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On soft multigroups

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Abstract. In the present paper, a notion of soft multigroups (in short mgroups), identity soft mgroups, absolute soft mgroups, soft abelian mgroups, soft factor mgroups are introduced and some of theirs important properties are studied. Some results regarding soft multi homomorphic image and pre image of a soft mgroup is discussed. The fundamental homomorphism theorem is established in soft mgroup setting.

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1. Introduction

Multisets (msets in short) as Knuth notes [25], first suggested by N. G. de Bruijn [16] in a private communication to him, is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. Owing to its aptness, it has replaced a variety of terms viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fireset used in different contexts but conveying synonymity with mset. This set theory has various applications in mathematics and computer science, overview of which can be obtained in [41]. Many authors like Yager [43], Miyamoto [32], Hickman [22], Blizard [5], Girish and John [18, 19] etc. have studied the properties of multisets. In 1999, Molodtsov [33] proposed a new approach, viz. soft set theory, for modeling vagueness and uncertainties, and in 2002-03, Maji et al. [28, 29] worked on some mathematical aspects of soft sets. Some authors have also generalized the notion of multisets in the settings of fuzzy sets [44] and soft sets [33] to form fuzzy multisets [27], soft multisets [30, 2], multisoft sets [21], connectedness and compactness on soft multi topological spaces in soft multi spaces have been studied in [42, 37]. Multisets, fuzzy multisets and soft multisets have been applied in multiple type of scenario's such as in information retrieval on the web, multicriteria decision making, knowledge representation in data based systems, biological systems and membrane computing [24, 26, 31, 34, 38, 43]. More works on multisets and soft multisets can be found in [3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 20]. Again the theory of groups is one of the most important algebraic structures in modern mathematics. Several authors have introduced the notion of group in fuzzy sets, intuitionistic L-fuzzy sets, soft sets, fuzzy soft sets, multi sets settings and to form fuzzy groups [39], intuitionistic L-fuzzy groups [17], soft groups [1], fuzzy soft groups [35], multi groups [36], fuzzy multigroups [40] etc. As a continuation of this it is natural to investigate the properties of group structures in soft multiset settings. Here we have introduced a notion of group, its subsystem and morphisms in soft multiset setting and studied their properties. The organization of the rest of this paper is as follows: In section 2, some preliminary definitions and results regarding soft sets, multisets, soft multisets and multi groups have been discussed. In section 3, the notion of soft multigroup has been introduced. Several properties regarding soft multigroups are studied and notions like soft abelian multigroup, soft factor multigroup etc are also defined. The fundamental homomorphism theorem is established in soft multi group setting.

2. Preliminaries

In this section, some basic definitions and results of soft sets, multisets (msets in short), multi groups (mgroups) and soft multi sets are given.

Soft sets:

Following Molodtsov [33], Maji et al. [28] and Kharal et. al. [23] some definitions and preliminary results of soft sets are presented in this section. Unless otherwise stated, X will be assumed to be an initial universal set and E will be taken to be a set of parameters. Let $P(X)$ denote the power set of X and $S(X, E)$ denote the set of all soft sets over X under the parameter set E.

Definition 2.1 ([33, 28]). A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

Definition 2.2 ([28]). Let (F, A) and (G, B) be two soft sets over X. Then their union is a soft set (H, C) over X where $C = A \cup B$ and for all $\alpha \in C$,

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A/B \\ G(\alpha) & \text{if } \alpha \in B/A \\ F(\alpha) \cup G(\alpha) & \text{if } \alpha \in A \cap B \end{cases}$$

This relationship is written as $(F, A) \mathbin{\widetilde{\cup}} (G, B) = (H, C)$.

Definition 2.3 ([28]). Let (F, A) and (G, B) be two soft sets over X. Then their intersection is a soft set (H, C) over X where $C = A \cap B$ and for all $\alpha \in C$, $H(\alpha) = F(\alpha) \cap G(\alpha)$. This relationship is written as $(F, A) \mathbin{\widetilde{\cap}} (G, B) = (H, C)$.

Definition 2.4 ([23]). Let $S(X, A)$ and $S(Y, B)$ be the families of all soft sets over X and Y respectively. The mapping $f_\varphi : S(X, A) \to S(Y, B)$ is called a soft mapping from X to Y, where $f : X \to Y$ and $\varphi : A \to B$ are two mappings. Also

(i) the image of a soft set $(F, A) \in S(X, A)$ under the mapping f_φ is denoted by $f_\varphi[(F, A)] = (f_\varphi(F), B)$, and is defined by

$$[f_\varphi(F)](\beta) = \begin{cases} \bigcup_{\alpha \in \varphi^{-1}(\beta)} \left[f[F(\alpha)] \right] & \text{if } \varphi^{-1}(\beta) \neq \phi \\ \phi & \text{otherwise} \end{cases}$$

for all $\beta \in B$.

(ii) the inverse image of a soft set $(G, B) \in S(Y, B)$ under the mapping f_φ is denoted by $f_\varphi^{-1}[(G, B)] = (f_\varphi^{-1}(G), A)$, and is defined by $[f_\varphi^{-1}(G)](\alpha) = f^{-1}[G[\varphi(\alpha)]]$, for all $\alpha \in A$.

(iii) If f_φ is bijective, then the inverse soft mapping, denoted by $(f_\varphi)^{-1}$, is defined by $(f_\varphi)^{-1} = f_{\varphi^{-1}}^{-1}$.

Note: If $(f_\varphi)^{-1}$ exists, then $f_\varphi^{-1}[(G, B)] = f_{\varphi^{-1}}^{-1}[(G, B)]$.

(iv) The soft mapping f_φ is called injective (surjective) if f and φ are both injective (surjective).

(v) The soft mapping f_φ is said to be constant, if f is constant.

(vi) The soft mapping f_φ is identity soft mapping, if f and φ are both classical identity mappings.

Multi sets:

Definition 2.5 ([19])**.** An mset M drawn from the set X is represented by a Count function C_M defined as $C_M : X \to \mathbb{N}$, where $\mathbb{N}$ represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrence of the element x in the mset M. The presentation of the mset M drawn from $X = \{x_1, x_2, \ldots, x_n\}$ will be as $M = \{x_1/m_1, x_2/m_2, \ldots, x_n/m_n\}$ where m_i is the number of occurrences of the element $x_i,\ i = 1, 2, \ldots, n$ in the mset M.

Also here for any positive integer w, $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times and $[X]^\infty$ is the set of all msets whose elements are in X such that there is no limit on the number of occurrences of an element in a mset. As in [19], $[X]^w$ and $[X]^\infty$ will be referred to as mset spaces. $MS(X)$ denote the set of all msets drawn from X.

