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Near semigroups on nearness approximation spaces

Ebubekir İnan, Mehmet Ali Öztürk

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Abstract. In this paper, our approach is to study properties of approximations on nearness approximation spaces. Afterwards, our aim is to define near semigroups and near ideals on near approximation spaces. We introduce some properties of these nearness structures. The nearness of sets and, in particular, the nearness of algebraic structures on nearness approximation spaces fits within the milieu of fuzzy sciences theory.

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Corresponding Author: Ebubekir İnan (einan@adiyaman.edu.tr)

1. INTRODUCTION

Rough sets were introduced in 1982 [13]. An algebraic approach to rough sets has been given by Iwinski [7]. Afterwards, rough subgroups were introduced by Biswas and Nanda [1]. Rough ideal in a semigroup were introduced by Kuroki [8]. In 2004 and 2006, Davvaz investigated the concept of roughness of rings and modules [2, 3] (and other algebraic approaches to rough sets in [9, 23, 22, 24]).

In 2007, near set theory and nearness approximation spaces were introduced by J. F. Peters as a generalization of rough set theory [15, 16, 19]. In this theory, Peters utilizes the features of objects to define the nearness of objects [17] and, consequently, the classification of the universal set with respect to the available information of the objects. More recent work considers generalized approach theory in the study of the nearness of nonempty sets that resemble each other and a topological framework for the study of nearness and apartness of sets [10, 18, 20]. Nonempty sets are near, provided the sets resemble each other descriptively. It is the resemblance of sets that places near set theory in the fuzzy sciences theory milieu, since membership of a set in a family of near sets depends on a comparison of object descriptions that are usually not exact and such inexact descriptions establish the resemblance of each set in a family of sets that are descriptively near each other. In 2013, Öztürk and İnan

[11] combined the soft sets approach with near set theory, which gives rise to the new concepts of soft nearness approximation spaces (SNAS), soft lower and upper approximations.

Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real-valued function representing a feature of physical objects. But in this paper, in a more general setting that includes data mining, probe functions φ_i would be defined to allow for non-numerical values, i.e., let $\varphi_i : X \longrightarrow V$, where V is the value set for the range of φ_i [21]. This more general definition of $\varphi_i \in \mathcal{F}$ is also better in setting forth the algebra and logic.

In 2012, İnan and Öztürk [5, 6] investigated the basic concepts of the algebraic structures of the near set theory. They introduced the concept of near groups, weak cosets, near normal subgroups, homomorphism of near groups on nearness approximation spaces. Moreover, in 2014, Öztürk et al. [12] introduced near group of weak cosets on nearness approximation spaces. In this article, our aim is to improve the concept of nearness semigroup theory, which extends the notion of a semigroup to include the algebraic structures of near sets. Also, we introduce some properties of aproximations and these algebraic structures.

2. Preliminaries

In this section we give some definitions and properties regarding near sets [15].

Objects are known by their descriptions. An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \longrightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$, a vector containing measurements associated with each functional value $\varphi_i(x)$, where the description length $|\Phi| = L$.

Object Description: $\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), ..., \varphi_i(x), ..., \varphi_L(x))$.

Sample objects $X \subseteq \mathcal{O}$ are near each other if and only if the objects have similar descriptions. Recall that each φ defines a description of an object. Then let Δ_{φ_i} denote $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$ where $x, x' \in \mathcal{O}$. The difference Δ_φ leads to a definition of the indiscernibility relation $\sim_B$ introduced by Z. Pawlak [14].

Let $x, x' \in \mathcal{O}$, $B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \ \forall \varphi_i \in B, \ \ \Delta_{\varphi_i} = 0 \ \}$$

is called the indiscernibility relation on $\mathcal{O}$ where description length $i \leq |\Phi|$.

Let $B \subseteq \mathcal{F}$ be a set of functions representing features of objects $x, x' \in \mathcal{O}$. Objects x, x' are called minimally near each other, if there exists $\varphi_i \in B$ such that $x \sim_{\{\varphi_i\}} x'$, $\Delta_{\varphi_i} = 0$. This is called the "Nearness Description Principle - NDP". The objects in a class $[x]_B \in \xi_B$ are near objects [15].

