frac{1}{3} & 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2} & \frac{1}{3} < x \leq 1, \end{cases}$$

Let δ be the fuzzy topology induced by $\{\mu, \nu\} \cup \{c_X : 0 \leq c \leq 1\}$. It is ready to see that δ is normal but $i(\delta)$ is not normal. Note that δ is not zero-dimensional.

We modify the construction methods applied in the proofs of Proposition 3.10 by controling required maps to get the following generalization.

Theorem 3.14. *Every T_1 and normal fuzzy topology δ on a set X is zero-dimensional.*

Proof. We prove that for every $\mu \in \delta$ and every fuzzy point $x_\alpha \in \mu$ there exists a clopen fuzzy set f with $x_\alpha \in f \leq \mu$. Let $\alpha \leq r < \mu(x)$ be a rational number and let $r_0,\ r_1, \dots$ be a sequence consisting of all rational numbers in the interval $[0, r]$ with $r_0 = 0$ and $r_1 = r$. For every r_n we may choose open fuzzy set ν_n such that $\nu_m \geq \overline{\nu}_n$ whenever $r_m < r_n$. Let ν_1 be an open fuzzy set such that $x_r \in \nu_1 \leq \overline{\nu}_1 \leq \mu$. Put $\nu_0 = \mu$. Suppose that ν_n has been constructed. Let r_l and r_m be the nearest ones to r_{n+1} between $r_0,\ r_1, \dots,\ r_n$ from left and right. We have $\overline{\nu}_m \leq \nu_l$ and there exists $\nu_{n+1} \in \delta$ such that $\overline{\nu}_m \leq \nu_{n+1} \leq \overline{\nu}_{n+1} \leq \nu_l$. This complete our inductive task. Now we define $f = \bigvee_{n \geq 0} \nu_n \wedge r_n$. Clearly, f is an open fuzzy set that is contained in μ and that $f(x) \geq \alpha$. One may directly verify that $1 - f = \bigvee_{n \geq 0}[(1 - \overline{\nu}_n) \wedge (1 - r_n)]$. So f is a closed fuzzy set too. □

The converse of Theorem 3.14 is not true. For example if (X, τ) is the Niemytski plane then it is well known that X is a Tychonoff space which is not normal; see [15]. So by Proposition 3.12 $(X, \omega(\tau))$ is zero-dimensional.

Corollary 3.15. *Every pseudo-metrizable fuzzy topological space is zero-dimensional.*

In view of Proposition 2.5 and following the same method in [15, Lemma 1.5.15.] we have the following characterization.

Proposition 3.16. *A fuzzy topological space X is normal if and only if for every closed fuzzy set μ and every open fuzzy set ν that contains μ there exists a countable family $\{\eta_n\}_{n=1}^{\infty}$ of open fuzzy sets such that $\overline{\eta}_n \leq \nu$ for every n, and that $\mu = \bigvee_{n=1}^{\infty} \eta_n$.*

Corollary 3.17. *Any T_1 regular fuzzy space with a countable base is zero-dimensional.*

A subset of a topological space is called a C-set if it can be written as an intersection of clopen subsets of the space. By [13, Proposition 6.1] a space X is almost zero-dimensional (AZD) if and only if for every $x \in X$ and every neighborhood U of x there exists a C-set neighborhood V of x with $V \subset U$. This concept is originally introduced by Oversteegen and Tymchatyn in the realm of separable metrizable spaces [21], and they showed among others that the dimension of such space is at most one; see [1] for a simpler proof and generalization. It is clear that every zero-dimensional space is almost zero-dimensional. Almost zero-dimensionality is clearly hereditary and preserved under products. Note that every σ-compact almost zero-dimensional space is zero-dimensional; see [3] for more informations.

Regarding above conditions it is natural to define an analogy for almost zero-dimensionality in fuzzy case as follows.

Definition 3.18. A fuzzy topological space is called almost zero-dimensional if it is T_1 and for every fuzzy point x_α and every open fuzzy set μ contains x_α there exists a fuzzy neighborhood ν of x_α with $x_\alpha \in \nu \leq \mu$ such that ν is an intersection of clopen fuzzy sets.

According to the definition any almost zero-dimensional fuzzy space is T_1 and regular. So by Corollary 3.17 we have the following result.