Definition 2.6 ([19])**.** Let M_1 and M_2 be two msets drawn from a set X. Then M_1 is said to be submset of M_2 if $C_{M_1}(x) \le C_{M_2}(x),\ \forall\ x \in X$. This relation is denoted by $M_1 \subseteq M_2$. M_1 is said to be *equal* to M_2 if $C_{M_1}(x) = C_{M_2}(x),\ \forall\ x \in X$. It is denoted by $M_1 = M_2$.

Definition 2.7 ([19])**.** Let $\{M_i; i \in I\}$ be a nonempty family of msets drawn from the set X. Then

(a) Their intersection, denoted by $\bigcap_{i \in I} M_i$ where $C_{\bigcap_{i \in I} M_i}(x) = \bigwedge_{i \in I} C_{M_i}(x),\ \forall\ x \in X$.

(b) Their union, denoted by $\bigcup_{i \in I} M_i$ where $C_{\bigcup_{i \in I} M_i}(x) = \bigvee_{i \in I} C_{M_i}(x),\ \forall\ x \in X$.

(c) The complement of any mset M_i in $[X]^w$, denoted by M_i^c where $C_{M_i^c}(x) = w - C_{M_i}(x),\ \forall\ x \in X$.

Definition 2.8 ([36])**.** Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then

(i) the image of a mset $M \in [X]^w$ under the mapping f is denoted by $f(M)$, where

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

(ii) the inverse image of a mset $N \in [Y]^w$ under the mapping f is denoted by $f^{-1}(N)$, where $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

Proposition 2.9 ([36]). *Let X, Y and Z be three nonempty sets and $f : X \to Y$, $g : Y \to Z$ be two mappings. If $M_i \in [X]^w$, $N_i \in [Y]^w$, $i \in I$ then*

(i) $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$.
(ii) $f[\bigcup_{i\in I} M_i] = \bigcup_{i\in I} f[M_i]$.
(iii) $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$.
(iv) $f^{-1}[\bigcup_{i\in I} M_i] = \bigcup_{i\in I} f^{-1}[M_i]$.
(v) $f^{-1}[\bigcap_{i\in I} M_i] = \bigcap_{i\in I} f^{-1}[M_i]$.
(vi) $f(M_i) \subseteq N_j \Rightarrow M_i \subseteq f^{-1}[N_j]$.
(vii) $g[f(M_i)] = [gf](M_i)$ *and* $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.

Proposition 2.10 ([36]). *Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. If $M \in [X]^w$ and $N \in [Y]^w$, then*

(i) $M \widetilde{\subseteq} f^{-1}[f(M)]$;
(ii) $f^{-1}[f(M)] = M$, *if f is injective;*
(iii) $f[f^{-1}(N)] \widetilde{\subseteq} N$;
(iv) $f[f^{-1}(N)] = N$, *if f is surjective.*

Definition 2.11 ([36]). An mset containing only one element x of n times is called a singleton mset and it is denoted by n_x.

Multigroups

Throughout this section, let X be a group and e be the identity element of X. Also throughout the rest of the paper we assume that msets are taken from $[X]^w$.

Definition 2.12 ([36]). Let $A,\ B \in [X]^w$. Then we define $A \circ B$ and A^{-1} as follows: $C_{A\circ B}(x) = \bigvee\{C_A(y) \wedge C_B(z);\ y,\ z \in X \text{ and } yz = x\}$ and $C_{A^{-1}}(x) = C_A(x^{-1})$.

Definition 2.13. [36] Let X be a group. A multiset G over X is said to be a multigroup over X if the Count function G or C_G satisfies the following two conditions.

(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)],\ \forall\ x, y \in X$;
(ii) $C_G(x^{-1}) \geq C_G(x),\ \forall\ x \in X$.

The set of all multigroups over X is denoted by $MG(X)$.

Proposition 2.14 ([36]). *Let $G \in MG(X)$. Then*

(i) $C_G(e) \geq C_G(x),\ \forall\ x \in X$;
(ii) $C_G(x^n) \geq C_G(x),\ \forall\ x \in X$;
(iii) $C_G(x^{-1}) = C_G(x),\ \forall\ x \in X$;
(iv) $G = G^{-1}$.

Proposition 2.15 ([36]). *Let X, Y be two groups and $f : X \to Y$ be a homomorphism. If $G \in MG(X)$, then $f(G) \in MG(Y)$ and if $H \in MG(Y)$, then $f^{-1}(H) \in MG(X)$.*

Definition 2.16 ([36])**.** Let $G \in MG(X)$. Then define $G_* = \{x \in X;\ C_G(x) = C_G(e)\}$ and $G^* = \{x \in X;\ C_G(x) > 0\}$.

Proposition 2.17. [36] *Let $G \in MG(X)$. Then G_* and G^* are subgroups of X.*

Definition 2.18. [36] A mgroup G over X is called *abelian* over X if $C_G(xy) = C_G(yx),\ \forall\ x, y \in X$. Let $AMG(X)$ denote the set of all abelian mgroups over X.

Proposition 2.19 ([36])**.** *Let $G \in AMG(X)$. Then $G^*,\ G_*$ are normal subgroups of X.*

Proposition 2.20 ([36])**.** *If $\{G_i,\ i \in I\}$ be a family of multigroups over a group X, then their intersection $\cap_{i\in I} G_i$ is a multigroup over X.*

Proposition 2.21 ([36])**.** *Let $X,\ Y$ be two groups and $f : X \to Y$ be a homomorphism. If $G \in MG(X)$ and $H \in MG(Y)$, then $f(G) \in MG(Y)$ and $f^{-1}(H) \in MG(X)$.*

Proposition 2.22 ([36])**.** *Let $G \in MG(X)$ and P be a normal subgroup of X. Define $\hat{G} \in MS(X/P)$ such that $C_{\hat{G}}(xP) = \bigvee\{C_G(z);\ z \in xP\}, \forall\ x \in X$. Then $\hat{G} \in MG(X/P)$.*

Definition 2.23. The mgroup $\hat{G}$, as in Proposition 2.18, is called the factor mgroup of the mgroup G over X relative to the normal subgroup P of X and is denoted by G/P.

Proposition 2.24. *Let $X,\ Y$ be two groups, $f : X \to Y$ be a homomorphism and $G \in AMG(X),\ H \in AMG(Y)$. Then*

(i) *$f(G) \in AMG(Y)$, if f is onto;*
(ii) *$f^{-1}(H) \in AMG(X)$, if f is one-one.*

Soft multi sets

Following Tokat and Osmanoglu [42] some definitions and preliminary results of soft msets are presented in this section in our form. Unless otherwise stated, X is an universal set, M an mset over X and $M^* = \{x \in X :\ C_M(x) > 0\}$, is called the support set or root set of M. Again the set of all submsets of M, denoted by $P^*(M)$, is called the power mset of M.