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other, if the sets contain objects with at least partial matching descriptions.

Let $X, X' \subseteq \mathcal{O}$, $B \subseteq \mathcal{F}$. Set X is called *near* X', if there exists $x \in X$, $x' \in X'$, $\varphi_i \in B$ such that $x \sim_{\{\varphi_i\}} x'$.

Symbol	*Interpretation*
B	$B \subseteq \mathcal{F}$,
r	$\binom{\lvert B \rvert}{r}$, i.e. , $\lvert B \rvert$ probe functions $\varphi_i \in B$ taken r at a time,
B_r	$r \leq \lvert B \rvert$ probe functions in B,
$\sim_{B_r}$	Indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, equivalence class,
$\mathcal{O}/\sim_{B_r}$	$\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\} = \xi_{\mathcal{O},B_r}$, quotient set,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^* X \diagdown N_r(B)_* X = \left\{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\right\}$.

Table 1: Nearness Approximation Space Symbols

A nearness approximation space (NAS) is a tuple $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the approximation space NAS is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbourhoods) $N_r(B)$, and overlap function ν_{N_r}.

A semigroup is an algebraic structure on a nonempty set S together with an associative binary operation. That means, a semigroup is a set S together with a binary operation "$\cdot$" that satisfies:

(i) For all $a, b \in S$, $a \cdot b \in S$.

(ii) For all $a, b, c \in S$, the equation $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds in S.

A nonempty subset A of a semigroup S is said to be a subsemigroup of S, if $a \cdot b \in A$ for all $a, b \in A$, i.e., $A^2 \subseteq A$.

A nonempty subset A of a semigroup S is said to be a left (resp. right) ideal of S if $SA \subseteq A$ (resp. $AS \subseteq A$). A is said to be an ideal of S, provided it is both a left ideal and a right ideal of S [4].

3. Some Properties of Approximations

Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let "." be a binary operation defined on $\mathcal{O}$. In this section, we will use "xy" instead of "$x \cdot y$" for $x, y \in \mathcal{O}$.

Definition 3.1. Let $X \subseteq \mathcal{O}$ and $B_r \subseteq \mathcal{F}$, $r \leq \lvert B \rvert$. A indiscernibility relation $\sim_{B_r}$ on $\mathcal{O}$ is called a complete indiscernibility relation $\sim_{B_r}$ on perceptual objects $\mathcal{O}$, if $[x]_{B_r} [y]_{B_r} = [xy]_{B_r}$ for all $x, y \in X$.

Theorem 3.2. *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and $X, Y \subset \mathcal{O}$, then the following statements hold;*

(1) $N_r(B)_*(X) \subseteq X \subseteq N_r(B)^*(X)$,
(2) $N_r(B)^*(X \cup Y) = N_r(B)^*(X) \cup N_r(B)^*(Y)$,
(3) $N_r(B)_*(X \cap Y) = N_r(B)_*(X) \cap N_r(B)_*(Y)$,
(4) $X \subseteq Y$ *implies* $N_r(B)_*(X) \subseteq N_r(B)_*(Y)$,
(5) $X \subseteq Y$ *implies* $N_r(B)^*(X) \subseteq N_r(B)^*(Y)$,
(6) $N_r(B)_*(X \cup Y) \supseteq N_r(B)_*(X) \cup N_r(B)_*(Y)$,
(7) $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*(X) \cap N_r(B)^*(Y)$.

Proof. (1) Let $x \in N_r(B)_*(X)$, then $x \in [x]_{B_r} \subseteq X$ and $N_r(B)_*(X) \subseteq X$. Let $x \in X$ then $x \in [x]_{B_r}$ and we have $[x]_{B_r} \cap X \neq \varnothing$. Then observe that $x \in N_r(B)^*(X)$. Hence, $X \subseteq N_r(B)^*(X)$.