Proposition 3.19. *Every almost zero-dimensional fuzzy space with a countable base is zero-dimensional.*

We conclude this section by constructing an interesting example of a zero-dimensional fuzzy space on the basis of Erdős space and in the view of Corollary 3.17. Consider the Hilbert space ℓ^2 consisting of all square summable sequences $x = (x_0, x_1, ...)$ of the real numbers. Erdős [16] introduced the closed subspace of ℓ^2 consisting of all $x \in \ell^2$ such that every coordinate x_i is in the convergent sequence $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. This space is now known as Complete Erdős space. The set of all rational sequences in ℓ^2 is known Erdős space $\mathfrak{E}$. Both spaces are one-dimensional and universal elements for the class of almost zero-dimensional spaces, that is, every almost zero-dimensional space can be imbedded in $\mathfrak{E}$ or $\mathfrak{E}_c$. For a number of characterizations and generalizations see [12, 2].

The following fact is essential for us.

Proposition 3.20. *Every closed ball in $\mathfrak{E}$ is a closed and co-dense subset with respect to product topology. In other words, the norm is a lower semi-continuous mapping with respect to product topology.*

Proof. Let $B = \{x \in \mathfrak{E} : \|x\| \leq r\}$ and let y be a point in $\mathfrak{E}$ with $\|y\|^2 = \sum_{i=0}^{\infty} |y_i|^2 > r^2$. There exists an integer k suth that $\sum_{i=0}^{k} |y_i|^2 > r^2$. We may choose $\delta > 0$ so that the inequalities $|z_i - y_i| < \delta$ for all $0 \leq i \leq k$ imply $\sum_{i=0}^{k} |z_i|^2 > r^2$. Now

$$W = (\prod_{i=0}^{k}(y_i - \delta, y_i + \delta) \times \prod_{i>k} \mathbb{R}) \cap \mathfrak{E}$$

is an open neighborhood of y with respect to product topology which any point of it has the norm larger than r. The rest of statements in the proposition are some trivialities. $\square$

Recall that the Cantor set is an universal for all separable metrizable zero-dimensional spaces, that is, every such space can be imbedded into Cantor set. Specially the countable product of rationals, $\mathbb{Q}^\omega$, can be imbedded into C. In the following example we follow the method in [23] and use essentially this fact that every non-empty clopen set of Erdős space is unbounded [16].

Example 3.21. Let φ be an imbedding of $E = \ell^2 \cap \mathbb{Q}^\omega$ with product topology, into the Cantor ternary set C such that $\varphi(0) = 0$ and let $X = \varphi(E) \setminus \{0\}$. Let $\mathcal{B} = \{W_k : k \in \mathbb{N}\}$ be a base for X consisting of nonempty clopen sets. Note that every $\varphi^{-1}(W_k)$ is unbounded in $\mathfrak{E}$. Define $\psi : X \to [0, 1]$ by $\psi(x) = \frac{\|q\|}{1+\|q\|}$, where $x = \varphi(q)$ and $\|.\|$ is the norm of ℓ^2. Let $P = \mathbb{Q} \cap (0, 1)$. For every $r \in P$ put $U_r = \{x \in X : \psi(x) < r\}$ and $f_r = \psi^{1-r} \wedge \chi_{U_r}$. It is clear that $0 < \psi < f_r < f_s$ whenever $r < s$. Note that every fuzzy point x_α is contained in some f_r. Evidenty $\varphi^{-1}(U_r) = \{q : \|q\| < \frac{r}{1-r}\}$ is open and bounded in $\mathfrak{E}$ and so U_r is not open in X. Let

$$\{c_0, c_1, c_t : t \in P\} \cup \{\chi_{W_k} : k \in \mathbb{N}\} \cup \{f_r : r \in P\}$$

be a subbase for a fuzzy topology δ on X which is obviously countable. It is ready to see that δ is T_1.

