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Fuzzy totally e-continuous functions

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Abstract. In this paper, fuzzy e-open sets and fuzzy e-closed sets are used to define and investigate a new class of functions called fuzzy totally e-continuous functions. Relationships between the new class and other classes of functions are established.

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1. Introduction

The concept of fuzzy has invaded almost all branches of mathematics with the introduction of fuzzy sets by Zadeh [14] of 1965. The theory of fuzzy topological spaces was introduced and developed by Chang [5]. Since then many fuzzy topologists have extended various notions in classical topology to fuzzy topological spaces. In classical topology the class of totally continuous functions was introduced in [9] and as a consequence of this e-open sets and e-continuous was introduced and studied in [7]. The fuzzyfied concept of the totally continuous function is introduced as perfectly fuzzy continuous functions and studied along with other functions in [3]. In 2014 Seenivasan [13] introduced the concept of fuzzy e-continuity and fuzzy e-open sets in fuzzy topological space. The purpose of this paper is to introduce the concept of fuzzy totally e-continuous functions and investigated its basic properties.

2. Preliminaries

Throughout this paper X, Y and Z are always mean fuzzy topological spaces. The class of all fuzzy sets on a universe X will be denoted by I^X. A fuzzy topology on a nonempty set X is a family δ of fuzzy subsets of X which satisfies the following three conditions:

(i) $0, 1 \in \delta$,
(ii) If $g, h \in \delta$, their $g \wedge h \in \delta$,

(iii) $f_i \in \delta$ for each $i \in I$, then $\bigvee_{i \in I} f_i \in \delta$.

The pair (X, δ) is called a fuzzy topological space [5]. Let λ be a fuzzy subset of a space X. The fuzzy closure of λ, fuzzy interior of λ, fuzzy δ-closure of λ and the fuzzy δ-interior of λ are denoted by $Cl(\lambda)$, $Int(\lambda)$, $Cl_\delta(\lambda)$ and $Int_\delta(\lambda)$ respectively. A fuzzy subset λ of space X is called fuzzy regular open [2] (resp. fuzzy regular closed) if $\lambda = Int(Cl(\lambda))$ (resp. $\lambda = Cl(Int(\lambda))$. Now $Cl(\lambda)$ and $Int(\lambda)$ are defined as follows $Cl(\lambda) = \wedge\{\mu : \mu \geq \lambda, \mu$ is fuzzy closed in $X\}$ and $Int(\lambda) = \vee\{\mu : \mu \leq \lambda$, μ is fuzzy open in $X\}$. The fuzzy δ-interior of a fuzzy subset λ of X is the union of all fuzzy regular open sets contained in λ. A fuzzy subset λ is called fuzzy δ-open [10] if $\lambda = Int_\delta(\lambda)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $\lambda = Cl_\delta(\lambda)$).

Definition 2.1 ([11])**.** A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda (0 < \lambda \leq 1)$ we denote this fuzzy point by x_λ, where the point x is called its support.

Definition 2.2 ([11])**.** The fuzzy point x_λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $x_\lambda \in A$, iff $\lambda \leq A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A.

Definition 2.3 ([13])**.** A fuzzy set λ of a fuzzy topological space X is said to be fuzzy e-open if $\lambda \leq Cl(Int_\delta \lambda) \vee Int(Cl_\delta \lambda)$, where $Cl(\lambda) = \wedge\{\mu : \mu \geq \lambda$, μ is fuzzy closed in $X\}$ and $Int(\lambda) = \vee\{\mu : \mu \leq \lambda$, μ is fuzzy open in $X\}$. If λ is fuzzy e-open, then $1 - \lambda$ is fuzzy e-closed.

Definition 2.4 ([13])**.** Let X be a fuzzy topological space and λ be any fuzzy set in X. The fuzzy e-closure of λ in X is denoted by $eCl(\lambda)$ defined as follows: $eCl(\mu) = \wedge\{\lambda : \lambda \geq \mu,\ \lambda$ is a fuzzy e-closed set of $X\}$. Similarly we can define $eInt(\lambda)$.

Remark 2.5. For a fuzzy set λ of X, $1 - eInt(\lambda) = eCl(1 - \lambda)$.

