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T-fuzzy KU-subalgebras of KU-algebras

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Abstract. In this paper, using t-norm T, the notion of (imaginable) T-fuzzy subalgebras of KU-algebras are introduced and investigated their related results. Images and preimages of KU-subalgebras under homomorphism are investigated. Using level subsets of KU-algebras, some characterization theorems are given. The direct product and T-product of T-fuzzy subalgebras of KU-algebras are discussed.

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1. INTRODUCTION

Triangular norms (t-norms for short) were introduced by Schweizer and Sklar in [17, 18], following some ideas of Menger in the context of probabilistic metric spaces [9] (as statistical metric spaces were called after 1964). With the development of t-norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in fuzzy set theory, t-norms have been widely used for fuzzy operations, fuzzy logic and fuzzy relation equations [19]. In recent years, a systematic study concerning the properties and related matters of t-norms have been made by Klement et al. [7, 8].

Prabpayak and Leerawat [14] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties in [15]. Mostafa et al. [10, 12] introduced the notion of fuzzy (n-fold) KU-ideals of KU-algebras. Akram et al. [2] and Yaqoob et al. [20] introduced the notion of cubic KU-subalgebras and KU-ideals in KU-algebras. They discussed relationship between a cubic KU-subalgebra and a cubic KU-ideal. Gulistan et al. [5] applied the concept of soft set theory to KU-algebra. Muhiuddin [13] applied the bipolar-valued fuzzy set theory to KU-algebras, and introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. He considered the specifications of a bipolar fuzzy KU-subalgebra, a bipolar fuzzy

KU-ideal in KU-algebras and discussed the relations between a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal and provided conditions for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal. Gulistan et al. [4] studied (α, β)-fuzzy KU-ideals in KU-algebras and discussed some special properties. Mostafa and Kareem [11] introduced n-fold KU-ideals and obtained some related results. Akram et al. [1] introduced the notion of interval-valued $(\tilde{\theta}, \tilde{\delta})$-fuzzy KU-ideals of KU-algebras and obtained some related properties.

The objective of this paper is to introduce the concept of (imaginable) triangular norm to KU-subalgebras of KU-algebras. Section 2 proceeds with a recapitulation of all required definitions and properties. In Section 3, concepts and operations of T-fuzzy KU-subalgebras of KU-algebras are proposed and discussed their properties in details. In Section 4, direct product and T-product of T-fuzzy KU-subalgebras of KU-algebras are introduced. Finally, in Section 5, a conclusion of the proposed work is given.

2. Preliminaries

In this section, some elementary aspects that are necessary for the main part of the paper are included.

Definition 2.1 ([14])**.** . By a KU-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following axioms: for any $x, y, z \in X$,

1. $(x * y) * ((y * z) * (x * z)) = 0$,
2. $x * 0 = 0$,
3. $0 * x = x$,
4. $x * y = 0 = y * x$ implies $x = y$.

We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $y * x = 0$.

Definition 2.2 ([10])**.** In a KU-algebra, the following axioms are true: for any $x, y, z \in X$,

(i) $z * z = 0$,
(ii) $z * (x * z) = 0$,
(iii) $x \leq y$ imply $y * z \leq x * z$,
(iv) $z * (y * x) = y * (z * x)$,
(v) $y * ((y * x) * x) = 0$.

An example of KU-algebra is given below.

Example 2.3. Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	a
b	0	c	0	c	d
c	0	a	b	0	a
d	0	0	0	0	0

It is easy to see that X is a KU-algebra.

A non-empty subset S of a KU-algebra X is called a KU-subalgebra [14] of X if $x * y \in S$, for all $x, y \in S$. From this definition it is observed that, if a subset S of a KU-algebra satisfies only the closure property, then S becomes a KU-subalgebra.