(2)

$$\begin{aligned} x \in N_r(B)^*(X \cup Y) &\Leftrightarrow [x]_{B_r} \cap (X \cup Y) \neq \varnothing \\ &\Leftrightarrow \left([x]_{B_r} \cap X\right) \cup \left([x]_{B_r} \cap Y\right) \neq \varnothing \\ &\Leftrightarrow [x]_{B_r} \cap X \neq \varnothing \text{ or } [x]_{B_r} \cap Y \neq \varnothing \\ &\Leftrightarrow x \in N_r(B)^*(X) \text{ or } x \in N_r(B)^*(Y) \\ &\Leftrightarrow x \in N_r(B)^*(X) \cup N_r(B)^*(Y). \end{aligned}$$

Thus $N_r(B)^*(X \cup Y) = N_r(B)^*(X) \cup N_r(B)^*(Y)$.

(3)

$$\begin{aligned} x \in N_r(B)_*(X \cap Y) &\Leftrightarrow [x]_{B_r} \subseteq X \cap Y \\ &\Leftrightarrow [x]_{B_r} \subseteq X \text{ and } [x]_{B_r} \subseteq Y \\ &\Leftrightarrow x \in N_r(B)_*(X) \text{ and } x \in N_r(B)_*(Y) \\ &\Leftrightarrow x \in N_r(B)_*(X) \cap N_r(B)_*(Y). \end{aligned}$$

Hence, $N_r(B)_*(X \cap Y) = N_r(B)_*(X) \cap N_r(B)_*(Y)$.

(4) Let $X \subseteq Y$, then $X \cap Y = X$, consequently by (3) we have $N_r(B)_*(X) = N_r(B)_*(X \cap Y) = N_r(B)_*(X) \cap N_r(B)_*(Y)$. Then observe that $N_r(B)_*(X) \subseteq N_r(B)_*(Y)$.

(5) Let $X \subseteq Y$, then $X \cup Y = Y$, consequently by (2) we have $N_r(B)^*(Y) = N_r(B)^*(X \cup Y) = N_r(B)^*(X) \cup N_r(B)^*(Y)$. This implies that $N_r(B)^*(X) \subseteq N_r(B)^*(Y)$.

(6) We know that $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, consequently by (4) we have $N_r(B)_*(X) \subseteq N_r(B)_*(X \cup Y)$ and $N_r(B)_*(Y) \subseteq N_r(B)_*(X \cup Y)$.

Hence, $N_r(B)_*(X) \cup N_r(B)_*(Y) \subseteq N_r(B)_*(X \cup Y)$.

(7) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, consequently by (5) we have $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*(X)$ and $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*(Y)$.

Hence, $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*(X) \cap N_r(B)^*(Y)$. □

Theorem 3.3. *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ *be a nearness approximation space. If* X *and* Y *are nonempty subsets of perceptual objects* $\mathcal{O}$, *then*

$$N_r(B)^*(X)\, N_r(B)^*(Y) \subseteq N_r(B)^*(XY).$$

Proof. Let $z \in N_r(B)^*(X)\, N_r(B)^*(Y)$, then we have $z = xy$ with $x \in N_r(B)^*(X)$ and $y \in N_r(B)^*(Y)$. Thus there exist elements $k, l \in \mathcal{O}$ such that $k \in [x]_{B_r} \cap X$ and $l \in [y]_{B_r} \cap Y$. Consequently, $k \in [x]_{B_r}$, $l \in [y]_{B_r}$, $x \in X$ and $y \in Y$. Since $\sim_{B_r}$ is an indiscernibility relation on $\mathcal{O}$, $kl \in [x]_{B_r} [y]_{B_r} \subseteq [xy]_{B_r}$. Since $kl \in XY$, we observe that $kl \in [xy]_{B_r} \cap XY$ and so $xy \in N_r(B)^*(XY)$.