Theorem 2.6 ([13])**.** *In a fuzzy topological space X, λ is a fuzzy e-closed (resp. fuzzy e-open) if and only if $\lambda = eCl(\lambda)$ (resp. $\lambda = eInt(\lambda)$).*

Definition 2.7. Let X and Y be two fuzzy topological spaces. Let $\lambda \in I^X$, $\mu \in I^Y$. Then $f(\lambda)$ is a fuzzy subset of Y, defined by $f(\lambda) : Y \to [0, 1]$

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} \lambda(x) & \text{if } f^{-1}(\{y\}) \neq \phi \\ 0 & \text{if } f^{-1}(\{y\}) = \phi. \end{cases}$$

and $f^{-1}(\mu)$ is a fuzzy subset of X, defined by $f^{-1}(\mu)(x) = \mu(f(x))$.

Definition 2.8. A function $f : X \to Y$ is said to be fuzzy totally continuous [4] (resp. fuzzy e-continuous [13]) if the inverse image of every fuzzy open set in Y is a fuzzy Clopen (resp. fuzzy e-open) set in X.

Definition 2.9 ([12])**.** A fuzzy topological space X is called fuzzy T_0-space if for any pair of distinct fuzzy points x_t and x_s, there exist a fuzzy open set λ such that $x_t \in \lambda$ and $x_s \notin \lambda$ or $x_t \notin \lambda$ and $x_s \in \lambda$.

Definition 2.10 ([12])**.** A fuzzy space X is said to be fuzzy T_2 (= fuzzy Hausdorff) if for each pair of fuzzy points x_α and y_β such that $x_\alpha \neq y_\beta$ in X, there exist disjoint fuzzy open sets λ and μ in X such that $x_\alpha \in \lambda$ and $y_\beta \in \mu$.

Definition 2.11 ([8])**.** A collection μ of fuzzy sets in a fuzzy space X is said to be cover of a fuzzy set η of X if $(\bigvee_{A\in\mu} A)(x) = 1$, for every $x \in s(\eta)$. A fuzzy cover μ of a fuzzy set η in a fuzzy space X is said to be have a finite subcover if there exists a finite subcollection $\rho = \{A_1, A_2, \ldots, A_n\}$ of μ such that $(\bigvee_{j=1}^{n} A_j)(x) \geq \eta(x)$, for every $x \in s(\eta)$, where $s(\eta)$ denotes the support of a fuzzy set η.

Definition 2.12 ([1])**.** A fuzzy topological space (X, τ) is said to be fuzzy Lindelöf, if for each family $\mathscr{B} \subset \tau$ and for each $\alpha \in I$ such that $\bigvee\limits_{\lambda\in\mathscr{B}} \lambda \geq \alpha$, there exist for each $\epsilon \in (0, \alpha]$ a countable subset $\mathscr{B}_0$ of $\mathscr{B}$ such that $\bigvee\limits_{\lambda\in\mathscr{B}_0} \lambda \geq \alpha - \epsilon$.

Definition 2.13 ([13])**.** A fuzzy topological space (X, τ) is said to be fuzzy e-T_1 if for each pair of distinct points x and y of X, there exists fuzzy e-open sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.

Definition 2.14 ([13])**.** A fuzzy topological space (X, τ) is said to be fuzzy e-T_2 (i.e., fuzzy e-Hausdorff) if for each pair of distinct points x and y of X, there exists disjoint fuzzy e-open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.15 ([13])**.** A fuzzy topological space (X, τ) is said to be fuzzy e-regular if for each closed set F of X and each $x \in X - F$, there exists disjoint fuzzy e-open sets U and V such that $x \in U$ and $F \leq V$.

Definition 2.16 ([13])**.** A fuzzy topological space (X, τ) is said to be fuzzy e-normal if for every two disjoint fuzzy closed sets A and B of X, there exist two disjoint fuzzy e-open sets U and V such that $A \leq U$ and $B \leq V$ and $U \wedge V = 0$.

3. Fuzzy Totally e-continuous Function

We have introduced the following definition

Definition 3.1. A function $f : X \to Y$ is said to be fuzzy totally e-continuous if the inverse image of every fuzzy open set in Y is a fuzzy e-Clopen (that is, fuzzy e-open and fuzzy e-closed) set in X.

It is clear that every fuzzy totally continuous function is fuzzy totally e-continuous and every fuzzy totally e-continuous function is fuzzy e-continuous.

However, none of the above implications are reversible as shown in the following examples.