Let $(X, *, 0)$ and $(Y, *', 0')$ be KU-algebras. A homomorphism is a mapping $f : X \to Y$ satisfying $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Theorem 2.4 ([15]). *Let f be a homomorphism of a KU-algebra X into a KU-algebra Y, then*
(1) *if 0 is the identity in X, then $f(0)$ is the identity in Y;*
(2) *if S is a KU-subalgebra of X, then $f(S)$ is a KU-subalgebra of Y;*
(3) *if S is a KU-subalgebra of $f(Y)$, then $f^{-1}(S)$ is a KU-subalgebra of X.*

We now review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x then a fuzzy set [21] A in X is defined as $A = \{< x, \alpha_A(x) >: x \in X\}$ where $\alpha_A(x)$ is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$. For any fuzzy sets A and B of a set X, we define $A \cap B = \min\{\alpha_A(x), \alpha_B(x)\}$for all $x \in X$.

Let A be a fuzzy set in X. For $\tilde{s} \in [0, 1]$, the set $U(\alpha_A : \tilde{s}) = \{x \in X : \alpha_A(x) \geq \tilde{s}\}$ is called an upper $\tilde{s}$-level [3] of A.

Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y. Then the inverse image [16] of B, denoted by $f^{-1}(B)$ in X and is given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$.

Conversely, let A be a fuzzy set in X with membership function α_A. Then the image [16] of A, denoted by $f(A)$ in Y and is given by

$$f(A)(y) = f(\alpha_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \alpha_A(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5 ([9]). A triangular norm (t-norm) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies:

(T1) boundary condition: $T(x, 1) = x$;
(T2) commutativity: $T(x, y) = T(y, x)$;
(T3) associativity: $T(x, T(y, z)) = T(T(x, y), z)$;
(T4) monotonicity: $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

for all $x, y, z \in [0, 1]$.

Some example [8] of t-norms are the minimum $T_M(x, y) = \min(x, y)$, the product $T_P(x, y) = x.y$ and the Lukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Also, it is well known [6, 7] that if T is a t-norm, then $T(x, y) \leq \min\{x, y\}$ for all $x, y \in [0, 1]$.

Lemma 2.6 ([6]). *Let T be a t-norm. Then $T(T(x, y), T(z, t)) = T(T(x, z), T(y, t))$ for all x, y, z and $t \in [0, 1]$.*

Definition 2.7. Let P be a t-norm. Denote by $\triangle_P$ the set of elements $x \in [0, 1]$ such that $P(x, x) = x$, that is, $\triangle_P = \{x \in [0, 1] : P(x, x) = x\}$.

A fuzzy set A in X is said to satisfy imaginable property with respect to P if $Im(\alpha_A) \subseteq \triangle_P$.

3. T-fuzzy KU-subalgebras of KU-algebras

In this section, T-fuzzy KU-subalgebras of KU-algebras are defined and some important properties are presented. In what follows, let X denote a KU-algebra unless otherwise specified.

Definition 3.1. Let A be a fuzzy set in X. Then the set A is a T-fuzzy KU-subalgebra over the binary operator $*$ if it satisfies

$$\text{(T1) } \alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \text{ for all } x, y \in X.$$

Let us illustrate this definition using the following examples.

Example 3.2. Let X=$\{0, a, b, c\}$ be a KU-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	b
b	0	b	0	a
c	0	0	0	0

Let $T_m : [0, 1] \times [0, 1] \to [0, 1]$ be functions defined by $T_m(x, y) = \max(x+y-1, 0)$ for all $x, y \in [0, 1]$. Then T_m is a t-norm. Define a fuzzy set A in X by $\alpha_A(0) = 0.9$, $\alpha_A(a) = \alpha_A(b) = 0.7$ and $\alpha_A(c) = 0.5$. Then A is a T_m-normed fuzzy KU-subalgebra of X.

Definition 3.3. A T-fuzzy KU-subalgebra A is called an imaginable T-fuzzy KU-subalgebra of X if α_A satisfy the imaginable property with respect to T.