Hence, $N_r(B)^*(X)\, N_r(B)^*(Y) \subseteq N_r(B)^*(XY)$. □

Theorem 3.4. *Let $\sim_{B_r}$ be a complete indiscernibility relation on $\mathcal{O}$. If X and Y are nonempty subsets of $\mathcal{O}$, then $N_r(B)_*(X)\, N_r(B)_*(Y) \subseteq N_r(B)_*(XY)$.*

Proof. Let $z \in N_r(B)_*(X)\, N_r(B)_*(Y)$, then $z = xy$ with $x \in N_r(B)_*(X)$ and $y \in N_r(B)_*(Y)$. Thus we have $[x]_{B_r} \subseteq X$ and $[y]_{B_r} \subseteq Y$. Since $\sim_{B_r}$ is a complete indiscernibility relation on $\mathcal{O}$, we have that $[xy]_{B_r} = [x]_{B_r}[y]_{B_r} \subseteq XY$ and so $xy \in N_r(B)_*(XY)$.

Consequently, $N_r(B)_*(X)\, N_r(B)_*(Y) \subseteq N_r(B)_*(XY)$. □

4. Near Semigroups and Near Ideals

Definition 4.1. Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let "$\cdot$" be a binary operation defined on $\mathcal{O}$.

A subset S of the set of perceptual objects $\mathcal{O}$ is called a near semigroup on nearness approximation space or shortly nearness semigroup, provided the following properties are satisfied:

(1) For all $x, y \in S$, $x \cdot y \in N_r(B)^*(S)$,

(2) For all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*(S)$.

Example 4.2. Let $\mathcal{O} = \{o, a, b, c, d, e, f, g, h, i\}$ be a set of perceptual objects and $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions. Values of the probe functions

$$\varphi_1 : \mathcal{O} \longrightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$\varphi_2 : \mathcal{O} \longrightarrow V_2 = \{\alpha_1, \alpha_2, \alpha_3\} \text{ and}$$
$$\varphi_3 : \mathcal{O} \longrightarrow V_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

are given in Table 2.

	o	a	b	c	d	e	f	g	h	i
φ_1	α_1	α_2	α_2	α_3	α_1	α_1	α_4	α_2	α_3	α_2
φ_2	α_3	α_3	α_1	α_1	α_2	α_2	α_2	α_3	α_1	α_3
φ_3	α_2	α_3	α_1	α_3	α_4	α_2	α_1	α_3	α_2	α_3

Table 2

Let "$\cdot$" be a binary operation of perceptual objects on $\mathcal{O}$ as in Table 3.

$\cdot$	o	a	b	c	d	e	f	g	h	i
o	o	a	o	o	o	o	o	g	o	o
a	a	a	b	c	c	c	f	g	h	i
b	o	b	b	o	o	o	f	o	o	o
c	c	b	c	c	d	o	o	g	h	o
d	o	o	o	d	d	e	o	o	h	i
e	o	e	o	c	e	e	o	g	h	o
f	o	a	o	c	f	f	o	f	b	i
g	g	a	b	c	d	d	e	g	h	o
h	o	a	h	h	o	o	e	h	o	o
i	i	i	o	o	d	o	o	g	o	i

Table 3

Let $S = \{d, e, f\}$ be a subset of perceptual objects and let "$\cdot$" be an operation on $S \subseteq \mathcal{O}$ as in Table 4.

$\cdot$	d	e	f
d	d	e	o
e	e	e	o
f	f	f	o

Table 4

$$\begin{aligned}
[o]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(o) = \alpha_1\} \\
&= \{o, d, e\} = [d]_{\varphi_1} = [e]_{\varphi_1}, \\
[a]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(a) = \alpha_2\} \\
&= \{a, b, g, i\} = [b]_{\varphi_1} = [g]_{\varphi_1} = [i]_{\varphi_1}, \\
[c]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(c) = \alpha_3\} \\
&= \{c, h\} = [h]_{\varphi_1}, \\
[f]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(f) = \alpha_4\} \\
&= \{f\}.
\end{aligned}$$

Hence, we get that $\xi_{\varphi_1} = \left\{[o]_{\varphi_1}, [a]_{\varphi_1}, [c]_{\varphi_1}, [f]_{\varphi_1}\right\}$.

