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Some types of compactness in double topological spaces

A. Kandil, S. A. El-Sheikh, M. M. Yakout, Shawqi A. Hazza

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Abstract. The aim of this paper is to construct the basic concepts related to compactness in double (intuitionistic) topological spaces. Here we introduce the concepts of double compact (D-compact), double compactness modulo double ideal ($\underline{\mathcal{I}}$-D-compactness), double quasi-H-closed (DQHC set), double quasi-H-closed modulo double ideal ($\underline{\mathcal{I}}$ -DQHC), DC-compact, $\underline{\mathcal{I}}$-DC-compact and obtain several preservation properties and some characterizations concerning compactness in these concepts.

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Keywords: double sets; double topology; D-ideals; D-compactness; $\underline{\mathcal{I}}$-D-compactness; DQHC; $\underline{\mathcal{I}}$-DQHC; DC-compact; $\underline{\mathcal{I}}$-DC-compact

Corresponding Author: Shawqi Ahmed Hazza (shawqialalimi@yahoo.com)

1. Introduction

After Atanassov [1, 2, 3, 4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets, Çoker [6] generalized topological structures in fuzzy topological spaces to intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. The concept of intuitionistic sets which are under the classical intuitionistic fuzzy sets was first given by Çoker in [5]. He studied topology on intuitionistic sets in [7]. In 2007, Kandil et. al [17] introduced the concept of Flou set. This is a discrete form of intuitionistic fuzzy sets, where all the ordinary sets are entirely the crisp sets. In this paper, we follow the terminology of Rodabaugh [9] that double set is more appropriate name than intuitionistic (Flou) set, and therefore, adopted the term double set for the intuitionistic(Flou) set and double topology for the intuitionistic (Flou) topology. Kandil [17] also introduced the concept of double topological spaces with double sets and investigated basic properties of continuous functions. He also examined separation axioms in double topological spaces. In 2010, Kandil [14] obtained a new double topology form the old (X, η), constructed by use of a double ideal (D-ideal, for short) on X, and described as follows.

Let $\underline{\mathcal{I}} \in DI(X)$ and let (X, η) be a double topological space. Consider the local function of $\underline{A}$ with respect to $\underline{\mathcal{I}}$ and η, denoted by $\underline{A}^*(\eta, \underline{\mathcal{I}})$, given by $x_t \in \underline{A}^* \Leftrightarrow (\underline{O}_{x_t} \cap \underline{H}^c) q \underline{A} \ \forall \ \underline{H} \in \underline{\mathcal{I}} \ \forall \ \underline{O}_{x_t} \in N^q_\eta(x_t)$. Then the operator $cl^* : D(X) \to D(X)$, defined by $cl^*(\underline{A}) = \underline{A} \cup \underline{A}^*$, is a D-closure operator and hence it generates a double topology $\eta^*(\underline{\mathcal{I}}) = \{\underline{A} \in D(X) : cl^*(\underline{A}^c) = \underline{A}^c\}$, which is finer than η. A double open base β for the double topology $\eta^*(\underline{\mathcal{I}})$ on X is given by $\beta(\eta, \underline{\mathcal{I}}) = \{\underline{G} \backslash \underline{A} : \underline{G} \in \eta, \ \underline{A} \in \underline{\mathcal{I}}\}$. In 2009, Kandil and et. al.[16] introduced the notion of CD-compact topological spaces(Flou-compact topological space), and studied some fundamental properties of this notion.

2. Preliminaries

The purpose of this section is merely to recall some known results concerning ideal, compactness, double sets, double ideals and double compact spaces.
For more information see [11, 12, 13, 14, 16, 17, 19, 22].

Definition 2.1 ([11])**.** A nonempty collection $\mathcal{I}$ of subsets of a nonempty set X is said to be an ideal on X, if it satisfies the following two conditions:
(i) $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$ (hereditary),
(ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ (finite additivity).

Definition 2.2 ([8])**.** A topological space (X, τ) is said to be compact or τ-compact iff every open cover of X has a finite sub-cover.

Definition 2.3 ([19])**.** Let $\mathcal{I}$ be an ideal on a topological space (X, τ). A cover $\{G_\alpha : \alpha \in \Omega\}$ of X is said to be an $\mathcal{I}$-cover if there exists a finite subset Ω_0 of Ω such that $\{G_\alpha : \alpha \in \Omega_0\}$ covers X except, perhaps, for some subset which belongs to the ideal $\mathcal{I}$, i.e. $X \backslash \bigcup_{\alpha \in \Omega_0} G_\alpha \in \mathcal{I}$.

Definition 2.4 ([19])**.** A topological space (X, τ) with an ideal $\mathcal{I}$ is said to be $\mathcal{I}$-compact or compact modulo $\mathcal{I}$, if every open covering of X is an $\mathcal{I}$-cover.