Example 3.4. Consider T_m be a t-norm and X=$\{0, a, b, c\}$ be a KU-algebra in Example 3.2. Define a fuzzy set A in X by $\alpha_A(x) = 1$, if $x \in \{0, a, b\}$ and $\alpha_A(x) = 0$, if $x = c$. It is easy to check that $\alpha_A(x * y) \geq T_m\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$. Also, $Im(\alpha_A) \subseteq \triangle_{T_m}$. Hence, A is an imaginable T_m-fuzzy KU-subalgebra of X.

Proposition 3.5. *If A is an imaginable T-fuzzy KU-subalgebra of X, then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 3.6. *Let A be an imaginable T-fuzzy KU-subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \alpha_A(x_n) = 1$ then $\alpha_A(0) = 1$.*

Proof. By Proposition 3.5, $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, therefore, $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer n. Consider, $1 \geq \alpha_A(0) \geq \lim_{n\to\infty} \alpha_A(x_n) = 1$. Hence, $\alpha_A(0) = 1$. □

The intersection of any two T-fuzzy KU-subalgebras is also a T-fuzzy KU-subalgebra, which is proved in the following theorem.

Theorem 3.7. *Let A_1 and A_2 be two T-fuzzy KU-subalgebras of X. Then $A_1 \cap A_2$ is a T-fuzzy KU-subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now,

$$\begin{aligned}
\alpha_{A_1 \cap A_2}(x * y) &= \min\{\alpha_{A_1}(x * y), \alpha_{A_2}(x * y)\}, \\
&\geq \min\{T\{\alpha_{A_1}(x), \alpha_{A_1}(y)\}, T\{\alpha_{A_2}(x), \alpha_{A_2}(y)\}\} \\
&\geq T\{\min\{\alpha_{A_1}(x), \alpha_{A_2}(x)\}, \min\{\alpha_{A_1}(y), \alpha_{A_2}(y)\}\} \\
&= T\{\alpha_{A_1 \cap A_2}(x), \alpha_{A_1 \cap A_2}(y)\}
\end{aligned}$$

Hence, $A_1 \cap A_2$ is a T-fuzzy KU-subalgebra of X. □

The above theorem can be generalized as follows.

Corollary 3.8. *Let $\{A_i : i = 1, 2, 3, 4, \ldots\}$ be a family of T-fuzzy KU-subalgebras of X. Then $\bigcap A_i$ is also a T-fuzzy KU-subalgebra of X, where $\bigcap A_i = \{< x, \min \alpha_{A_i}(x) >: x \in X\}$.*

The set $\{x \in X : \alpha_A(x) = \alpha_A(0)\}$ is denoted by I_{α_A}.

Theorem 3.9. *Let A be an imaginable T-fuzzy KU-subalgebra of X. Then the set I_{α_A} is a KU-subalgebra of X.*

Proof. Let $x, y \in I_{\alpha_A}$. Then $\alpha_A(x) = \alpha_A(0) = \alpha_A(y)$ and so, $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} = T\{\alpha_A(0), \alpha_A(0)\} = \alpha_A(0)$. By using Proposition 3.5, we know that $\alpha_A(x * y) \leq \alpha_A(0)$. Hence, $\alpha_A(x * y) = \alpha_A(0)$ or equivalently $x * y \in I_{\alpha_A}$. Therefore, the set I_{α_A} is a KU-subalgebra of X. □

As is well known, the characteristic function of a set is a special fuzzy set. Suppose A is a non-empty subset of X. By χ_A we denote the characteristic function of A, that is,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.10. *If A is a KU-subalgebra of X, then the characteristic function χ_A is a T-fuzzy KU-subalgebra of X.*