$$\begin{aligned}
[o]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(o) = \alpha_3\} \\
&= \{o, a, g, i\} = [a]_{\varphi_2} = [g]_{\varphi_2} = [i]_{\varphi_2}, \\
[b]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(b) = \alpha_1\} \\
&= \{b, c, h\} = [c]_{\varphi_2} = [h]_{\varphi_2}, \\
[d]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(d) = \alpha_2\} \\
&= \{d, e, f\} = [e]_{\varphi_2} = [f]_{\varphi_2}.
\end{aligned}$$

Thus we have that $\xi_{\varphi_2} = \left\{[o]_{\varphi_2}, [b]_{\varphi_2}, [d]_{\varphi_2}\right\}$.

$$\begin{aligned}
[o]_{\varphi_3} &= \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(o) = \alpha_2\} \\
&= \{o, e, h\} = [e]_{\varphi_3} = [h]_{\varphi 3}, \\
[a]_{\varphi_3} &= \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(a) = \alpha_3\} \\
&= \{a, c, g, i\} = [c]_{\varphi_3} = [g]_{\varphi_3} = [i]_{\varphi_3}, \\
[b]_{\varphi_3} & \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(b) = \alpha_1\} \\
&= \{b, f\} = [f]_{\varphi_3}, \\
[d]_{\varphi_3} &= \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(d) = \alpha_4\} \\
&= \{d\}.
\end{aligned}$$

So we obtain that $\xi_{\varphi_3} = \left\{[o]_{\varphi_3}, [a]_{\varphi_3}, [b]_{\varphi_3}, [d]_{\varphi_3}\right\}$.
Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$.
Then, we can write

$$\begin{aligned}
N_1(B)^*(S) &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= \{o, d, e\} \cup \{f\} \cup \{d, e, f\} \cup \{o, e, h\} \cup \{b, f\} \cup \{d\} \\
&= \{o, b, d, e, f, h\}.
\end{aligned}$$

From Definition 4.1, subset S of perceptual objects $\mathcal{O}$ is a nearness semigroup.

We defined a collection of partitions $N_1(B)$, where $N_1(B) = \{\xi_{\mathcal{O},B_1} \mid B_1 \subseteq B\}$. Families of neighborhoods are constructed for each combination of probe functions in B using $\binom{|B|}{1}$, i.e., $|B|$ probe functions taken 1 at a time. We can give an example for $r = 2$.

Example 4.3. Let $\mathcal{O} = \{o, a, b, c, d, e, f, g, h, i\}$ be a set of perceptual objects and $B = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subseteq \mathcal{F}$ be a set of probe functions. Values of the probe functions

$$\begin{array}{l} \varphi_1 : \mathcal{O} \longrightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \\ \varphi_2 : \mathcal{O} \longrightarrow V_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \\ \varphi_3 : \mathcal{O} \longrightarrow V_3 = \{\alpha_1, \alpha_3, \alpha_5\} \text{ and} \\ \varphi_4 : \mathcal{O} \longrightarrow V_4 = \{\alpha_3, \alpha_{4,}\alpha_5\} \end{array}$$

are given in Table 5.

	o	a	b	c	d	e	f	g	h	i
φ_1	α_1	α_2	α_2	α_5	α_1	α_1	α_2	α_4	α_1	α_3
φ_2	α_4	α_2	α_3	α_5	α_4	α_5	α_3	α_5	α_2	α_2
φ_3	α_1	α_5	α_1	α_1	α_1	α_1	α_3	α_1	α_3	α_1
φ_4	α_5	α_5	α_3	α_4	α_5	α_3	α_3	α_5	α_3	α_5

Table 5

Let "$\cdot$" be a binary operation of perceptual objects on $\mathcal{O}$ as in Table 6.

$\cdot$	o	a	b	c	d	e	f	g	h	i
o	o	a	o	o	o	o	o	g	o	o
a	a	a	b	c	c	c	f	g	h	i
b	o	b	b	o	o	o	f	o	o	o
c	c	b	c	c	d	o	o	g	h	o
d	o	o	o	d	d	e	o	o	h	i
e	o	e	o	c	e	e	o	g	h	o
f	o	a	o	c	f	f	o	f	b	i
g	g	a	b	c	d	d	e	g	h	o
h	o	a	h	h	o	o	e	h	o	o
i	i	i	o	o	d	o	o	g	o	i

Table 6

Let $S = \{d, e, f\}$ be a subset of perceptual objects. Then, "$\cdot$" be an operation of perceptual objects on $S \subseteq \mathcal{O}$ as in Table 7.

