Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 2, (August 2015), pp. 193–202
ISSN 2093–9310
http://www.afmi.or.kr

Fuzzy divisible semigroups

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Received 29 October 2014; Revised 19 December 2014; Accepted 18 January 2015

Abstract. In this paper we define fuzzy divisible (weakly divisible) S-acts and fuzzy P-injective S-act. We prove that every weakly divisible fuzzy S-act is fuzzy P-injective. We also prove that every fuzzy S-act can be embedded in a divisible fuzzy S-act.

2010 AMS Classification: 03E72, 20M30.

Keywords: Fuzzy divisible S-acts, Fuzzy P-injective S-acts, Right S-cancellative fuzzy S-acts.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by L. A. Zadeh in his definitive paper [28] of 1965, provides a natural framework for generalizing several basic notions of algebra. Thus, for example, Rosenfeld [24] formulated the elements of the theory of fuzzy groups. On the other hand, Kuroki initiated the theory of fuzzy semigroups in his papers [14, 15] (see also [20]), and Ahsan et al. [2] developed a theory of acts over monoids, called an S-system over a monoid S in that paper. Similarly fuzzy subrings and fuzzy ideals of a ring were investigated in [17, 19, 21] and Golan [11] initiated the study of fuzzy modules (see also [22]), while the paper [18] deals with the category of fuzzy modules. The notion of fuzzy semirings and semimodules over them were initially investigated by Ahsan et al. [4] and later investigated by several authors (see, for example [1, 10, 12, 29]). Fo the fuzzification of Γ-semigroups see [8, 9, 26, 27]. It is observed that the notion of divisibility, which plays a very important role in Algebra, has not been studied in a fuzzy context as such, even though some specific results on fuzzy divisible modules can be found in the paper [23].

Let us recall that if G is an additive abelian group then it is well-known that G is closed under the multiplication of integers. Is G also closed under the division by nonzero integer? The answer is clearly no, not in general. This motivates the definition of "divisibility" in abelian groups. Thus let x be an element of an additive abelian group G and let n be a nonzero integer then x is said to be 'divisible by

n' if there exists an element y in G such that $ny = x$. If all the elements of G are divisible by every nonzero integer then G is called a "divisible abelian group". Now an abelian group may be viewed as a $\mathbb{Z}$-module where $\mathbb{Z}$, the ring of integers, may be viewed as an integral domain. Thus the well-known fact in abelian groups that an abelian group G is divisible $\Leftrightarrow$ G is injective may equivalently be stated, using the language of module theory, as follows: Every $\mathbb{Z}$-module is divisible $\Leftrightarrow$ it is injective. This has then led to the concept of injective modules over arbitrary rings. For the definitions, various characterizations and properties of this important concept and the topics of rings, modules and Homological algebra, we refer the readers to the sources [7, 16, 25].

In 1991, Ahsan et al. [3] investigated the concept of divisibility in the more general case of monoids and their representations called S-acts where S is a monoid. Note that an S-act over a monoid S is a non-additive generalization of modules over rings and the theory of S-acts has led to the development of a non-additive and noncommutative Homological Algebra. For a detailed study of this subject we refer to [6] and [13].

Let M be an S-act over a monoid S. Then M is called "divisible" or more strictly speaking "S-divisible" if for all $m \in M$ and $a \in S$ there exists $n \in M$ such that $m = na$.

It has been proved in 6.1 of [3] that the the S-act M is divisible $\Leftrightarrow$ M is P-injective. For the definition and properties of divisible and P-injective S-acts, we refer to the book [6]. One object of the present paper is to initiate the concept of divisibility of S-acts over monoids S in a fuzzy contexts and see whether the investigations of divisible S-acts made so for can be extended to the more general context of fuzzy sets. Further investigations on this theme, we believe, will lead to the applications of these notions to fuzzy automata theory and fuzzy computing.