Proof. Let $x, y \in X$. We consider here four cases:
Case (i) If $x, y \in A$ then $x * y \in A$ since A is a KU-subalgebra of X. Then

$$\chi_A(x * y) = 1 \geq T\{\chi_A(x), \chi_A(y)\}.$$

Case (ii) If $x, y \notin A$, then $\chi_A(x) = 0 = \chi_A(y)$. Thus

$$\chi_A(x * y) \geq 0 = \min\{0, 0\} \geq T\{0, 0\} = T\{\chi_A(x), \chi_A(y)\}.$$

Case (iii) If $x \in A$ and $y \notin A$ then $\chi_A(x) = 1$, $\chi_A(y) = 0$. Thus

$$\chi_A(x * y) \geq 0 = T\{0, 1\} = T\{1, 0\} = T\{\chi_A(x), \chi_A(y)\}.$$

Case (iv) If $x \notin A$ and $y \in A$ then by the same argument as in Case (iii), we conclude that $\chi_A(x * y) \geq T\{\chi_A(x), \chi_A(y)\}$.

Therefore, the characteristic function χ_A is a T-fuzzy KU-subalgebra of X. □

Theorem 3.11. *Let A be a non-empty subset of X. If χ_A satisfies $(T1)$, then A is a KU-subalgebra of X.*

Proof. Suppose that χ_A satisfy $(T1)$. Let $x, y \in A$. Then it follows from $(T1)$ that $\chi_A(x * y) \geq T\{\chi_A(x), \chi_A(y)\} = T\{1, 1\} = 1$ so that $\chi_A(x * y) = 1$, i.e., $x * y \in A$. Hence, A is a KU-subalgebra of X. □

Theorem 3.12. *Let P be a KU-subalgebra of X and A be a fuzzy set in X defined by*

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in P \\ \tau & \text{otherwise} \end{cases}$$

for all $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then A is a T_m-fuzzy KU-subalgebra of X, where T_m is the t-norm in Example 3.2. In particular if $\lambda = 1$ and $\tau = 0$ then A is an imaginable T_m-fuzzy KU-subalgebra of X. Moreover, $I_{\alpha_A} = P$.

Proof. Let $x, y \in X$. We consider here three cases:
Case (i) If $x, y \in P$ then $T_m(\alpha_A(x), \alpha_A(y)) = T_m(\lambda, \lambda) = \max(2\lambda - 1, 0)$
$= \begin{cases} 2\lambda - 1 & \text{if } \lambda > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \leq \lambda = \alpha_A(x * y)$.
Case (ii) If $x \in P$ and $y \notin P$ (or, $x \notin P$ and $y \in P$) then $T_m(\alpha_A(x), \alpha_A(y))$
$= T_m(\lambda, \tau) = \max(\lambda + \tau - 1, 0) = \begin{cases} \lambda + \tau - 1 & \text{if } \lambda + \tau > 1 \\ 0 & \text{otherwise} \end{cases} \leq \tau = \alpha_A(x * y)$.
Case (iii) If $x, y \notin P$ then $T_m(\alpha_A(x), \alpha_A(y)) = T_m(\tau, \tau) = \max(2\tau - 1, 0)$
$= \begin{cases} 2\tau - 1 & \text{if } \tau > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \leq \tau = \alpha_A(x * y)$.

Hence, A is a T_m-fuzzy KU-subalgebra of X.

Assume that $\lambda = 1$ and $\tau = 0$. Then $T_m(\lambda, \lambda) = \max(\lambda + \lambda - 1, 0) = 1 = \lambda$ and $T_m(\tau, \tau) = \max(\tau + \tau - 1, 0) = 0 = \tau$. Thus $\lambda, \tau \in \triangle_{T_m}$ i.e., $Im(\alpha_A) \subseteq \triangle_{T_m}$. So, A is an imaginable T_m-fuzzy KU-subalgebra of X.