Now we prove that X is a fuzzy regular space. It suffices to show that for every f_r and every fuzzy point $x_\alpha \in f_r$ there exists a C-set neighborhood μ such that $x_\alpha \in \mu \leq f_r$. First notice that for every $r \in P$ if $\bar{U}_r = \{x \in X : \psi(x) \leq r\}$ then the fuzzy set $\chi_{\bar{U}_r}$ is a C-set by Proposition 3.20. Let t be a number with $\alpha < t < f_r(x)$ and then choose a number s with $s < \alpha$ such that for every y with $\psi(y) > s$ we have $t < f_r(y)$. Now by another appeal to Proposition 3.20 if W_j is a clopen such that $W_j \cap \bar{U}_s = \varnothing$ then $\mu = t\chi_{\bar{U}_t \cap W_j}$ is a fuzzy C-set neighborhood as required.

References

[1] M. Abry and J. J. Dijkstra, On topological Kadec norms, Math. Ann. 332 (2005) 759–765.
[2] M. Abry and J. J. Dijkstra, Universal spaces for almost n-dimensionality, Proc. Amer. Math. Soc. 135 (2007) 2623–2628.
[3] M. Abry, J. J. Dijkstra and J. van Mill, Sums of almost zero-dimensional spaces, Topology Proc. 29 (2005) 1–12.
[4] D. Adnadjevic, Dimension of fuzzy spaces, Fuzzy Sets and Systems 26 (1988) 85–92.
[5] D. Adnadjevic, Some properties of the dimension $F - Ind$ of $GF - Spaces$, Fuzzy Sets and Systems 54 (1993) 221–224.
[6] N. Ajmal and J. K. Kohli, Zero dimensional and strongly zero dimensional fuzzy topological spaces, Fuzzy Sets and Systems 61 (1994) 231–237.
[7] T. Baiju and Sunil Jacob John, Covering dimension and normality in L-topological spaces, Mathematical Sciences 6(35) (2012) 1–6.
[8] N. Bourbaki, Elements of Mathematics: General Topology, Part 2, Addison–Wesley 1966.
[9] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[10] E. Cuchillo-Ibanez and J. Tarres, On zero dimensionality in fuzzy topological spaces, Fuzzy Sets and Systems 82 (1996) 361–367.
[11] E. Cuchillo-Ibanez and J. Tarres, On the boundary of fuzzy sets, Fuzzy Sets and Systems 89 (1997) 113–119.
[12] J. J. Dijkstra and J. van Mill, Characterizing complete Erdős space, Canad. J. Math. 61 (2009) 124–140.
[13] J. J. Dijkstra, J. van Mill and J. Steprāns, Complete Erdős space is unstable, Math. Proc. Cambridge Philos. Soc. 137 (2004) 465–473.
[14] R. Engelking, Dimension Theory, North-Holland, Amsterdam 1978.
[15] R. Engelking, General Topology, Heldermann Verlag, Berlin, second ed. 1989.
[16] P. Erdős, The dimension of the rational points in Hilbert space, Ann. of Math. 41 (1940) 734–736.
[17] M. Katětov, On real-valued functions in topological spaces, Fund. Math. 38 (1951) 85–91.
[18] A. S. Kechris, Classical Descriptive Set Theory, Springer Verlag, New York 1995.
[19] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.
[20] R. Lowen, A comparison of different compactness notions in fuzzy topological spaces, J. Math. Anal. Appl. 64 (1978) 446–454.
[21] L. G. Oversteegen and E. D. Tymchatyn, On the dimension of certain totally disconnected spaces, Proc. Amer. Math. Soc. 122 (1994) 885–891.
[22] N. Palaniappan, Fuzzy Topology, Narosa Publishing House, New Delhi 2002.
[23] J. H. Roberts, The rational points in Hilbert space, Duke Math. 23 (1956) 489–491.
[24] R. Srivastava, S. N. Lal, and A. K. Srivastava, Fuzzy Hausdorff topological spaces, J. Math. Anal. Appl. 81 (1981) 497–506.
[25] H. Tong, Some charactrizations of normal and perfectly normal spaces, Duke Math. 19 (1952) 289–292.
[26] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338–353.

Mohammad Abry (`mabry@du.ac.ir`)
School of Mathematics and Computer science, university of Damghan , postal code 3671641167, Damghan, Iran

Jafar Zanjani (`j_zanjani@std.du.ac.ir`)

School of Mathematics and Computer science, university of Damghan , postal code 3671641167, Damghan, Iran