2. Preliminaries

Throughout this paper, S will denote a monoid, that is, a semigroup with identity 1. A nonempty subset I of S is called right (left) ideal of S if $as \in I$ ($sa \in I$) for all $a \in I$ and $s \in S$. A nonempty subset I of S is called two-sided ideal or simply an ideal of S if it is both a right and a left ideal of S. Let $a \in S$ then the smallest right (left) ideal of S which contains a is called a principal right (left) ideal of S generated by a. By a right S-act M_S we mean a nonempty set M and a function $M \times S \to M$, such that, if ms denotes the image of (m, s) for $m \in M$ and $s \in S$, then the following conditions hold:

$$(ms)t = m(st) \text{ and } m1 = m \text{ for all } m \in M \text{ and } 1, s, t \in S.$$

From the above definition, it follows that the semigroup S is a right S-act over itself, denoted by S_S. More generally, if I is a right ideal of S, then I is a right S-act through the action $(a, s) \mapsto as$ for all $a \in I$ and $s \in S$, which is induced by the multiplication in S. An S-subact N_S of M_S is a subset of M such that $ns \in N$ for all $n \in N$ and $s \in S$. Let A and B be two right S-acts. A mapping $f : A \to B$ is called an S-homomorphism if $f(as) = f(a)s$ for all $a \in A$ and $s \in S$. An S-act with one generating element is called cyclic.

A function f from a nonempty set X to the unit interval $[0,1]$ is called a fuzzy subset of X. For fuzzy subsets λ and μ of X, $\lambda \leq \mu$ means that for all $x \in X$, $\lambda(x) \leq \mu(x)$.

Let M_S be a right S-act. A function $\lambda : M \to [0,1]$ is called a fuzzy subact of M if $\lambda(ms) \geq \lambda(m)$ for all $m \in M$ and $s \in S$.

Lemma 2.1. *Let M_S be a right S-act and A a nonempty subset of M. Then the characteristic function δ_A of A is a fuzzy subact of M if and only if A is an S-subact of M.*

Proof. Straightforward. □

Let $m \in M$ and $t \in (0,1]$. The fuzzy subset m_t of M defined by

$$m_t(x) = \begin{cases} t & \text{if } x = m \\ 0 & \text{otherwise} \end{cases} \text{ for all } x \in M,$$

is called a fuzzy point with support m and value t. A fuzzy point m_t is said to belong to a fuzzy subset λ of M, written as $m_t \in \lambda$ if $\lambda(m) \geq t$.

3. Fuzzy divisibility and fuzzy P-injectivity

Recall that an element x of a right S-act Q is said to be S-divisible in Q if for every $a \in S$, there exists $y \in Q$ such that $x = ya$ (cf. [3]). The right S-act Q is called S-divisible if $Qa = Q$ for all $a \in S$ (cf. [3]). Can we formulate the concept of divisibility in more general context than an S-act over a semigroup or a monoid S? More general algebraic structure than a semigroup or an S-act is a groupoid G and a G-act over a groupoid G. The definition of a G-act over a groupoid G can be formulated as follows:

Let G be a groupoid and let M be a set. Then a G-act may be defined as a mapping $\phi : M \times G \to M$ such that the image of the pair (m, g) $(m \in M, g \in G)$ may be denoted by mg. Thus every groupoid G is a G-act. The following example of a groupoid G (and therefore a G-act) shows that every G-act does not admit the concept of divisibility.

Example 3.1. Let $A = \{1, 0, x\}$ be a groupoid with the following table:

$\cdot$	1	0	x
1	0	0	x
0	1	0	0
x	x	x	0

Then 1 is not divisible by x, because there doesnot exist $y \in A$ such that $1 = yx$.

Note that the above example shows that divisibility cannot be defined in finite structures. However, considering the set $\mathbb{Z}$ of integers with the binary operation of classical multiplication. Then of course $(\mathbb{Z}, \cdot)$ is a groupoid G (and therefore a G-act, where $G = \mathbb{Z}$). Then the G-act $\mathbb{Z}$ does not admit divisibility since for example, if we consider $2 \in \mathbb{Z}$ then there exists no integer y in $\mathbb{Z}$ such that the equation $3y = 2$ has a solution in $\mathbb{Z}$, so 2 is not divisible by 3. Thus the notion of divisibility is not meaningful concept even in some infinite groupoids or G-acts.

Definition 3.2. A fuzzy subact λ of a right S-act M is called weakly divisible if for each $m_t \in \lambda$ and $s \in S$ there exists $y_p \in \lambda$ such that $\lambda(m) = \lambda(ys)$.