Definition 2.25. Let X be an universal set, M be an mset over $X,\ E$ be a set of parameters and $A \subseteq E$. Then a pair (F, A) is called a soft mset over M where F is a mapping given by $F : A \to P^*(M)$. Let $SM(M_X, E)$ denote the set of all soft msets over M under the parameter set E.

Definition 2.26. Let $(F, A),\ (G, B)$ be two soft msets over M. Then (F, A) is said to be soft submset of (G, B) if $A \subseteq B$ and $F(\alpha) \subseteq G(\alpha),\ \forall\ \alpha \in A$.

Definition 2.27. Let $(F, A),\ (G, B)$ be two soft msets over M. Then

(i) their intersection, denoted by $(F, A)\ \widetilde{\cap}\ (G, B) = (F\widetilde{\cap}G, A \cap B)$, is defined by $(F\widetilde{\cap}G)(\alpha) = [F(\alpha) \cap G(\alpha)],\ \forall\ \alpha \in (A \cap B)$;

(ii) their union, denoted by $(F, A) \,\widetilde{\cup}\, (G, B) = (F\widetilde{\cup}G, A \cup B)$, is defined by $\forall\ \alpha \in (A \cup B)$,

$$(F\widetilde{\cup}G)(\alpha) = \begin{cases} F(\alpha) & \text{if } \ \alpha \in (A - B) \\ G(\alpha) & \text{if } \ \alpha \in (B - A) \\ F(\alpha) \cup G(\alpha) & \text{if } \ \alpha \in (A \cap B) \end{cases}$$

(iii) their AND, denoted by $(F, A) \,\widetilde{\wedge}\, (G, B) = (F\widetilde{\wedge}G, A \times B)$, is defined by $(F\widetilde{\wedge}G)(\alpha, \beta) = [F(\alpha) \cap G(\beta)],\ \forall\ (\alpha, \beta) \in (A \times B)$;

(iv) the complement of (F, A), denoted by $(F, A)^c = (F^c, A)$, is defined by $F^c(\alpha) = M - F(\alpha),\ \forall\ \alpha \in A$, where $C_{M-F(\alpha)}(x) = C_M(x) - C_{F(\alpha)}(x),\ \forall\ x \in M^*$.

Definition 2.28. Let $M,\ N$ be two msets over $X,\ Y$ respectively and $E,\ K$ be two set of parameters. Let $SM(M_X, E)$ and $SM(N_Y, K)$ be the families of all soft msets over M and N respectively. The mapping $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ is called a soft multi mapping from M to N, where $f : M^*(\subseteq X) \to N^*(\subseteq Y)$ and $\varphi : E \to K$ are two mappings. Also for $A \subseteq E$ and $B \subseteq K$,

(i) the *image* of a soft mset $(F, A) \in SM(M_X, E)$ under the mapping f_φ is denoted by $f_\varphi[(F, A)] = (f_\varphi(F), K)$, and is defined by $\forall\ \beta \in K,\ y \in N^*$ and $A' = [\varphi^{-1}(\beta) \cap A]$,

$$C_{[f_\varphi(F)](\beta)}(y) = \begin{cases} \vee_{\alpha\in A'}\ \ \vee_{x\in f^{-1}(y)}\, C_{F(\alpha)}(x) & \text{if } A' \neq \phi,\ f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

(ii) the *inverse image* of a soft mset $(G, B) \in SM(N_Y, K)$ under the mapping f_φ is denoted by $f_\varphi^{-1}[(G, B)] = (f_\varphi^{-1}(G), E)$, and is defined by $\forall\ \alpha \in E,\ x \in M^*$,

$$C_{[f_\varphi^{-1}(G)](\alpha)}(x) = \begin{cases} C_{G[\varphi(\alpha)]}f(x) & \text{if } \varphi(\alpha) \in B \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.29. *Let $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft multi mapping, $(F, A),\ (F_i, A_i)$ soft msets in $SM(M_X, E)$ and $(G, B),\ (G_i, B_i)$ soft msets in $SM(N_Y, K)$. Then the following properties are satisfied.*

(i) $f_\varphi(\widetilde{\phi}) = \widetilde{\phi},\ f_\varphi(\widetilde{X}) \,\widetilde{\subseteq}\, \widetilde{Y}$;
(ii) $(f_\varphi)^{-1}(\widetilde{\phi}) = \widetilde{\phi},\ (f_\varphi)^{-1}(\widetilde{Y}) = \widetilde{X}$;
(iii) $f_\varphi[(F_1, A_1) \,\widetilde{\cup}\, (F_2, A_2)] = f_\varphi[(F_1, A_1)] \,\widetilde{\cup}\, f_\varphi[(F_2, A_2)]$;
(iv) $(f_\varphi)^{-1}[(F_1, A_1) \,\widetilde{\cup}\, (F_2, A_2)] = (f_\varphi)^{-1}[(F_1, A_1)] \,\widetilde{\cup}\, (f_\varphi)^{-1}[(F_2, A_2)]$;
(v) $f_\varphi[(F_1, A_1) \,\widetilde{\cap}\, (F_2, A_2)] \,\widetilde{\subseteq}\, f_\varphi[(F_1, A_1)] \,\widetilde{\cap}\, f_\varphi[(F_2, A_2)]$;
(vi) $(f_\varphi)^{-1}[(F_1, A_1) \,\widetilde{\cap}\, (F_2, A_2)] = (f_\varphi)^{-1}[(F_1, A_1)] \,\widetilde{\cap}\, (f_\varphi)^{-1}[(F_2, A_2)]$;
(vii) *If $(F_1, A_1) \,\widetilde{\subseteq}\, (F_2, A_2)$, then $f_\varphi[(F_1, A_1)] \,\widetilde{\subseteq}\, f_\varphi[(F_2, A_2)]$;*
(viii) *If $(G_1, B_1) \,\widetilde{\subseteq}\, (G_2, B_2)$, then $(f_\varphi)^{-1}[(G_1, B_1)] \,\widetilde{\subseteq}\, (f_\varphi)^{-1}[(G_2, B_2)]$.*

3. Soft multigroups

Throughout this section, unless otherwise stated, X will be assumed to be a group, M be a mgroup over X and $E,\ K$ be the set of parameters, $A_i \subseteq E,\ B_i \subseteq K,\ i \in \Delta$.

Definition 3.1. Let X be a group, M be a mgroup over X and $A \subseteq E$ be a set of parameters. A soft mset (F, A) drawn from M is said to be a soft multigroup (shortly soft mgroup) over M iff $F(\alpha)$ is a submgroup of $M, \forall\ \alpha \in A$.