Also, $I_{\alpha_A} = \{x \in X, \alpha_A(x) = \alpha_A(0)\} = \{x \in X, \alpha_A(x) = 1\} = P$. Therefore, $I_{\alpha_A} = P$. □

By using the level subsets of KU-algebras, we can characterize the T-fuzzy KU-subalgebras as follows:

Theorem 3.13. *Let A be a T-fuzzy KU-subalgebra of X and $\tilde{s} \in [0, 1]$. Then if $\tilde{s} = 1$, the upper level set $U(\alpha_A : \tilde{s})$ is either empty or a KU-subalgebra of X.*

Proof. Let $\tilde{s} = 1$ and $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s} = 1$ and $\alpha_A(y) \geq \tilde{s} = 1$. It follows that $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y)) \geq T(1, 1) = 1$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X when $s = 1$. □

Theorem 3.14. *If A is an imaginable T-fuzzy KU-subalgebra of X, then the upper $\tilde{s}$-level of A is KU-subalgebra of X.*

Proof. Assume that $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s}$ and $\alpha_A(y) \geq \tilde{s}$. It follows that $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \geq T(\tilde{s}, \tilde{s}) = \tilde{s}$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X. □

Theorem 3.15. *Let A be a fuzzy set in X such that the set $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X for every $\tilde{s} \in [0, 1]$. Then A is a T-fuzzy KU-subalgebra of X.*

Proof. Let for every $\tilde{s} \in [0,1]$, $U(\alpha_A : \tilde{s})$ be a subalgebra of X. In contrary, let $x_0, y_0 \in X$ be such that $\alpha_A(x_0 * y_0) < T\{\alpha_A(x_0), \alpha_A(y_0)\}$. Let us consider,

$$\tilde{s}_0 = \frac{1}{2}\Big[\alpha_A(x_0 * y_0) + T\{\alpha_A(x_0), \alpha_A(y_0)\}\Big].$$

Then $\alpha_A(x_0 * y_0) < \tilde{s}_0 \leq T\{\alpha_A(x_0), \alpha_A(y_0)\} \leq \min\{\alpha_A(x_0), \alpha_A(y_0)\}$ and so $x_0 * y_0 \notin U(\alpha_A : \tilde{s})$ but $x_0, y_0 \in U(\alpha_A : \tilde{s})$. This is a contradiction and hence α_A satisfies the inequality $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$. □

Theorem 3.16. *Let $f : X \to Y$ be a homomorphism of KU-algebras. If $B = \{< x, \alpha_B(x) >: x \in Y\}$ is a T-fuzzy KU-subalgebra of Y, then the pre-image $f^{-1}(B) = \{< x, f^{-1}(\alpha_B)(x) >: x \in X\}$ of B under f is a T-fuzzy KU-subalgebra of X.*

Proof. Assume that B is a T-fuzzy KU-subalgebra of Y and let $x, y \in X$. Then $f^{-1}(\alpha_B)(x * y) = \alpha_B(f(x * y)) = \alpha_B(f(x) * f(y)) \geq T\{\alpha_B(f(x), \alpha_B(f(y))\} = T\{f^{-1}(\alpha_B)(x), f^{-1}(\alpha_B)(y)\}$. Therefore, $f^{-1}(B)$ is a T-fuzzy KU-subalgebra of X. □

Theorem 3.17. *Let $f : X \to Y$ be a homomorphism from a KU-algebra X onto a KU-algebra Y. If A is an imaginable T-fuzzy KU-subalgebra of X, then the image $f(A)$ of A under f is a T-fuzzy KU-subalgebra of Y.*

Proof. Let A be an imaginable T-fuzzy KU-subalgebra of X. By Theorem 3.14, $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X for every $\tilde{s} \in [0,1]$. Therefore, by Theorem 2.4, $f(U(\alpha_A : \tilde{s}))$ is a KU-subalgebra of Y. But $f(U(\alpha_A : \tilde{s})) = U(f(\alpha_A) : \tilde{s})$. Hence, $U(f(\alpha_A) : \tilde{s})$ is a KU-subalgebra of Y for every $\tilde{s} \in [0,1]$. By Theorem 3.15, $f(A)$ is a T-fuzzy KU-subalgebra of Y. □

4. Product of T-fuzzy KU-subalgebras

In this section, the direct product and T-normed product of fuzzy KU-subalgebras of KU-algebras with respect to t-norm are presented and several properties are studied. Before going into the product of fuzzy KU-subalgebras of KU-algebras, we first define some kind of product of fuzzy subsets.