A fuzzy subact λ of a right S-act M is called divisible if for each $m_t \in \lambda$ and $s \in S$ there exists $y_p \in \lambda$ such that $m = ys$.

Clearly, every divisible fuzzy subact λ of a right S-act M is weakly divisible but the converse is not true.

Example 3.3. Consider the semigroup $S = \{1, 0, a, b\}$ as a right S-act over itself.

	1	0	a	b
1	1	0	a	b
0	0	0	0	0
a	a	0	a	b
b	b	0	a	b

Then S is not divisible as a right S-act, because for $a \in S$ and $b \in S$ there does not exist $x \in S$ such that $a = xb$.

A fuzzy subset $\lambda : S \to [0, 1]$ is a fuzzy subact of S if and only if

(i) $\lambda(0) \geq \lambda(x)$ for all $x \in S$,

(ii) $\lambda(a) = \lambda(b)$ and

(iii) $\lambda(x) \geq \lambda(1)$ for all $x \in S$.

Let λ be the fuzzy subact of S defined by $\lambda(1) = \lambda(0) = \lambda(a) = \lambda(b) = 1$. Then λ is a weakly divisible fuzzy subact of S, but is not divisible.

Lemma 3.4. *Let M be a right S-act and A be a nonempty subset of M. If A is a divisible S-subact of M, then the characteristic function δ_A of A is divisible.*

Proof. Suppose that A is a divisible S-subact of M. Then by Lemma 2.1, δ_A is a fuzzy subact of M. Let $m_t \in \delta_A$ for some $t \in (0, 1]$ and $s \in S$. Then $\delta_A(m) \geq t > 0$. Thus $\delta_A(m) = 1$ and so $m \in A$. Since A is divisible so there exists $y \in A$ such that $m = ys$. Since $y \in A$, so $y_p \in \delta_A$ for each $p \in (0, 1]$. Thus δ_A is a divisible fuzzy subact of M. □

Corollary 3.5. *If A is a divisible S-subact of M then δ_A is a weakly divisible fuzzy subact of M.*

Proof. By Lemma 3.4, δ_A is divisible. Since every divisible fuzzy subact is weakly divisible, so δ_A is weakly divisible. □

The converse of the Corollary 3.5, is not true.

Example 3.6. Consider the semigroup given in Example 3.3. S is not S–divisible, but δ_S ($\delta_S(1) = \delta_S(0) = \delta_S(a) = \delta_S(b) = 1$) the characteristic function of S is weakly divisible.

Proposition 3.7. *Let M be a right S-act and A be a nonempty subset of M. Then δ_A is divisible fuzzy subact of M if and only if A is divisible.*

Proof. Suppose that δ_A is divisible fuzzy subact of M. Then by Lemma 2.1, A is an S-subact of M. Let $a \in A$ and $s \in S$ then $a_t \in \delta_A$ for all $t \in (0, 1]$, so there exists $x_p \in \delta_A$ such that $a = xs$. Since $x_p \in \delta_A$, so $\delta_A(x) \geq p > 0$, that is $\delta_A(x) = 1$. Hence $x \in A$. This shows that A is divisible. If A is divisible S-subact of M then by Lemma 3.4, δ_A is divisible fuzzy subact of M. □

Definition 3.8. Let M be a right S-act and λ a fuzzy subact of M, then the pair (M, λ) is called a fuzzy S-act.

Definition 3.9. Let (M, λ) and (N, μ) be two fuzzy S-acts. An S-homomorphism $f : M \to N$ is called a fuzzy S-homomorphism from $(M, \lambda) \to (N, \mu)$ if $\mu(f(m)) \geq \lambda(m)$ for all $m \in M$.

A fuzzy S-act (M, λ) is called a retract of a fuzzy S-act (N, μ) if there exist fuzzy S-homomorphisms $p : (N, \mu) \to (M, \lambda)$ and $q : (M, \lambda) \to (N, \mu)$ such that $p \circ q = 1_M$.

Definition 3.10. A fuzzy S-act (M, λ) is called divisible (weakly divisible) fuzzy S-act if M is divisible S-act and λ is divisible (weakly divisible) fuzzy subact of M.