Example 3.2. Let $X = \{a, b, c\}$, $Y = \{p, q\}$ $\tau = \{0, 1, \alpha, \beta, \gamma\}$ and $\sigma = \{0, 1, \delta\}$ where $\alpha = \frac{0.5}{a} + \frac{0}{b} + \frac{0}{c}$, $\beta = \frac{0}{a} + \frac{0.5}{b} + \frac{0}{c}$, $\gamma = \frac{0.5}{a} + \frac{0.5}{b} + \frac{0}{c}$ and $\delta = \frac{0.5}{p} + \frac{0}{q}$. Define $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = p$, $f(b) = f(c) = q$. Then f is fuzzy totally e-continuous. However, it is not fuzzy totally continuous, since for the fuzzy open set δ of (Y, σ), $f^{-1}(\delta) = \alpha$, which is not fuzzy closed in (X, τ).

Example 3.3. Let $I = [0,1]$ and μ_1, μ_2 and μ_3 be fuzzy sets of I defined as

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_2(x) = \mu_3(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4} \\ -4x+2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Clearly, $\tau_1 = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2\}$ and $\tau_2 = \{0, 1, \mu_3\}$ are fuzzy topologies on I. Let $f : (I, \tau_1) \to (I, \tau_2)$ be defined by $f(x) = x$ for each $x \in I$. Then f is fuzzy totally e-continuous but not fuzzy totally continuous, since for the fuzzy open set μ_3 of (I, τ_2), $f^{-1}(\mu_3) = \mu_3$, is fuzzy open in (I, τ_1) but not fuzzy closed in (I, τ_1).

Example 3.4. Let $X = Y = \{a, b, c\}$, $\tau = \{0, 1, \alpha, \beta, \gamma, \delta\}$ and $\sigma = \{0, 1, \lambda\}$ where$\alpha = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{c}$, $\beta = \frac{0.6}{a} + \frac{0.5}{b} + \frac{0.5}{c}$, $\gamma = \frac{0.6}{a} + \frac{0.5}{b} + \frac{0.4}{c}$, $\delta = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.4}{c}$ and $\lambda = \frac{0.6}{a} + \frac{0.9}{b} + \frac{0.5}{c}$. Define $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Clearly f is fuzzy e-continuous but not fuzzy totally e-continuous, since for the fuzzy open set λ of (Y, σ), $f^{-1}(\lambda) = \lambda$ is fuzzy e-open in (X, τ) but not fuzzy e-closed in (X, τ).

Example 3.5. Let $I = [0,1]$ and μ_1,μ_2 and μ_3 be fuzzy sets of I in Example 3.3 Consider fuzzy topologies $\tau_1 = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2\}$ and $\tau_2 = \{0, 1, \mu_1 \vee \mu_2\}$ are fuzzy topologies on I. Let $f : (I, \tau_1) \to (I, \tau_2)$ be defined by $f(x) = x$ for each $x \in I$. Then f is fuzzy e-continuous but not fuzzy totally e-continuous, since for the fuzzy open set $\mu_1 \vee \mu_2$ of (I, τ_2), $f^{-1}(\mu_1 \vee \mu_2) = \mu_1 \vee \mu_2$ is fuzzy e-open in (X, τ_1) but not fuzzy e-closed in (X, τ_1).

Definition 3.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly fuzzy e-continuous if and only if $f^{-1}(\lambda)$ is fuzzy e-Clopen whenever $\lambda \in I^Y$.

It is evident that every strongly fuzzy e-continuous function is fuzzy totally e-continuous. However, the converse is not true as shown in the following example.

Example 3.7. Let $X = Y = \{a, b, c\}$, $\tau = \{0, 1, \alpha, \beta, \gamma, \delta\}$ and $\sigma = \{0, 1, \lambda\}$ where $\alpha = \frac{0.4}{a} + \frac{0.6}{b} + \frac{0.5}{c}$, $\beta = \frac{0.6}{a} + \frac{0.4}{b} + \frac{0.4}{c}$, $\gamma = \frac{0.6}{a} + \frac{0.6}{b} + \frac{0.5}{c}$, $\delta = \frac{0.4}{a} + \frac{0.4}{b} + \frac{0.4}{c}$ and $\lambda = \frac{0.4}{a} + \frac{0.5}{b} + \frac{0.5}{c}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy totally e-continuous but not strongly fuzzy e-continuous, since for the fuzzy set $\mu = \frac{0.6}{a} + \frac{0.4}{b} + \frac{0.4}{c} \in I^Y$, $f^{-1}(\mu) = \mu$ is fuzzy e-open in (X, τ) but not fuzzy e-closed in (X, τ).