Example 3.2. Let $X = \{e,\ x,\ y,\ z\}$ be Klein's 4-group, $M = \{e,\ e,\ e, x,\ x,\ y,\ y,\ y,\ z,\ z\}$ be a mgroup over X and $A = \{\alpha_1, \alpha_2\}$. Let (F, A) be a mset drawn from M, defined by $F(\alpha_1) = \{e,\ e,\ e,\ x,\ y,\ y,\ z\}$ and $F(\alpha_2) = \{e,\ e,\ x,\ x,\ y,\ y,\ z,\ z\}$. Now

$$\begin{aligned}
C_{F(\alpha_1)}(ex) = C_{F(\alpha_1)}(x) = 1 \geq [C_{F(\alpha_1)}(e) \wedge C_{F(\alpha_1)}(x)],\\
C_{F(\alpha_1)}(ey) = C_{F(\alpha_1)}(y) = 2 \geq [C_{F(\alpha_1)}(e) \wedge C_{F(\alpha_1)}(y)],\\
C_{F(\alpha_1)}(ez) = C_{F(\alpha_1)}(z) = 1 \geq [C_{F(\alpha_1)}(e) \wedge C_{F(\alpha_1)}(z)],\\
C_{F(\alpha_1)}(xy) = C_{F(\alpha_1)}(z) = 1 \geq [C_{F(\alpha_1)}(x) \wedge C_{F(\alpha_1)}(y)],\\
C_{F(\alpha_1)}(yz) = C_{F(\alpha_1)}(x) = 1 \geq [C_{F(\alpha_1)}(y) \wedge C_{F(\alpha_1)}(z)],\\
C_{F(\alpha_1)}(zx) = C_{F(\alpha_1)}(y) = 2 \geq [C_{F(\alpha_1)}(z) \wedge C_{F(\alpha_1)}(x)],\\
C_{F(\alpha_1)}(x^2) = C_{F(\alpha_1)}(e) = 3 \geq [C_{F(\alpha_1)}(x) \wedge C_{F(\alpha_1)}(x)],\\
C_{F(\alpha_1)}(y^2) = C_{F(\alpha_1)}(e) = 3 \geq [C_{F(\alpha_1)}(y) \wedge C_{F(\alpha_1)}(y)],\\
C_{F(\alpha_1)}(z^2) = C_{F(\alpha_1)}(e) = 3 \geq [C_{F(\alpha_1)}(z) \wedge C_{F(\alpha_1)}(z)],\\
C_{F(\alpha_1)}(e^2) = C_{F(\alpha_1)}(e) = 3 \geq [C_{F(\alpha_1)}(e) \wedge C_{F(\alpha_1)}(e)]
\end{aligned}$$

and

$$\begin{aligned}
C_{F(\alpha_1)}(x^{-1}) = C_{F(\alpha_1)}(x) = 1,\ C_{F(\alpha_1)}(y^{-1}) = C_{F(\alpha_1)}(y) = 2,\\
C_{F(\alpha_1)}(z^{-1}) = C_{F(\alpha_1)}(z) = 1,\ C_{F(\alpha_1)}(e^{-1}) = C_{F(\alpha_1)}(e) = 3.
\end{aligned}$$

Therefore $F(\alpha_1)$ is a submgroup of M.
Similarly we can show that $F(\alpha_2)$ is also a submgroup of M.
Therefore (F, A) is a soft mgroup over M.

Theorem 3.3. *Let M be a mgroup over a group X and $(F_1, A),\ (F_2, B)$ be two soft mgroups over M. Then their intersection $(F_1, A)\ \widetilde{\cap}\ (F_2, B)$ is a soft mgroup over M.*

Proof. Let $\alpha \in (A \cap B)$.
Since (F_1, A) be a soft mgroup, it follows that $F_1(\alpha)$ is a msubgroup of M and hence $C_{F_1(\alpha)}(xy) \geq [C_{F_1(\alpha)}(x) \wedge C_{F_1(\alpha)}(y)]$ and $C_{F_1(\alpha)}(x^{-1}) = C_{F_1(\alpha)}(x), \forall x,\ y \in X$.
Similarly $F_2(\alpha)$ is a msubgroup of M and hence $C_{F_2(\alpha)}(xy) \geq C_{F_2(\alpha)}(x) \wedge C_{F_2(\alpha)}(y)$ and $C_{F_2(\alpha)}(x^{-1}) = C_{F_2(\alpha)}(x), \forall x,\ y \in X$.

$$\begin{aligned}
\text{Now}\ \ C_{(F_1 \widetilde{\cap} F_2)(\alpha)}(xy) &= C_{F_1(\alpha)}(xy) \wedge C_{F_2(\alpha)}(xy)\\
&\geq C_{F_1(\alpha)}(x) \wedge C_{F_1(\alpha)}(y) \wedge C_{F_2(\alpha)}(x) \wedge C_{F_2(\alpha)}(y)\\
&= C_{(F_1 \widetilde{\cap} F_2)(\alpha)}(x) \wedge C_{(F_1 \widetilde{\cap} F_2)(\alpha)}(y),\ \forall\ x,\ y \in X.
\end{aligned}$$

$$\begin{aligned}
\text{and}\ \ C_{(F_1 \widetilde{\cap} F_2)(\alpha)}(x^{-1}) &= C_{F_1(\alpha)}(x^{-1}) \wedge C_{F_2(\alpha)}(x^{-1}) = C_{F_1(\alpha)}(x) \wedge C_{F_2(\alpha)}(x)\\
&= C_{(F_1 \widetilde{\cap} F_2)(\alpha)}(x),\ \forall\ x \in X.
\end{aligned}$$

Therefore $(F_1\ \widetilde{\cap}\ F_2)(\alpha)$ is a msubgroup of $M,\ \forall\ \alpha \in (A \cap B)$.
Therefore $(F_1, A)\widetilde{\cap}(F_2, B)$ is a soft mgroup over X. □

Remark 3.4. Union of two soft mgroups may not be a soft mgroup. This is shown by the following example.