Proposition 3.11 ([3]). *A retract of an S-divisible S-act is S-divisible.*

Proposition 3.12. *A retract of a divisible fuzzy S-act is divisible.*

Proof. Let (M, λ) be a divisible fuzzy S-act and (N, μ) be a retract of (M, λ). Then there exist fuzzy S-homomorphisms $f : (N, \mu) \to (M, \lambda)$ and $g : (M, \lambda) \to (N, \mu)$ so that $g \circ f = 1_N$.

By Proposition 3.11, N is S-divisible S-act. Let $x_t \in \mu$ and $s \in S$. Then $\mu(x) \geq t$. As $f(x) \in M$ and $\lambda(f(x)) \geq \mu(x) \geq t$, so $f(x)_t \in \lambda$. Since λ is divisible fuzzy subact of M so there exists $y_p \in \lambda$ such that $f(x) = ys$. Thus $g(f(x)) = g(ys) \Rightarrow x = g(y)s$.

Also $\mu(g(y)) \geq \lambda(y) \geq p \Rightarrow (g(y))_p \in \mu$. This shows that $\mu$is a divisible fuzzy subact of N. Hence (N, μ) is a divisible fuzzy S-act. □

Definition 3.13. Let f be a mapping from a set X into a set Y and μ, ν be fuzzy subsets of X and Y respectively. The fuzzy subsets $f(\mu)$ and $f^{-1}(\nu)$ of Y and X respectively are defined by:

$$f(\mu)(y) = \begin{cases} \bigvee \{\mu(x) : x \in X \text{ and } f(x) = y\} & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$.

$f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$ are called, respectively, the image of μ under f and the pre image of ν under f.

Proposition 3.14. *Let M and N be S-acts and f an S-homomorphism from M into N. Then*

a) If λ is a fuzzy subact of M then $f(\lambda)$ is a fuzzy subact of N.
b) If μ is a fuzzy subact of N then $f^{-1}(\mu)$ is a fuzzy subact of M.

Proof. a) Let $n \in N$ then

$$f(\lambda)(n) = \begin{cases} \bigvee \{\lambda(m) : m \in M \text{ and } f(m) = n\} & \text{if } f^{-1}(n) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

If $f(\lambda)(n) = 0$ then $f(\lambda)(ns) \geq f(\lambda)(n)$.
If $f(\lambda)(n) = \bigvee_{m \in f^{-1}(n)} \lambda(m)$, then $f(m) = n$ and so $f(ms) = f(m)s = ns$.
Thus $f^{-1}(ns) \neq \phi$ and $ms \in f^{-1}(ns)$ for all $m \in f^{-1}(n)$. Hence
$f(\lambda)(ns) = \bigvee_{m \in f^{-1}(ns)} \lambda(m) \geq \bigvee_{x \in f^{-1}(n)} \lambda(xs) \geq \bigvee_{x \in f^{-1}(n)} \lambda(x) = f(\lambda)(n)$.
Thus $f(\lambda)(ns) \geq f(\lambda)(n)$. This shows that $f(\lambda)$ is a fuzzy subact of N.

b) Let μ be a fuzzy subact of N. Then for all $m \in M$ and $s \in S$,
$f^{-1}(\mu)(ms) = \mu(f(ms)) = \mu(f(m)s) \geq \mu(f(m)) = f^{-1}(\mu)(m)$.
Hence $f^{-1}(\mu)$ is a fuzzy subact of M. □

Lemma 3.15. *Let A be a subact of a right S-act M and η be a fuzzy subact of A. Then the fuzzy subset $\widetilde{\eta}$ of M defines by*

$$\widetilde{\eta}(m) = \begin{cases} \eta(m) & \text{if } m \in A \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy subact of M.