Recall that a fuzzy topological space (X, τ) is called a discrete fuzzy topological space if $\tau = I^X$.

Example 3.8. Let $I = [0,1]$ and μ_1 and μ_2 and μ_3 be fuzzy sets of I in Example 3.3 Consider fuzzy topologies $\tau_1 = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2\}$ $\tau_2 = \{0, 1, \mu_3\}$ are topologies on I. Let $f : (I, \tau_1) \to (I, \tau_2)$ be defined by $f(x) = x$ for each $x \in I$. Then f is fuzzy totally e-continuous but not strongly fuzzy e-continuous, since for the fuzzy set $\mu_1 \vee \mu_2$ in (I, τ_2), $f^{-1}(\mu_1 \vee \mu_2) = \mu_1 \vee \mu_2$ is fuzzy e-open in (X, τ_1) but not fuzzy e-closed in (X, τ_1).

Theorem 3.9. *Every fuzzy totally e-continuous function onto a discrete fuzzy topological space is strongly fuzzy e-continuous.*

Proof. Obvious. □

Definition 3.10. A fuzzy topological space (X, τ) is said to be fuzzy e-connected if there does not exist fuzzy e-open sets λ and μ such that $\lambda + \mu = 1$, $\lambda \neq 0$ and $\mu \neq 0$.

Theorem 3.11. *If f is fuzzy totally e-continuous function from a fuzzy e-connected space X into any fuzzy topological space Y, then Y is indiscrete fuzzy topological space.*

Proof. Suppose Y is not indiscrete. Then Y has a proper ($\neq 0$ and $\neq 1$) fuzzy open set λ (say). Then by hypothesis on f, $f^{-1}(\lambda)$ is a proper fuzzy e-Clopen subset of X, which is a contradiction to the assumption that X is fuzzy e-connected. □

Definition 3.12. A fuzzy topological space X is fuzzy e-T_2 if for any pair of distinct fuzzy points x_t and x_s, there exist fuzzy e-open sets λ and μ such that $x_t \in \lambda$, $x_s \in \mu$ and $eCl(\lambda) \leq 1 - eCl(\mu)$.

Theorem 3.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be an injective fuzzy totally e-continuous function. If Y is fuzzy T_0, then X is fuzzy e-T_2.*

Proof. Let x_t and y_s be any two distinct fuzzy points of X. Then $f(x_t) \neq f(y_s)$. Since Y is fuzzy T_0, there exists a fuzzy open set say λ in Y such that $f(x_t) \in \lambda$ and $f(y_s) \notin \lambda$. This mean $x_t \in f^{-1}(\lambda)$ and $y_s \notin f^{-1}(\lambda)$. Since f is fuzzy totally e-continuous, $f^{-1}(\lambda)$ is fuzzy e-Clopen set of X. Also $x_t \in f^{-1}(\lambda)$ and $y_s \in 1 - f^{-1}(\lambda)$. Now put $\mu = 1 - f^{-1}(\lambda)$. Then $f^{-1}(\lambda) = eCl(f^{-1}(\lambda))$ and $eCl(1 - f^{-1}(\lambda)) = eCl(\mu) = 1 - f^{-1}(\lambda)$ (since $f^{-1}(\lambda)$ is fuzzy e-closed) and $eCl(f^{-1}(\lambda)) = f^{-1}(\lambda) = 1 - eCl(1 - f^{-1}(\lambda)) = 1 - eCl(\mu) \leq 1 - eCl(\mu)$. □

Theorem 3.14. *If $f : X \to Y$ is fuzzy totally e-continuous and $g : Y \to Z$ is fuzzy continuous, $g \circ f : X \to Z$ is a fuzzy totally e-continuous function.*

Proof. Obvious. □

Theorem 3.15. *Let $p_i : X_1 \times X_2 \to X_i (i = 1, 2)$ be the projection of $X_1 \times X_2$ on X_i. If $f : X \to X_1 \times X_2$ is fuzzy totally e-continuous, then $p_i \circ f$ is also fuzzy totally e-continuous.*

Proof. This follows directly from Theorem 3.14 □

Theorem 3.16. *Let $f : X_1 \to X_2$ be a function. If the graph $g : X_1 \to X_1 \times X_2$ of f is fuzzy totally e-continuous, then f is also fuzzy totally e-continuous.*

Proof. This follows directly from Theorem 3.15 □

Definition 3.17 ([13])**.** A mapping $f : X \to Y$ is said to be a fuzzy e-irresolute if $f^{-1}(\lambda)$ is fuzzy e-open in X for every fuzzy e-open set λ in Y.