Definition 4.1. Let $A_1 = \{< x, \alpha_{A_1}(x) >: x \in X\}$ and $A_2 = \{< x, \alpha_{A_2}(x) >: x \in X\}$ be two fuzzy subsets of X. Then the T-product of A_1 and A_2 denoted by $[A_1.A_2]_T = \{< x, [\alpha_{A_1}.\alpha_{A_2}]_T(x) >: x \in X\}$ and is defined by $[\alpha_{A_1}.\alpha_{A_2}]_T(x) = T(\alpha_{A_1}(x), \alpha_{A_2}(x))$ for all $x \in X$.

Theorem 4.2. *Let A_1 and A_2 be two T-fuzzy KU-subalgebras of X. If T^* is a t-norm which dominates T, i.e., $T^*(T(a,b), T(c,d)) \geq T(T^*(a,c), T^*(b,d))$ for all a, b, c and $d \in [0,1]$, then the T^*-product of A_1 and A_2, $[A_1.A_2]_{T^*}$ is a T-fuzzy KU-subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
[\alpha_{A_1}.\alpha_{A_2}]_{T^*}(x * y) &= T^*(\alpha_{A_1}(x * y), \alpha_{A_2}(x * y)) \\
&\geq T^*(T(\alpha_{A_1}(x), \alpha_{A_1}(y)), T(\alpha_{A_2}(x), \alpha_{A_2}(y))) \\
&\geq T(T^*(\alpha_{A_1}(x), \alpha_{A_2}(x)), T^*(\alpha_{A_1}(y), \alpha_{A_2}(y))) \\
&= T([\alpha_{A_1}.\alpha_{A_2}]_{T^*}(x), [\alpha_{A_1}.\alpha_{A_2}]_{T^*}(y)).
\end{aligned}$$

Hence, $[A_1.A_2]_{T^*}$ is a T-fuzzy KU-subalgebra of X. □

Let $f : X \to Y$ be an epimorphism of KU-algebras. Let T, T^* be t-norms such that T^* dominates T. If A_1 and A_2 be two T-fuzzy KU-subalgebras of Y, then the T^*-product of A_1 and A_2, $[A_1.A_2]_{T^*}$ is a T-fuzzy KU-subalgebra of Y. Since every epimorphic pre-image of a T-fuzzy KU-subalgebra is a T-fuzzy KU-subalgebra, the pre-images $f^{-1}(A_1)$, $f^{-1}(A_2)$ and $f^{-1}([A_1.A_2]_{T^*})$ are T-fuzzy KU-subalgebras of X. The next theorem provides the relation between $f^{-1}([A_1.A_2]_{T^*})$ and the T^*-product $[f^{-1}(A_1).f^{-1}(A_2)]_{T^*}$ of $f^{-1}(A_1)$ and $f^{-1}(A_2)$.

Theorem 4.3. *Let $f : X \to Y$ be an epimorphism of KU-algebras. Let T, T^* be t-norms such that T^* dominates T. Let A_1 and A_2 be two T-fuzzy KU-subalgebras of Y. If $[A_1.A_2]_{T^*}$ is the T^*-product of A_1 and A_2 and $[f^{-1}(A_1).f^{-1}\ (A_2)]_{T^*}$ is the T^*-product of $f^{-1}(A_1)$ and $f^{-1}(A_2)$, then*

$$f^{-1}([\alpha_{A_1}.\alpha_{A_2}]_{T^*}) \quad = \quad [f^{-1}(\alpha_{A_1}).f^{-1}(\alpha_{A_2})]_{T^*}.$$