$\cdot$	d	e	f
d	d	e	o
e	e	e	o
f	f	f	o

Table 7

$$\begin{aligned}[a]_{\{\varphi_1,\varphi_2\}} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_2(x') = \varphi_1(a) = \varphi_2(a) = \alpha_2\} \\ &= \{a\}, \\ [c]_{\{\varphi_1,\varphi_2\}} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_2(x') = \varphi_1(c) = \varphi_2(c) = \alpha_5\} \\ &= \{c\}.\end{aligned}$$

Hence, we get that $\xi_{(\varphi_1,\varphi_2)} = \xi_{(\varphi_2,\varphi_1)} = \left\{[a]_{\{\varphi_1,\varphi_2\}}, [c]_{\{\varphi_1,\varphi_2\}}\right\}$.

$$\begin{aligned}[o]_{\{\varphi_1,\varphi_3\}} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_3(x') = \varphi_1(o) = \varphi_3(o) = \alpha_1\} \\ &= \{o, d, e\} = [d]_{\{\varphi_1,\varphi_3\}} = [e]_{\{\varphi_1,\varphi_3\}}.\end{aligned}$$

Thus we have that $\xi_{(\varphi_1,\varphi_3)} = \xi_{(\varphi_3,\varphi_1)} = \left\{[o]_{\{\varphi_1,\varphi_3\}}\right\}$.

$$\begin{aligned}[f]_{\{\varphi_2,\varphi_3\}} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_3(x') = \varphi_2(f) = \varphi_3(f) = \alpha_3\} \\ &= \{f\}.\end{aligned}$$

Hence, we have that $\xi_{(\varphi_2,\varphi_3)} = \xi_{(\varphi_3,\varphi_2)} = \left\{[f]_{\{\varphi_2,\varphi_3\}}\right\}$.

$$\begin{aligned}[b]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_4(x') = \varphi_2(b) = \varphi_4(b) = \alpha_3\} \\ &= \{b, f\} = [f]_{\{\varphi_2,\varphi_4\}}, \\ [g]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_4(x') = \varphi_2(g) = \varphi_4(g) = \alpha_5\} \\ &= \{g\}.\end{aligned}$$

So we get that $\xi_{(\varphi_2,\varphi_4)} = \xi_{(\varphi_4,\varphi_2)} = \left\{[b]_{\{\varphi_2,\varphi_4\}}, [g]_{\{\varphi_2,\varphi_4\}}\right\}$.

$$\begin{aligned}[a]_{\{\varphi_3,\varphi_4\}} &= \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_4(x') = \varphi_3(a) = \varphi_4(a) = \alpha_5\} \\ &= \{a\}, \\ [f]_{\{\varphi_3,\varphi_4\}} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_4(x') = \varphi_2(g) = \varphi_4(g) = \alpha_3\} \\ &= \{f, h\} = [h]_{\{\varphi_3,\varphi_4\}},\end{aligned}$$

Hence, we obtain that $\xi_{(\varphi_3,\varphi_4)} = \xi_{(\varphi_4,\varphi_3)} = \left\{[a]_{\{\varphi_3,\varphi_4\}}, [f]_{\{\varphi_3,\varphi_4\}}\right\}$.
Therefore, for $r = 2$, a set of partitions of $\mathcal{O}$ is
$N_2(B) = \left\{\xi_{(\varphi_1,\varphi_2)}, \xi_{(\varphi_1,\varphi_3)}, \xi_{(\varphi_2,\varphi_3)}, \xi_{(\varphi_2,\varphi_4)}, \xi_{(\varphi_3,\varphi_4)}\right\}$.
Then, we can write

$$\begin{aligned}N_2(B)^*(S) &= \bigcup_{[x]_{\{\varphi_i,\varphi_j\}} \cap S \neq \varnothing} [x]_{\{\varphi_i,\varphi_j\}} \\ &= \{o, d, e\} \cup \{f\} \cup \{b, f\} \cup \{f, h\} \\ &= \{o, b, d, e, f, h\}.\end{aligned}$$

From Definition 4.1, subset S of perceptual objects $\mathcal{O}$ is a nearness semigroup.