Theorem 3.18. *If $f : X \to Y$ is fuzzy e-irresolute and $g : Y \to Z$ is fuzzy totally e-continuous function, then $g \circ f : X \to Z$ is a fuzzy e-continuous function.*

Proof. Straightforward. □

Definition 3.19. A function $f : X \to Y$ is said to be fuzzy totally e-open if the image of every fuzzy open subset of X is fuzzy e-Clopen of Y.

Theorem 3.20. *If $f : X \to Y$ is fuzzy open and $g : Y \to Z$ is fuzzy totally e-open, then $g \circ f$ is fuzzy totally e-open.*

Proof. Straightforward. □

Definition 3.21. A function $f : X \to Y$ is said to be fuzzy almost e-open if the image of every fuzzy e-Clopen subset of X is fuzzy open subset of Y.

Theorem 3.22. *If $f : X \to Y$ is an onto fuzzy almost e-open and fuzzy totally e-continuous function and $g : Y \to Z$ is a function such that $g \circ f$ is fuzzy totally e-continuous, then g is fuzzy continuous.*

Proof. Let λ be a fuzzy open subset of Z. Then $(g \circ f)^{-1}(\lambda)$ is a fuzzy e-Clopen subset of X. But $f(g \circ f)^{-1}(\lambda) = g^{-1}(\lambda)$ is a fuzzy open subset of Y. Hence, g is fuzzy continuous. □

Theorem 3.23. *If $f : X \to Y$ is fuzzy totally e-continuous function and A is fuzzy open crisp subset of X, then $f_A : A \to Y$ is also fuzzy totally e-continuous.*

Proof. Let λ be a fuzzy open subset of Y, then $f^{-1}(\lambda)$ is fuzzy e-Clopen in X. Now, $f^{-1}(\lambda)$ is fuzzy e-Clopen in X and A is a fuzzy open crisp subset of a fuzzy topological space X, then $f^{-1}(\lambda) \cap A$ is fuzzy e-Clopen set in A. But, $f^{-1}(\lambda) \cap A = f_A^{-1}(\lambda)$ is fuzzy e-Clopen set in A. Hence f_A is fuzzy totally e-continuous. □

Definition 3.24. A function $f : X \to Y$ is called slightly fuzzy e-continuous if $f^{-1}(\lambda)$ is fuzzy e-closed in X for every fuzzy Clopen set λ of Y.

Theorem 3.25. *If $f : X \to Y$ is a fuzzy e-irresolute and $g : Y \to Z$ is slightly fuzzy e-continuous, then $g \circ f : X \to Z$ is slightly fuzzy e-continuous.*

Proof. Let λ be any fuzzy clopen set of Z. Then by hypothesis on g, $g^{-1}(\lambda)$ is fuzzy e-closed in Y. Now $(g \circ f)^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda))$ and therefore by hypothesis on f, $(g \circ f)^{-1}(\lambda)$ is fuzzy e-closed. This proves that $g \circ f$ is slightly fuzzy e-continuous. □

Definition 3.26. A fuzzy space X is said to be:

(i) fuzzy mildly e-compact if every fuzzy e-Clopen cover of X has a finite subcover;
(ii) fuzzy mildly countably e-compact if every fuzzy e-Clopen countably cover of X has a finite subcover;
(iii) fuzzy mildly e-Lindelöf if every cover of X by fuzzy e-Clopen sets has a countable subcover;
(iv) fuzzy compact [6] if every fuzzy open cover of X has a finite subcover;
(v) fuzzy countably compact [6] if every countable fuzzy open covering of X contains a finite subcollection that covers X.