Example 3.5. Let $X = K_4 = \{e,\ x,\ y,\ z\}$ be the Klein's 4-group and $M = \{e,\ e,\ x,\ x,\ y,\ y,\ z,\ z\}$ be a mgroup over X and $E = \{\alpha_1, \alpha_2, \alpha_3\}$, $A_1 = \{\alpha_1, \alpha_2\},\ A_2 = \{\alpha_1, \alpha_3\}$.
Let $(F_1, A_1) = \big\{\{e,\ e,\ x\},\ M\big\}$ and $(F_2, A_2) = \big\{\{e,\ e,\ y\},\ M\big\}$ be two soft msets.
Then $(F_1, A_1),\ (F_2, A_2)$ are soft mgroups.
Clearly $(F_1, A_1)\ \widetilde{\cup}\ (F_2, A_2) = \big\{\{e,\ e,\ x,\ y\},\ M,\ M\big\}$.
Again $C_{[F_1\widetilde{\cup}F_2](\alpha_1)}(z) = C_{[F_1\widetilde{\cup}F_2](\alpha_1)}(xy) = 0 \ngeq \bigwedge[C_{[F_1\widetilde{\cup}F_2](\alpha_1)}(x),\ C_{[F_1\widetilde{\cup}F_2](\alpha_1)}(y)] = 1$.
Thus $[F_1\widetilde{\cup}F_2](\alpha_1)$ is not a multigroup over X and hence not a submgroup of M.
Therefore $(F_1, A)\ \widetilde{\cup}\ (F_2, B)$ is not a soft mgroup over M.

Theorem 3.6. *If (F_1, A) and (F_2, B) be two soft mgroups over M, then $(F_1, A)\ \widetilde{\wedge}\ (F_2, B)$ is a soft mgroup over M.*

Proof. Let $(\alpha_1, \alpha_2) \in (A \times B)$.
Since (F_1, A) be a soft mgroup, it follows that $F_1(\alpha_1)$ is a msubgroup of M and hence $C_{F_1(\alpha_1)}(xy) \geq [C_{F_1(\alpha_1)}(x) \wedge C_{F_1(\alpha_1)}(y)]$ and $C_{F_1(\alpha_1)}(x^{-1}) = C_{F_1(\alpha_1)}(x), \forall x,\ y \in X$.
Similarly $F_2(\alpha_2)$ is a msubgroup of M and hence $C_{F_2(\alpha_2)}(xy) \geq C_{F_2(\alpha_2)}(x) \wedge C_{F_2(\alpha_2)}(y)$ and $C_{F_2(\alpha_2)}(x^{-1}) = C_{F_2(\alpha_2)}(x), \forall x,\ y \in X$.

$$\begin{aligned}
\text{Now}\ \ C_{(F_1\widetilde{\wedge}F_2)(\alpha_1,\alpha_2)}(xy) &= C_{F_1(\alpha_1)}(xy) \wedge C_{F_2(\alpha_2)}(xy) \\
&\geq C_{F_1(\alpha_1)}(x) \wedge C_{F_1(\alpha_1)}(y) \wedge C_{F_2(\alpha-2)}(x) \wedge C_{F_2(\alpha_2)}(y) \\
&= C_{(F_1\widetilde{\wedge}F_2)(\alpha_1,\alpha_2)}(x) \wedge C_{(F_1\widetilde{\wedge}F_2)(\alpha_1,\alpha_2)}(y),\ \forall\ x,\ y \in X.
\end{aligned}$$

$$\begin{aligned}
\text{and}\ \ C_{(F_1\widetilde{\wedge}F_2)(\alpha_1,\alpha_2)}(x^{-1}) &= C_{F_1(\alpha_1)}(x^{-1}) \wedge C_{F_2(\alpha_2)}(x^{-1}) = C_{F_1(\alpha_1)}(x) \wedge C_{F_2(\alpha_2)}(x) \\
&= C_{(F_1\widetilde{\wedge}F_2)(\alpha_1,\alpha_2)}(x),\ \forall\ x \in X.
\end{aligned}$$

Therefore $(F_1\ \widetilde{\wedge}\ F_2)(\alpha_1, \alpha_2)$ is a msubgroup of M, $\forall\ (\alpha_1, \alpha_2) \in (A \times B)$.
Therefore $(F_1, A)\widetilde{\wedge}(F_2, B)$ is a soft mgroup over X. □

Definition 3.7. The soft multi mapping $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ is called

(i) a soft multi homomorphism (mhomomorphism) if the mapping $f : M^* \to N^*$ is a homomorphism;
(ii) a soft into mhomomorphism if the mapping $\varphi : E \to K$ is into and the mapping $f : M^* \to N^*$ is an into homomorphism;
(iii) a soft onto mhomomorphism if the mapping $\varphi : E \to K$ is onto and the mapping $f : M^* \to N^*$ is an onto homomorphism;
(iv) a soft multi isomorphism (misomorphism) if the mapping $\varphi : E \to K$ is bijective and the mapping $f : M^* \to N^*$ is an isomorphism.

Theorem 3.8. *Let (F, A) be a soft mgroup over M and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft mhomomorphism. If $[C_{F(\alpha_1)}(x_1) \wedge C_{F(\alpha_2)}(x_2)] \leq [C_{F(\alpha_1)}(x_2) \vee C_{F(\alpha_2)}(x_1)]$, $\forall\ \alpha_1,\ \alpha_2 \in A$ and $\forall\ x_1, x_2 \in X$, then $f_\varphi[(F, A)] = (f_\varphi(F), K)$ is soft mgroup over N.*

Proof. Since (F, A) be a soft mgroup over M, it follows that $F(\alpha)$ is a submgroup of M, $\forall\ \alpha \in A$.
Let $\beta \in K$ and $y_1,\ y_2 \in N^*$.
Case-I: Let $[\varphi^{-1}(\beta) \cap A] \neq \phi$ and $f^{-1}(y_1 y_2) \neq \phi$. Then

$$\begin{aligned}
C_{[f_\varphi(F)](\beta)}(y_1y_2) &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(y_1y_2)} C_{F(\alpha)}(x) \\
&\geq \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} C_{F(\alpha)}(x_1x_2) \\
&\geq \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} [C_{F(\alpha)}(x_1) \wedge C_{F(\alpha)}(x_2)] \\
&= [\vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x_1\in f^{-1}(y_1)} C_{F(\alpha)}(x_1)] \wedge \\
&\qquad\qquad [\vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x_2\in f^{-1}(y_2)} C_{F(\alpha)}(x_2)], \\
&\quad \big(\text{since } [C_{F(\alpha_1)}(x_1)\ \wedge\ C_{F(\alpha_2)}(x_2)] \leq [C_{F(\alpha_1)}(x_2)\ \vee\ C_{F(\alpha_2)}(x_1)]\big) \\
&= C_{[f_\varphi(F)](\beta)}(y_1)\ \wedge\ C_{[f_\varphi(F)](\beta)}(y_2).
\end{aligned}$$