Definition 4.4. Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space, S be a nearness semigroup and I a nonempty subset of S. If $N_r(B)^*(I)$ is a left (right, two sided) ideal of S, then I is called a nearness left (right, two sided) ideal of S.

Theorem 4.5. *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ *be a nearness approximation space. Then*

(1) If S is a semigroup of perceptual objects $\mathcal{O}$, then S is a near semigroup on nearness approximation space.

(2) If I is a left (right, two-sided) ideal of nearness semigroup S, then I is a nearness left (right, two-sided) ideal of nearness semigroup S.

Proof. (1) Let S be a semigroup of perceptual objects $\mathcal{O}$. Since $S \subseteq \mathcal{O}$, by Theorem 3.2.(1), we have that $\varnothing \neq S \subseteq N_r(B)^*(S)$.

This means that $x \cdot y \in N_r(B)^*(S)$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*(S)$ for all $x, y, z \in S$. Hence, S is a nearness semigroup on nearness approximation space.

(2) Let I be a left ideal of nearness semigroup S, that is, $SI \subseteq I$. We know that $S \subseteq N_r(B)^*(S)$. Then, by Theorems 3.3 and 3.2.(5), we have that

$$\begin{aligned} S\left(N_r(B)^*(I)\right) &\subseteq N_r(B)^*(S)\, N_r(B)^*(I) \\ &\subseteq N_r(B)^*(SI) \subseteq N_r(B)^*(I). \end{aligned}$$

This means that $N_r(B)^*(I)$ is a left ideal of nearness semigroup S, and so I is a nearness left ideal of nearness semigroup S. Also, we can easily show that I is a near right ideal of nearness semigroup S. Hence, I is a nearness left, right or two-sided ideal of nearness semigroup S. □

The Theorem 4.5 shows that the notion of a nearness semigroup (left ideal, right ideal, two-sided ideal) is an extended notion of an ordinary semigroup (left ideal, right ideal, two-sided ideal).

Theorem 4.6. *Let $\sim_{B_r}$ be a complete indiscernibility relation on $\mathcal{O}$, $S \subseteq \mathcal{O}$ be a semigroup, $A \subseteq S$ and $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space.*

(1) Let A be a subsemigroup of semigroup S. Then $N_r(B)_*(A)$ is, if it is nonempty, a subsemigroup of S.

(2) Let I be a left (right, two-sided) ideal of S. Then $N_r(B)_*(I)$ is, if it is nonempty, a left (right, two-sided) ideal of $N_r(B)_*(S)$.

Proof. (1) Let A be a subsemigroup of semigroup S, then by Theorems 3.4 and 3.2.(4), we have that

$$\begin{aligned} N_r(B)_*(A)\, N_r(B)_*(A) &\subseteq N_r(B)_*(AA) \\ &\subseteq N_r(B)_*(A). \end{aligned}$$

This means that $N_r(B)_*(A)$ is, if it is nonempty, a subsemigroup of $S \subseteq \mathcal{O}$.

(2) Let I be a left ideal of S, i.e., $SI \subseteq I$. Then, by Theorems 3.4 and 3.2.(4), we have that

$$\begin{aligned} N_r(B)_*(S)\, N_r(B)_*(I) &\subseteq N_r(B)_*(SI) \\ &\subseteq N_r(B)_*(I). \end{aligned}$$

This means that $N_r(B)_*(I)$ is, if it is nonempty, a left ideal of $N_r(B)_*(S)$. The other cases can be seen in a similar way. □

Definition 4.7. Let I be a subset of $S \subseteq \mathcal{O}$. If $N_r(B)^*(I)$ is a bi-ideal of S, then I is called a nearness bi-ideal of S.