Theorem 3.27. *Let $f : X \to Y$ be a fuzzy totally e-continuous surjective function. Then the following statements hold:*

(i) *If X is a fuzzy mildly e-compact, then Y is fuzzy compact.*
(ii) *If X is a fuzzy mildly e-Lindelöf, then Y is fuzzy Lindelöf.*
(iii) *If X is a fuzzy mildly countably e-compact, then Y is fuzzy countably compact.*

Proof. (i) Let $\{\lambda_\alpha : \alpha \in I\}$ be any fuzzy open cover of Y. Since f is fuzzy totally e-continuous, then $\{f^{-1}(\lambda_\alpha) : \alpha \in I\}$ is fuzzy e-Clopen cover of X. Since X is fuzzy mildly e-compact, there exists a finite subset I_0 of I such that $\bigvee\{f^{-1}(\lambda_\alpha) : \alpha \in I_0\} = 1$. Thus, we have $\bigvee\{\lambda_\alpha : \alpha \in I_0\} = 1$ and Y is fuzzy compact.
The other proofs are similarly. □

Definition 3.28. A fuzzy space X is said to be fuzzy T_1 [12] (resp. fuzzy e-co-T_1) if for each pair of distinct fuzzy points x_α and y_β of X there exist fuzzy open (resp. fuzzy e-Clopen) sets λ and μ containing x_α and y_β, respectively such that $y_\beta \notin \lambda$ and $x_\alpha \notin \mu$.

Theorem 3.29. *If $f : X \to Y$ is a fuzzy totally e-continuous injective function and Y is fuzzy T_1, then X is fuzzy e-co-T_1.*

Proof. Suppose that Y is fuzzy T_1. For any two distinct fuzzy points x_α and y_β in X, there exist fuzzy open sets λ and μ in Y such that $f(x_\alpha) \in \lambda$, $f(y_\beta) \notin \lambda$, $f(x_\alpha) \notin \mu$ and $f(y_\beta) \in \mu$. Then $x_\alpha \in f^{-1}(\lambda)$, $y_\beta \notin f^{-1}(\lambda)$, $x_\alpha \notin f^{-1}(\mu)$ and $y_\beta \in f^{-1}(\mu)$. This shows that X is fuzzy e-co-T_1. □

Definition 3.30. A fuzzy space X is said to be fuzzy e-co-T_2 (= fuzzy e-co-Hausdorff) if for each pair of fuzzy points x_α and y_β such that $x_\alpha \neq y_\beta$ in X, there exist disjoint fuzzy e-Clopen sets λ and μ in X such that $x_\alpha \in \lambda$ and $y_\beta \in \mu$.

Theorem 3.31. *If $f : X \to Y$ is a fuzzy totally e-continuous injective function and Y is fuzzy T_2, then X is fuzzy e-co-T_2.*

Proof. Suppose that Y is fuzzy T_2 space. For any pair of distinct fuzzy points x_α and y_β in X, there exists disjoint fuzzy open sets λ and μ in Y such that $f(x_\alpha) \in \lambda$ and $f(y_\beta) \in \mu$. Since f is fuzzy totally e-continuous function, we have $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy e-Clopen sets in X containing x_α and y_β, respectively. By definition $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = f^{-1}(0) = 0$, and hence X is fuzzy e-co-T_2. □

Definition 3.32. A fuzzy space X is called fuzzy regular (resp. fuzzy e-co-regular) if for each fuzzy open (resp. fuzzy e-Clopen) set λ and each fuzzy point $x_\alpha \notin \lambda$, there exist disjoint fuzzy open sets μ and ρ such that $\lambda \leq \mu$ and $x_\alpha \in \rho$.

Definition 3.33. A fuzzy space X is called fuzzy normal (resp. fuzzy e-co-normal) if for every pair of disjoint fuzzy open (resp. fuzzy e-Clopen) set λ_1 and λ_2 in X, there exist disjoint fuzzy open sets μ and η such that $\lambda_1 \leq \mu$ and $\lambda_2 \leq \eta$.

Theorem 3.34. *If $f : X \to Y$ is a fuzzy totally e-continuous injective fuzzy open function and X is a fuzzy e-co-regular space, then Y is fuzzy regular.*

Proof. Let λ be a fuzzy open set of Y and a fuzzy point $y_\beta \notin \lambda$. Take $y_\beta = f(x_\alpha)$. Since f is fuzzy totally e-continuous, $f^{-1}(\lambda)$ is a fuzzy e-Clopen set of X. Take $\mu = f^{-1}(\lambda)$. We have $x_\alpha \notin \mu$. Since X is fuzzy e-co-regular, there exist disjoint fuzzy open sets η and ρ in X such that $\mu \leq \eta$ and $x_\alpha \in \rho$. We obtain that