Case-II: Let $f^{-1}(y_1y_2) = \phi$. Then at least one of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ is ϕ.
Let $f^{-1}(y_1) = \phi$. Then $C_{[f_\varphi(F)](\beta)}(y_1) = 0$.
Therefore $C_{[f_\varphi(F)](\beta)}(y_1y_2) = 0 = C_{[f_\varphi(F)](\beta)}(y_1)\ \wedge\ C_{[f_\varphi(F)](\beta)}(y_2)$.
Case-III: Let $[\varphi^{-1}(\beta) \cap A] = \phi$.
Then clearly $C_{[f_\varphi(F)](\beta)}(y_1y_2) = 0 = C_{[f_\varphi(F)](\beta)}(y_1)\ \wedge\ C_{[f_\varphi(F)](\beta)}(y_2)$.
Therefore, $C_{[f_\varphi(F)](\beta)}(y_1y_2) \geq [C_{[f_\varphi(F)](\beta)}(y_1)]\ \wedge\ [C_{[f_\varphi(F)](\beta)}(y_2)]$.
Again let $y \in N^*$ and $\beta \in K$ such that $[\varphi^{-1}(\beta) \cap A] \neq \phi$. Then,

$$\begin{aligned}
C_{[f_\varphi(F)](\beta)}(y^{-1}) &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(y^{-1})} C_{F(\alpha)}(x) \\
&= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x^{-1}\in f^{-1}(y)} C_{F(\alpha)}(x^{-1}) \\
&= C_{[f_\varphi(F)](\beta)}(y).
\end{aligned}$$

Also if $[\varphi^{-1}(\beta) \cap A] = \phi$, then $C_{[f_\varphi(F)](\beta)}(y^{-1}) = 0 = C_{[f_\varphi(F)](\beta)}(y)$.
Therefore, $f_\varphi[(F, A)](\beta)$ is a soft submgroup of N, $\forall\ \beta \in K$.
Therefore, $f_\varphi[(F, A)] = (f_\varphi(F), K)$ is soft mgroup over N. □

Theorem 3.9. *Let (G, B) be a soft mgroup over N and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft mhomomorphism. Then $f_\varphi^{-1}[(G, B)] = (f_\varphi^{-1}(G), E)$ is soft mgroup over M.*

Proof. Since (G, B) be a soft mgroup over N, it follows that $G(\beta)$ is a submgroup of N, $\forall\ \beta \in B$.
Let $\alpha \in E$ and $x_1,\ x_2,\ x \in M^*$.
If $\varphi(\alpha) \notin B$, then $C_{[f_\varphi^{-1}(G)](\alpha)}(x_1x_2) = 0 = [C_{[f_\varphi^{-1}(G)](\alpha)}(x_1)] \wedge [C_{[f_\varphi^{-1}(G)](\alpha)}(x_2)]$ and $C_{[f_\varphi^{-1}(G)](\alpha)}(x^{-1}) = 0 = C_{[f_\varphi^{-1}(G)](\alpha)}(x)$.

Again if $\varphi(\alpha) \in B$, then

$$\begin{aligned} C_{[f_\varphi^{-1}(G)](\alpha)}(x_1x_2) &= C_{G(\varphi(\alpha))}[f(x_1x_2)] \\ &= C_{G(\varphi(\alpha))}[f(x_1)f(x_2)] \\ &\geq [C_{G(\varphi(\alpha))}f(x_1)] \wedge [C_{G(\varphi(\alpha))}f(x_2)] \\ &= [C_{[f_\varphi^{-1}(G)](\alpha)}(x_1)] \wedge [C_{[f_\varphi^{-1}(G)](\alpha)}(x_2)] \end{aligned}$$

$$\begin{aligned} \text{and } C_{[f_\varphi^{-1}(G)](\alpha)}(x^{-1}) &= C_{G(\varphi(\alpha))}[f(x^{-1})] = C_{G(\varphi(\alpha))}[f(x)]^{-1} \\ &= C_{G(\varphi(\alpha))}[f(x)] = C_{[f_\varphi^{-1}(G)](\alpha)}(x). \end{aligned}$$

Therefore, $f_\varphi^{-1}[(G,B)](\alpha)$ is a soft submgroup of M, $\forall\ \alpha \in E$.
Therefore, $f_\varphi^{-1}[(G,B)] = (f_\varphi^{-1}(G), E)$ is soft mgroup over M. □

Definition 3.10. Let (F, A) be a soft mgroup over M. Then

(i) (F, A) is said to be an identity soft mgroup over M if $F(\alpha) = [C_M(e)]_e$, $\forall\ \alpha \in A$, where e is the identity element of X.
(ii) (F, A) is said to be an absolute soft mgroup over M if $F(\alpha) = M$, $\forall\ \alpha \in A$.

Theorem 3.11. *Let (F, A) be a soft mgroup over M and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft onto mhomomorphism.*

(i) *If (F, A) be an identity soft mgroup over M, then $[f_\varphi(F), \varphi(A)] = [f_\varphi(F), B]$ is an identity soft mgroup over $f(M)$.*
(ii) *If (F, A) be an absolute soft mgroup over M, then $[f_\varphi(F), \varphi(A)] = [f_\varphi(F), B]$ is an absolute soft mgroup over $f(M)$.*

Proof. (i) Since (F, A) is an identity soft mgroup over M, it follows that $F(\alpha) = [C_M(e_X)]_{e_X}$, $\forall\ \alpha \in A$. Now for any $\beta \in B$,

$$\begin{aligned} C_{[f_\varphi(F)](\beta)}(e_Y) &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(e_Y)}\ C_{F(\alpha)}(x) = \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]} C_{F(\alpha)}(e_X) \\ &= C_M(e_X) = \vee\{C_M(w) :\ w \in X,\ f(w) = e_Y\} \\ &= C_{f(M)}(e_Y), \text{since } f \text{ is a homomorphism and hence } e_X \in f^{-1}(e_Y). \end{aligned}$$

Again $C_{[f_\varphi(F)](\beta)}(y) = \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(y)}\ C_{F(\alpha)}(x) = 0,\ \forall\ y(\neq e_Y) \in Y$.
Therefore, $[f_\varphi(F)](\beta) = [C_{f(M)}(e_Y)]_{e_Y}$, $\forall\ \beta \in B$.
Therefore, $[f_\varphi(F), \varphi(A)] = [f_\varphi(F), B]$ is an identity soft mgroup over $f(M)$.