Proof. Let $m \in M$ if $\widetilde{\eta}(m) = 0$ then $\widetilde{\eta}(ms) \geq \widetilde{\eta}(m)$. If $\widetilde{\eta}(m) = \eta(m)$, then $m \in A$ and so $ms \in A$ for all $s \in S$. Hence $\widetilde{\eta}(ms) = \eta(ms) \geq \eta(m) = \widetilde{\eta}(m)$. Hence $\widetilde{\eta}$ is a fuzzy subact of M. □

Recall that a right S-act Q is called PM-injective (M is a fixed right S-act) if each S-homomorphism from a cyclic S-subact aS ($a \in M$) of M to Q extends to an S-homomorphism from M to Q. In particular, Q is called P-injective S-act if Q is PS-injective (cf. [3]).

Definition 3.16. Let M be a P-injective S-act. A fuzzy S-act (M, λ) is called fuzzy P-injective if each fuzzy S-homomorphism $f : (aS, \mu) \to (M, \lambda)$ can be extended to a fuzzy S-homomorphism $\psi : (S, \widetilde{\mu}) \to (M, \lambda)$ for all $a \in S$.

Theorem 3.17 ([3])**.** *If Q is S-divisible then Q is P-injective.*

Theorem 3.18. *Every weakly divisible fuzzy S-act is fuzzy P-injective.*

Proof. Let (M, μ)be a weakly divisible fuzzy S-act. Then M is divisible right S-act and μ is a weakly divisible fuzzy subact of M. Then M is a P-injective S-act. Let $f : (aS, \lambda) \to (M, \mu)$ be a fuzzy S-homomorphism that is $f : aS \to M$ is an S-homomorphism and $\mu(f(x)) \geq \lambda(x)$ for all $x \in aS$.

Since M is P-injective so there exists an S-homomorphism $\phi : S \to M$ which extends f. This homomorphism ϕ is defined as if $f(a) = x \in M$ then there exists $y \in M$ such that $x = ya$. Define $\phi(1) = y$ and $\phi(s) = ys$. We show that ϕ is a fuzzy S-homomorphism, that is $\mu(\phi(s)) \geq \widetilde{\lambda}(s)$. If $\lambda(s) = 0$ then $\mu(\phi(s)) \geq \widetilde{\lambda}(s)$. If $\widetilde{\lambda}(s) = \lambda(s)$ then $s \in aS$, so $\phi(s) = f(s)$. Hence $\mu(\phi(s)) = \mu(f(s)) \geq \lambda(s) \geq \widetilde{\lambda}(s)$. Which completes the proof. □

Definition 3.19 ([3])**.** Let A be a right S-act. A is said to be right S-cancellative if A has the following property:

$$xs = x's \text{ for } x,\ x' \in A \text{ and } s \in S \text{ implies that } x = x'.$$

Definition 3.20. A fuzzy subact λ of a right S-act M is called right S-cancellative if $\lambda(xs) = \lambda(x's) \Longrightarrow \lambda(x) = \lambda(x')$ for all $x,\ x' \in M$ and $s \in S$.

It is not necessary that if M is a right S-cancellative then every fuzzy subact of M is right S-cancellative.

Example 3.21. Let $\mathbb{N}$ be the set of natural numbers, then $\mathbb{N}$ under usual multiplication of numbers is a cancellative semigroup. Consider $\mathbb{N}$ as a right $\mathbb{N}$-cancellative right $\mathbb{N}$-act. Consider the fuzzy subact λ of $\mathbb{N}$ defined by:

$$\lambda(x) = \begin{cases} 1 \text{ if } x \in 4\mathbb{N}, \\ 1/2 \text{ if } x \in 2\mathbb{N} - 4\mathbb{N}, \\ 0 \quad \text{otherwise} \end{cases}$$

Then λ is not a right $\mathbb{N}$-cancellative, because $\lambda(2.2) = \lambda(4.2) = 1$, but $\lambda(2) = 1/2 \neq 1 = \lambda(4)$.

Example 3.22. Consider the semigroup $S = \{0, 1, a, b, c\}$

	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	a
b	0	b	a	a	a
c	0	c	a	a	a

Then S is commutative non-cancellative semigroup. Consider S as a right S-act. A fuzzy subset λ of S is a fuzzy subact of S if and only if

(i) $\lambda(0) \geq \lambda(x)$ for all $x \in S$,

(ii) $\lambda(a) \geq \lambda(x)$ for all non zero x in S and

(iii) $\lambda(x) \geq \lambda(1)$ for all x in S.