Theorem 4.8. *Let $\sim_{B_r}$ be an indiscernibility relation on $\mathcal{O}$ and $S \subseteq \mathcal{O}$. If I is a bi-ideal of S, then it is a nearness bi-ideal of S.*

Proof. Let I be a bi-ideal of S. Then, by Theorems 3.3 and 3.2.(5), we have that

$$\begin{aligned}\left(N_r(B)^*(I)\right)(S)\left(N_r(B)^*(I)\right) &\subseteq N_r(B)^*(I)\,N_r(B)^*(S)\,N_r(B)^*(I)\\ &\subseteq N_r(B)^*(ISI)\\ &\subseteq N_r(B)^*(I).\end{aligned}$$

From Theorem 4.6.(1), we obtain that $N_r(B)^*(I)$ is a bi-ideal of S, that is, I is a nearness bi-ideal of S. □

Theorem 4.9. *Let $\sim_{B_r}$ be a complete indiscernibility relation on $\mathcal{O}$ and $S \subseteq \mathcal{O}$. If I is a bi-ideal of S, then $N_r(B)_*(I)$ is, if it is nonempty, a bi-ideal of $N_r(B)_*(S)$.*

Proof. Let I be a bi-ideal of S. Then, by Theorems 3.4 and 3.2.(6), we have

$$\begin{aligned}N_r(B)_*(I)\,N_r(B)_*(S)\,N_r(B)_*(I) &\subseteq N_r(B)_*(ISI)\\ &\subseteq N_r(B)_*(I).\end{aligned}$$

From Theorem 4.6.(1), we obtain that $N_r(B)_*(I)$ is, if it is nonempty, a bi-ideal of $N_r(B)_*(S)$. □

Theorem 4.10. *Let $\sim_{B_r}$ be an indiscernibility relation on $\mathcal{O}$ and $S \subseteq \mathcal{O}$. If I is a right ideal of S and J is a left ideal of S, then*

$$N_r(B)^*(IJ) \subseteq N_r(B)^*(I) \cap N_r(B)^*(J).$$

Proof. Let I be a right ideal of S and J be a left ideal of S, then $IJ \subseteq IS \subseteq I$ and $IJ \subseteq SJ \subseteq J$. Thus $IJ \subseteq I \cap J$. Hence, it follows from Theorem 3.2.(5) and (7) that

$$N_r(B)^*(IJ) \quad \subseteq N_r(B)^*(I \cap J) \subseteq N_r(B)^*(I) \cap N_r(B)^*(J).$$

□

Theorem 4.11. *Let $\sim_{B_r}$ be an indiscernibility relation on $\mathcal{O}$ and $S \subseteq \mathcal{O}$. If I is a right ideal of S and J is a left ideal of S, then*

$$N_r(B)_*(IJ) \subseteq N_r(B)_*(I) \cap N_r(B)_*(J).$$

Proof. Let I be a right ideal of S and J be a left ideal of S, then $IJ \subseteq IS \subseteq I$ and $IJ \subseteq SJ \subseteq J$. Thus $IJ \subseteq I \cap J$. Hence, it follows from Theorem 3.2.(3) and (4) that

$$N_r(B)_*(IJ) \quad \subseteq N_r(B)_*(I \cap J) \subseteq N_r(B)_*(I) \cap N_r(B)_*(J).$$

□

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Ebubekir İnan (einan@adiyaman.edu.tr)
Department of Mathematics, Faculty of Art and Science, Adıyaman University, 02040, Adıyaman, Turkey
Computational Intelligence Laboratory, Department of Electrical & Computer Engineering, University of Manitoba, Winnipeg, Manitoba, Canada

Mehmet Ali Öztürk (maozturk@adiyaman.edu.tr)
Department of Mathematics, Faculty of Art and Science, Adıyaman University, 02040, Adıyaman, Turkey
Computational Intelligence Laboratory, Department of Electrical & Computer Engineering, University of Manitoba, Winnipeg, Manitoba, Canada