(ii) Since (F, A) is an absolute soft mgroup over M, it follows that $F(\alpha) = M$, $\forall\ \alpha \in A$. Now for any $\beta \in B$ and $y \in Y$,

$$\begin{aligned} C_{[f_\varphi(F)](\beta)}(y) &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(y)}\ C_{F(\alpha)}(x) \\ &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]}\ \vee_{x\in f^{-1}(y)}\ C_M(x) \\ &= \vee_{\alpha\in[\varphi^{-1}(\beta)\cap A]} C_{f(M)}(y) = C_{f(M)}(y). \end{aligned}$$

Therefore, $[f_\varphi(F)](\beta) = f(M)$, $\forall\ \beta \in B$.
Hence $[f_\varphi(F), \varphi(A)] = [f_\varphi(F), B]$ is an absolute soft mgroup over $f(M)$. □

Theorem 3.12. *Let (G, B) be a soft mgroup over N and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft into mhomomorphism.*

(i) *If (G,B) be an identity soft mgroup over N and $\varphi^{-1}(B)\subseteq E$, then* $[f_\varphi^{-1}(G),\varphi^{-1}(B)]=[f_\varphi^{-1}(G),A]$ *is an identity soft mgroup over* $f^{-1}(N)$.

(ii) *If (G,B) be an absolute soft mgroup over N and $\varphi^{-1}(B)\subseteq E$, then* $[f_\varphi^{-1}(G),\varphi^{-1}(B)]=[f_\varphi^{-1}(G),A]$ *is an absolute soft mgroup over* $f^{-1}(N)$.

Proof. (i) Since (G,B) is an identity soft mgroup over N, it follows that $G(\beta)=[C_N(e_Y)]_{e_Y},\ \forall\ \beta\in B$. Now for any $\alpha\in A$,

$$\begin{aligned} C_{[f_\varphi^{-1}(G)](\alpha)}(e_X) &= C_{G(\varphi(\alpha))}f(e_X)=C_{G(\varphi(\alpha))}(e_Y)=C_N(e_Y)\\ &= C_{f^{-1}(N)}(e_X), \text{since } f \text{ is a homomorphism}\\ &\qquad\qquad \text{and hence } f(e_X)=e_Y. \end{aligned}$$

Again $C_{[f_\varphi^{-1}(G)](\alpha)}(x)=C_{G(\varphi(\alpha))}f(x)=0,\ \forall\ x(\neq e_X)\in X$.
Therefore, $[f_\varphi^{-1}(G)](\alpha)=[C_{f^{-1}(N)}(e_X)]_{e_X},\ \forall\ \alpha\in A$.
Therefore, $f_\varphi^{-1}[(G,B)]=[f_\varphi^{-1}(G),A]$ is an identity soft mgroup over $f^{-1}(N)$.

(ii) Since (G,B) be an absolute soft mgroup over N, it follows that $G(\beta)=N,\ \forall\ \beta\in B$. Now for any $\alpha\in A$ and $x\in X$,
$C_{[f_\varphi^{-1}(G)](\alpha)}(x)=C_{G(\varphi(\alpha))}f(x)=C_Nf(x)=C_{f^{-1}(N)}(x)$.
Therefore, $[f_\varphi^{-1}(G)](\alpha)=f^{-1}(N),\ \forall\ \alpha\in A$.
Therefore, $f_\varphi^{-1}[(G,B)]=[f_\varphi^{-1}(G),A]$ is an absolute soft mgroup over $f^{-1}(N)$. □

Definition 3.13. Let (F_1,A_1) and (F_2,A_2) be two soft mgroups over M. Then (F_1,A_1) is said to be a soft submgroup of (F_2,A_2), denoted by $(F_1,A_1)\widetilde{\leq}(F_2,A_2)$ if $A_1\subseteq A_2$ and $F_1(\alpha)$ is a submgroup of $F_2(\alpha),\ \forall\ \alpha\in A_1$.

Theorem 3.14. *Let (F_1,A_1) and (F_2,A_2) be two soft mgroups over M and (F_1,A_1) be a soft submgroup of (F_2,A_2). If $f_\varphi : SM(M_X,E)\to SM(N_Y,K)$ is a soft mhomomorphism, then $\big(f_\varphi(F_1),\varphi(A_1)\big)$ and $\big(f_\varphi(F_2),\varphi(A_2)\big)$ are both soft msubgroups over N and $\big(f_\varphi(F_1),\varphi(A_1)\big)$ is a soft submgroup of $\big(f_\varphi(F_2),\varphi(A_2)\big)$.*

Proof. From Theorem 3.8, we have, $\big(f(F_1),\varphi(A_1)\big)$ and $\big(f(F_2),\varphi(A_2)\big)$ are both soft mgroups over N.
Also clearly $\varphi(A_1)\subseteq\varphi(A_2)$. Now for any $\beta\in\varphi(A_1)$ and $y\in N^*$, we have,

$$\begin{aligned} C_{[f(F_1)](\beta)}(y) &= \bigvee_{\alpha\in[\varphi^{-1}(\beta)]}\ \bigvee_{x\in f^{-1}(y)} C_{F_1(\alpha)}(x)\\ &\leq \bigvee_{\alpha\in[\varphi^{-1}(\beta)]}\ \bigvee_{x\in f^{-1}(y)} C_{F_2(\alpha)}(x)=C_{[f(F_2)](\beta)}(y) \end{aligned}$$

Thus $[f(F_1)](\beta)$ is a submset of $[f(F_2)](\beta),\ \forall\ \beta\in\varphi(A_1)$.
Therefore, $\big(f(F_1),\varphi(A_1)\big)$ is a soft submgroup of $\big(f(F_2),\varphi(A_2)\big)$. □

Theorem 3.15. *Let (G_1,B_1) and (G_2,B_2) be two soft mgroups over N and (G_1,B_1) be a soft submgroup of (G_2,B_2). If $f_\varphi : SM(M_X,E)\to SM(N_Y,K)$ is a soft mhomomorphism and $\varphi^{-1}(B_2)\subseteq E$, then $\big(f_\varphi^{-1}(G_1),\varphi^{-1}(B_1)\big)$ and $\big(f_\varphi^{-1}(G_2),\varphi^{-1}(B_2)\big)$ are both soft mgroups over M and $\big(f_\varphi^{-1}(G_1),\varphi^{-1}(B_1)\big)$ is a soft submgroup of $\big(f_\varphi^{-1}(G_2),\varphi^{-1}(B_2)\big)$.*

Proof. The proof is similar to that of Theorem 3.14. □

Definition 3.16. A soft mgroup (F, A) over a mgroup M of a group X is called soft abelian mgroup if $F(\alpha)$ is an abelian submgroup of M, $\forall\ \alpha \in A$.