Consider the fuzzy subact λ which maps every element of S on 1. Then λ is right S-cancellative.

Definition 3.23. A fuzzy S-act (M, λ) is called right S-cancellative if M is right S-cancellative S-act and λ is a right S-cancellative fuzzy subact of M.

Theorem 3.24 ([3])**.** *If A is a retract of a right S-cancellative S-act B, then A is right S-cancellative.*

Theorem 3.25. *If (A, λ) is a retract of a right S-cancellative fuzzy S-act (B, μ), then (A, λ) is right S-cancellative.*

Proof. Let (A, λ) be a retract of (B, μ) then there exist fuzzy S-homomorphisms $f : (A, \lambda) \longrightarrow (B, \mu)$ and $g : (B, \mu) \longrightarrow (A, \lambda)$ such that $g \circ f = 1_A$. By Theorem 3.24, A is right S-cancellative. We show that λ is a right S-cancellative. First we show that $\lambda(x) = \mu(f(x))$ for all $x \in A$. Let $x \in A$, then $g(f(x)) = x$. Hence $\lambda(x) = \lambda(g(f(x))) \geq \mu(f(x)) \geq \lambda(x)$. Thus

$\lambda(x) = \mu(f(x)) \qquad (*)$

Let $x, x' \in A$ and $s \in S$ such that $\lambda(xs) = \lambda(x's)$. Then by $(*)$

$\mu(f(xs)) = \mu(f(x's))$

$\Longrightarrow \mu(f(x)s) = \mu(f(x')s)$

$\Longrightarrow \mu(f(x)) = \mu(f(x'))$ since μ is right S-cancellative

By $(*)$ then $\lambda(x) = \lambda'(x)$. Thus λ is a right S-cancellative. Hence (A, λ) is a right S-cancellative fuzzy S-act. $\square$

4. Embedding and arbitrary fuzzy S-act into a fuzzy divisible S-act

The following construction is taken from [3].

Let A be a right S-act. Consider the set $A \times S = \{(a, s) : a \in A \text{ and } s \in S\}$.

On this set define S-action by $(a, s)t = (at, s)$ for all $t \in S$. Then the set $A \times S$, together with this S-action, is a right S-act, which is denoted by $Q(A)$. Now we define a relation on $Q(A)$ as follows:

$$(x, a) \equiv (x', a') \Leftrightarrow xa' = x'a.$$

Lemma 4.1 ([3]). *If S is a commutative monoid and A is a right S -cancellative S-act, then the above relation $\equiv$ is an S -congruence on $Q(A)$.*

We may construct a factor S-act $\overline{Q(A)} = Q(A)/\equiv$. For each element $(x, a) \in Q(A)$, we shall denote by $\overline{(x, a)}$ the corresponding element of $\overline{Q(A)}$. Moreover, the S-action on $\overline{Q(A)}$ is defined by

$$\overline{(x, a)}s = \overline{(xs, a)} \text{ for all } s \in S.$$

Proposition 4.2 ([3]). *Let S be a commutative monoid and A a right S-cancellative S-act. Then $\overline{Q(A)}$ is S-divisible and A is embedded in $\overline{Q(A)}$.*

Let (A, λ) be a fuzzy S-act. Then $A \times S$ is a right S-act under the S-action

$$(a, s)t = (at, s) \text{ for all } s \in S$$

with the help of λ we define a fuzzy subact λ_1 of $A \times S$ by

$$\lambda_1 : A \times S \to [0, 1]$$
$$\lambda_1((a, s)) = \lambda(a).$$

Lemma 4.3. *λ_1 is a fuzzy subact of $A \times S$.*

Proof. Let $(a, s) \in A \times S$ and $t \in S$. Then
$\lambda_1((a, s)t) = \lambda_1((at, s) = \lambda(at) \geq \lambda(a) = \lambda_1((a, s))$.
So λ_1 is a fuzzy subact of $A \times S$. □

Lemma 4.4. *If S is a commutative monoid and A is a right S-cancellative S-act and λ be a fuzzy subact of A then*

$$\lambda_2 : \overline{Q(A)} \to [0, 1] \text{ defined by}$$
$$\lambda_2(\overline{(x, a)}) = \bigvee_{(y,b)\in\overline{(x,a)}} \lambda_1((y, b)) = \bigvee_{(y,b)\in\overline{(x,a)}} \lambda(y)$$

is a fuzzy subact of $\overline{Q(A)}$.