$\lambda = f(\mu) \leq f(\eta)$ and $y_\beta = f(x_\alpha) \in f(\rho)$ such that $f(\eta)$ and $f(\rho)$ are disjoint fuzzy open sets of Y. This shows that Y is fuzzy regular. □

Theorem 3.35. *If $f : X \to Y$ is a fuzzy totally e-continuous injective fuzzy open function and X is fuzzy e-co-normal space, then Y is fuzzy normal.*

Proof. Let λ_1 and λ_2 be disjoint fuzzy open sets in Y. Since f is fuzzy totally e-continuous, $f^{-1}(\lambda_1)$ and $f^{-1}(\lambda_2)$ are fuzzy e-Clopen sets in X. Let $\beta = f^{-1}(\lambda_1)$ and $\mu = f^{-1}(\lambda_2)$. We have $\beta \wedge \mu = 0$. Since X is fuzzy e-co-normal, there exist disjoint fuzzy open sets λ and ρ such that $\beta \leq \lambda$ and $\mu \leq \rho$. We obtain that $\lambda_1 = f(\beta) \leq f(\lambda)$ and $\lambda_2 = f(\mu) \leq f(\rho)$ such that $f(\lambda)$ and $f(\rho)$ are disjoint fuzzy open sets. Thus, Y is fuzzy normal. □

Definition 3.36. A graph $G(f)$ of a function $f : X \to Y$ is said to be fuzzy co-e-closed if for each $(x_\alpha, y_\beta) \in (X \times Y)\backslash G(f)$, there exist a fuzzy e-Clopen set λ in X containing x_α and a fuzzy open set μ in Y containing y_β such that $f(\lambda) \wedge \mu = 0$.

Theorem 3.37. *If $f : X \to Y$ is fuzzy totally e-continuous and Y is fuzzy Hausdorff, then $G(f)$ is fuzzy co-e-closed in $X \times Y$.*

Proof. Let $(x_\alpha, y_\beta) \in (X \times Y)\backslash G(f)$, then $f(x_\alpha) \neq y_\beta$. Since Y is fuzzy Hausdorff, there exist fuzzy open sets λ and μ in Y with $f(x_\alpha) \in \lambda$ and $y_\beta \in \mu$ such that $\lambda \wedge \mu = 0$. Since f is fuzzy totally e-continuous, there exists a fuzzy e-Clopen set η in X containing x_α such that $f(\eta) \leq \lambda$. Therefore, we obtain $y_\beta \in \mu$ and $f(\eta) \wedge \mu = 0$. This shows that $G(f)$ is fuzzy co-e-closed. □

Theorem 3.38. *Let $f : X \to Y$ has a fuzzy co-e-closed graph $G(f)$. If f is injective, then X is fuzzy e-T_1.*

Proof. Let x_α and y_β be any two distinct points of X. Then, we have $(x_\alpha, f(y_\beta)) \in (X \times Y)\backslash G(f)$. By definition of fuzzy co-e-closed graph, there exist a fuzzy e-Clopen set λ in X and a fuzzy open set μ in Y such that $x_\alpha \in \lambda$, $f(y_\beta) \in \mu$ and $f(\lambda) \wedge \mu = 0$; hence $\lambda \wedge f^{-1}(\mu) = 0$. Therefore, we have $y_\beta \notin \lambda$. This implies that X is fuzzy e-T_1. □

Theorem 3.39. *Let $f : X \to Y$ has a fuzzy co-e-closed graph $G(f)$. If f is injective fuzzy e-continuous, then X is fuzzy e-T_2.*

Proof. Let x_α and y_β be any two distinct points of X. Then, we have $(x_\alpha, f(y_\beta)) \in (X \times Y)\backslash G(f)$. By definition of fuzzy co-e-closed graph, there exist a fuzzy e-Clopen set λ in X and a fuzzy open set μ in Y such that $x_\alpha \in \lambda$, $f(y_\beta) \in \mu$ and $f(\lambda) \wedge \mu = 0$; since f is fuzzy e-continuous then $f^{-1}(\mu)$ is fuzzy e-open set in X such that $f^{-1}(\mu)(y_\beta) = \mu(f(y_\beta))$ and $\lambda \wedge f^{-1}(\mu) = 0$. Hence X is fuzzy e-T_2. □

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