Proof. For any $x \in X$, we get, $f^{-1}([\alpha_{A_1}.\alpha_{A_2}]_{T^*})(x) = [\alpha_{A_1}.\alpha_{A_2}]_{T^*}(f(x)) = T^*(\alpha_{A_1}(f(x)), \alpha_{A_2}(f(x))) = T^*([f^{-1}(\alpha_{A_1})](x), [f^{-1}(\alpha_{A_2})](x)) = [f^{-1}(\alpha_{A_1}).\ f^{-1}(\alpha_{A_2})]_{T^*}(x)$. □

Theorem 4.4. *Let $X = X_1 \times X_2$ be direct product of KU-algebras. If $A_1 = \{< x, \alpha_{A_1}(x) >: x \in X\}$ and $A_2 = \{< x, \alpha_{A_2}(x) >: x \in X\}$ are two T-fuzzy KU-subalgebras of X_1 and X_2 respectively, then $A = \{< x, \alpha_A(x) >: x \in X\}$ is a T-fuzzy KU-subalgebra of X defined by $\alpha_A = \alpha_{A_1} \times \alpha_{A_2}$ such that $\alpha_A(x_1, x_2) = (\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2) = T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two elements of X. Since X is a KU-algebra, we have,

$$\begin{aligned}
\alpha_A(x * y) &= \alpha_A((x_1, x_2) * (y_1, y_2)) = \alpha_A(x_1 * y_1, x_2 * y_2) \\
&= (\alpha_{A_1} \times \alpha_{A_2})(x_1 * y_1, x_2 * y_2) \\
&= T(\alpha_{A_1}(x_1 * y_1), \alpha_{A_2}(x_2 * y_2)) \\
&\geq T(T(\alpha_{A_1}(x_1), \alpha_{A_1}(y_1)), T(\alpha_{A_2}(x_2), \alpha_{A_2}(y_2))) \\
&= T(T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2)), T(\alpha_{A_1}(y_1), \alpha_{A_2}(y_2))) \\
&= T((\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2), (\alpha_{A_1} \times \alpha_{A_2})(y_1, y_2)) \\
&= T(\alpha_A(x), \alpha_A(y)).
\end{aligned}$$

Hence, $A = \{< x, \alpha_A(x) >: x \in X\}$ is a T-fuzzy KU-subalgebra of X. □

The relationship between T-fuzzy KU-subalgebras $A_1 \times A_2$ and $[A_1 \cdot A_2]_T$ can be viewed via the following diagram where $I = [0, 1]$ and $g : X \to X \times X$ is defined by $g(x) = (x, x)$. It is not difficult to see that $[A_1 \cdot A_2]_T$ is the preimage of $A_1 \times A_2$ under g.

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "{[\alpha_{A_1}\cdot\alpha_{A_2}]_T}"] & X\times X \arrow[dl, "\alpha_{A_1}\times\alpha_{A_2}"] \arrow[d, shift right=2, "\alpha_{A_1}"] \arrow[d, shift left=2, "\alpha_{A_2}"'] \\
I & I\times I \arrow[l, "T"]
\end{tikzcd}$$

5. Conclusions and future work

In this paper, notion of T-fuzzy KU-subalgebras of KU-algebras are introduced and investigated some of their useful properties. Images and preimages of KU-subalgebras under homomorphism are studied. Finally, direct products and T-products of T-fuzzy KU-subalgebras has been introduced and some important properties of it are studied.

It is our hope that this work would other foundations for further study of the theory of KU-algebras. In our future study of fuzzy structure of KU-algebra, may be the following topics should be considered: (i) to find T-fuzzy KU-ideals of KU-algebras, (ii) to find interval-valued T-fuzzy KU-subalgebras of KU-algebras, (iii) to find interval-valued T-fuzzy KU-ideals of KU-algebras, (iv) to find intuitionistic (T,S)-fuzzy KU-subalgebras of KU-algebras.

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