Proof. Let
$\overline{(x, a)} = \overline{(x_1, a_1)}$
$\lambda_2(\overline{(x, a)}) = \bigvee_{(y,b)\in\overline{(x,a)}} \lambda(y)$.
As $(y, b) \in \overline{(x, a)} = \overline{(x_1, a_1)} \Leftrightarrow (y, b) \in \overline{(x_1, a_1)}$.
Thus
$\lambda_2(\overline{(x, a)}) = \bigvee_{(y,b)\in\overline{(x,a)}} \lambda(y) = \lambda_2(\overline{(x_1, a_1)})$.
Hence λ_2 is well defined.
Furthermore $\overline{(x, a)}s = \overline{(xs, a)}$. If $(y, b) \in \overline{(x, a)}$ then
$(y, b) \equiv (x, a)$
$\Rightarrow ya = xb \Rightarrow yas = xbs \Rightarrow (ys)a = (xs)b$
$\Rightarrow (ys, b) \equiv (xs, a) \Rightarrow (ys, b) \in \overline{(x, a)}s$.
Thus

$$\bigvee_{(z,c)\in\overline{(x,a)}s}\lambda(z) \geq \bigvee_{(y,b)\in\overline{(x,a)}}\lambda(ys) \geq \bigvee_{(y,b)\in\overline{(x,a)}}\lambda(y)$$

$\Rightarrow \lambda_2(\overline{(x,a)}s) \geq \lambda_2(\overline{(x,a)})$.

Thus λ_2 is a fuzzy subact of $\overline{Q(A)}$. □

Lemma 4.5. *λ_2 is divisible subact of $\overline{Q(A)}$.*

Proof. Let $(\overline{(x,a)})_t \in \lambda_2$ and $s \in S$ Then there exists $\overline{(x,as)} \in \overline{Q(A)}$ such that $\overline{(x,a)} = \overline{(x,as)}s$, because $\overline{(x,as)}s = \overline{(xs,as)}$ and $x(as) = (xs)a$. Also $\lambda_2(\overline{(x,a)}) \geq t$, because $(\overline{(x,a)})_t \in \lambda_2$. Thus

$$\lambda_2(\overline{(x,a)}) = \bigvee_{(y,b)\in\overline{(x,a)}}\lambda(y) \geq t.$$

But

$$\lambda_2(\overline{(x,as)}) = \bigvee_{(z,c)\in\overline{(x,as)}}\lambda(z) \geq \bigvee_{(y,bs)\in\overline{(x,a)}}\lambda(z) \geq t$$

$$\left(\begin{array}{l}\text{since if } (y,b)\in\overline{(x,a)} \text{ then } (y,b)\equiv(x,a)\Rightarrow ya=xb\\ \Rightarrow yas=xbs\Rightarrow(x,as)\equiv(y,bs)\\ \Rightarrow (y,bs)\in\overline{(x,as)}\end{array}\right)$$

$\Rightarrow (\overline{(x,as)})_t \in \lambda_2$.

Thus λ_2 is divisible subact of $\overline{Q(A)}$. Hence $(\overline{Q(A)}, \lambda_2)$ is a divisible fuzzy S-act. □

Theorem 4.6. *Any fuzzy S-act (A,λ) can be embedded into a divisible fuzzy S-act $\left(\overline{Q(A)}, \lambda_2\right)$.*

Proof. The mapping $q : A \to \overline{Q(A)}$ defined by $q(x) = \overline{(x,1)}$ is an S-monomorphism.

Also

$$\lambda_2(q(x)) = \lambda_2(\overline{(x,1)}) = \bigvee_{(y,a)\in\overline{(x,1)}}\lambda(y)$$

$\geq \lambda(x)$ because $(x,1) \in \overline{(x,1)}$.

Thus q is a fuzzy S-homomorphism. □

Acknowledgements. The Authors are very thankful to the learned referees and Editor in Chief Young Bae Jun for their suggestions to improve the present paper.

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