Theorem 3.17. *Let (F, A) be a soft abelian mgroup over M and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft onto mhomomorphism. Then $[f_\varphi(F), \varphi(A)]$ is soft abelian mgroup over N.*

Proof. Since (F, A) is a soft abelian mgroup over M, it follows that $C_{F(\alpha)}(x_1x_2) = C_{F(\alpha)}(x_2x_1),\ \forall\ \alpha \in A$.
Let $y_1,\ y_2 \in N^*$. Since f is onto, it follows that there exist $x_1,\ x_2 \in M^*$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
Now for any $\beta \in \varphi(A)$ and $y_1,\ y_2 \in N^*$, we have

$$\begin{aligned} C_{[f_\varphi(F)](\beta)}(y_1y_2) &= \vee_{\alpha\in\varphi^{-1}(\beta)} \vee_{x\in f^{-1}(y_1y_2)} C_{F(\alpha)}(x) \\ &= \vee_{\alpha\in\varphi^{-1}(\beta)} \vee_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} C_{F(\alpha)}(x_1x_2) \\ &= \vee_{\alpha\in\varphi^{-1}(\beta)} \vee_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} C_{F(\alpha)}(x_2x_1) \\ &= \vee_{\alpha\in\varphi^{-1}(\beta)} \vee_{x_2x_1\in f^{-1}(y_2y_1)} C_{F(\alpha)}(x_2x_1) \\ &= C_{[f_\varphi(F)](\beta)}(y_2y_1) \end{aligned}$$

Therefore, $[f_\varphi(F), \varphi(A)]$ is soft abelian mgroup over N. □

Theorem 3.18. *Let (G, B) be a soft abelian mgroup over N and $f_\varphi : SM(M_X, E) \to SM(N_Y, K)$ be a soft into mhomomorphism. Then $[f_\varphi^{-1}(G), \varphi^{-1}(B)]$ is soft abelian mgroup over M.*

Proof. The Proof is similar to that of Theorem 3.17. □

Definition 3.19. Let P be a normal subgroup of X and (F, A) be a soft mgroup over M. Define a mapping $\frac{F}{P}$ over A by $\frac{F}{P}(\alpha) =$ the factor mgroup $F(\alpha)/P$, $\forall\ \alpha \in A$, where $C_{[F(\alpha)/P]}(xP) = \vee\{C_{F(\alpha)}(z); z \in xP\}, \forall\ x \in M^*$. Thus for each $\alpha \in A$, $F(\alpha)/P$ is a submgroup of the factor mgroup M/P and hence $(\frac{F}{P}, A)$ is a soft mgroup over M/P and is called a soft factor mgroup of (F, A) w.r.t. P.

Definition 3.20. Let $M,\ N$ be two mgroups over the groups $X,\ Y$ respectively and $(F, A),\ (G, B)$ be two soft mgroups over $M,\ N$ respectively.

(i) Then (F, A) is said to be soft mhomomorphic to (G, B), denoted by $(F, A) \sim (G, B)$, if for each $\alpha \in A$, there exists $\beta \in B$ such that $f^{\alpha\beta} : [F(\alpha)]^* \to [G(\beta)]^*$ is a homomorphism with $f^{\alpha\beta}[F(\alpha)] = G(\beta)$;
(ii) Then (F, A) is said to be soft misomorphic to (G, B), denoted by $(F, A) \simeq (G, B)$, if for each $\alpha \in A$, there exists $\beta \in B$ such that $f^{\alpha\beta} : [F(\alpha)]^* \to [G(\beta)]^*$ is an isomorphism with $f^{\alpha\beta}[F(\alpha)] = G(\beta)$.

Theorem 3.21. *Let $M,\ N$ be two mgroups over the groups $X,\ Y$ respectively and $(F, A),\ (G, B)$ be two soft mgroup over $M,\ N$ respectively such that (F, A) is soft mhomomorphic to (G, B). If $f^{\alpha\beta} : [F(\alpha)]^* \to [G(\beta)]^*$ be the corresponding homomorphism for $(\alpha, \beta) \in (A \times B)$ and $K_{\alpha\beta}$ be the kernel of $f^{\alpha\beta}$, then there exists an isomorphism $\widehat{f^{\alpha\beta}} : [F(\alpha)]^*/K_{\alpha\beta} \to [G(\beta)]^*$ such that $\widehat{f^{\alpha\beta}}[F(\alpha)/K_{\alpha\beta}] = G(\beta)$.*

Proof. Since $(F,A) \sim (G,B)$, it follows that $\forall\ (\alpha,\beta) \in (A \times B)$, there exists an homomorphism $f^{\alpha\beta} : [F(\alpha)]^* \to [G(\beta)]^*$ such that $f^{\alpha\beta}[F(\alpha)] = G(\beta)$.
Again since $K_{\alpha\beta}$ is a normal subgroup of $[F(\alpha)]^*$, it follows that, there exists an onto homomorphism $\phi^{\alpha\beta} : [F(\alpha)]^* \to [F(\alpha)]^*/K_{\alpha\beta}$ such that $\phi^{\alpha\beta}(\xi) = \xi K_{\alpha\beta},\ \forall\ \xi \in [F(\alpha)]^*$.
Let us define a mapping $\widehat{f^{\alpha\beta}} : [F(\alpha)]^*/K_{\alpha\beta} \to [G(\beta)]^*$, where $\widehat{f^{\alpha\beta}}[\xi K_{\alpha\beta}] = f^{\alpha\beta}(\xi)$, $\xi \in [F(\alpha)]^*$.
Clearly $\widehat{f^{\alpha\beta}}\phi^{\alpha\beta} = f^{\alpha\beta}$ and $\widehat{f^{\alpha\beta}}$ is an isomorphism.

Now for each $y \in [G(\beta)]^*$,

$$\begin{aligned}
C_{\widehat{f^{\alpha\beta}}\left[F(\alpha)/K_{\alpha\beta}\right]}(y) &= \vee\{C_{F(\alpha)/K_{\alpha\beta}}(xK_{\alpha\beta})\ :\ \widehat{f^{\alpha\beta}}(xK_{\alpha\beta}) = y\} \\
&= \vee\big\{ \vee \{C_{F(\alpha)}(z)\ :\ z \in xK_{\alpha\beta}\}\ :\ \widehat{f^{\alpha\beta}}(xK_{\alpha\beta}) = y\big\} \\
&= \vee\{C_{F(\alpha)}(z)\ :\ z \in xK_{\alpha\beta},\ f^{\alpha\beta}(z) = y\} \\
&= C_{f^{\alpha\beta}[F(\alpha)]}(y) \\
&= C_{G(\beta)}(y).
\end{aligned}$$

Therefore $\widehat{f^{\alpha\beta}}\left[F(\alpha)/K_{\alpha\beta}\right] = G(\beta)$. □

4. Conclusion

After successful introduction of multi group theory in [36], we have attempted here to extend it in soft multi setting for the first time. As it works there is a scope of further research in developing topological algebraic structure in these settings.

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