Definition 2.5 ([18])**.** A topological space (X, τ) is said to be a quasi H - closed (QHC, for short) if for every open cover $\gamma (= \{G_\alpha : \alpha \in \Lambda\} \subseteq \tau)$ of X, there exists a finite sub-collection $\gamma^* (= \{G_i : i = 1, 2, 3, ..., n\})$ of γ such that $X = \cup_{i=1}^n cl_\tau G_i$. A Hausdorff quasi H - closed space is called H - closed (HC, for short).

Definition 2.6 ([21])**.** A topological space (X, τ) is said to be C-compact if for every closed set F and every τ-open cover γ of F, there exists a finite sub-collection $\{G_1, G_2, G_3, ..., G_n\}$ of γ such that $F \subseteq \cup_{i=1}^n cl_\tau(G_i)$.

Definition 2.7 ([10])**.** Let $\mathcal{I}$ be an ideal on a topological space (X, τ). A topological space (X, τ) is said to be $\mathcal{I}$-C-compact if for every closed set F and every τ-open cover γ of F, there exists a finite sub-collection $\{G_1, G_2, G_3, ..., G_n\}$ of γ such that $F \backslash \cup_{i=1}^n cl_\tau(G_i) \in \mathcal{I}$.

Definition 2.8 ([17])**.** Let X be a nonempty set:
(1) A double set $\underline{A}$ is an ordered pair $\underline{A} = (A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.
(2) $D(X) = \{(A_1, A_2) : (A_1, A_2) \in P(X) \times P(X), \ A_1 \subseteq A_2\}$ is the family of all double sets on X.

(3) Let $x \in X$. Then, the double sets $x_{0.5} = (\phi, \{x\})$ and $x_1 = (\{x\}, \{x\})$ are said to be double points in X.
$X_p = \{x_t : x \in X,\ t \in \{0.5, 1\}\}$ is the set of all double points of X.
(4) $x_1 \in \underline{A}$ iff $x \in A_1$, and $x_{0.5} \in \underline{A}$ iff $x \in A_2$, i.e. $\underline{A} = (\cup\{x_1 : x_1 \in \underline{A}\}) \cup (\cup\{x_{0.5} : x_{0.5} \in \underline{A}\})$
(5) Let $\eta_1, \eta_2 \subseteq P(X)$. Then the double product of η_1 and η_2, denoted by $\eta_1 \hat{\times} \eta_2$, is defined by $\eta_1 \hat{\times} \eta_2 = \{(A_1, A_2) : (A_1, A_2) \in \eta_1 \times \eta_2,\ A_1 \subseteq A_2\}$.
(6) The double set $\underline{X} = (X, X)$ is called the universal double set.
(7) The double set $\underline{\phi} = (\phi, \phi)$ is called the empty double set.
(8) The double set $\underline{A} = (A_1, A_2)$ is said to be a finite double set if A_2 is finite set.
(9) The double set $\underline{A} = (A_1, A_2)$ is said to be a countable if and only if A_2 is countable.
(10) The double set $\underline{A} = (A_1, A_2)$ is said to be a crisp double set if and only if $A_1 = A_2$.

Note that a double set in the sense of Çoker [5] is the form $\underline{A} = (A_1, A_2) \in P(X)$, where $A_1 \cap A_2 = \phi$. But $\underline{A} = (A_1, A_2) \in P(X)$ is a double set in the sense of Kandil et. al [17], where $A_1 \cap A_2$. then $\underline{A} = (A_1, A_2)$ is a double set in the sense of Çoker if and only if $\underline{A} = (A_1, A_2^c)$ is a double set in the sense of Kandil. And one can see that a one to one correspondence mapping between the two types. On the other hand, Kandil's notion simplify the concepts, specially in the case of intuitionistic fuzzy points or double fuzzy point see [20].

Definition 2.9 ([17])**.** Let $\underline{A} = (A_1, A_2)$, $\underline{B} = (B_1, B_2) \in D(X)$. Then:
(1) $\underline{A} = \underline{B} \Leftrightarrow A_i = B_i,\ i = 1, 2$.
(2) $\underline{A} \subseteq \underline{B} \Leftrightarrow A_i \subseteq B_i,\ i = 1, 2$.
(3) $\underline{A} \cap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$ and $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.
(4) If $\{A_\alpha : \alpha \in \Lambda\} \subseteq D(X)$ such that $\underline{A}_\alpha = (A_{1\alpha}, A_{2\alpha})$, then
$\bigcup_{\alpha\in\Lambda} \underline{A}_\alpha = (\bigcup_{\alpha\in\Lambda} A_{1\alpha}, \bigcup_{\alpha\in\Lambda} A_{2\alpha})$ and $\bigcap_{\alpha\in\Lambda} \underline{A}_\alpha = (\bigcap_{\alpha\in\Lambda} A_{1\alpha}, \bigcap_{\alpha\in\Lambda} A_{2\alpha})$.
(5) $\underline{A}^c = (A_2^c, A_1^c)$, where $\underline{A}^c$ is the complement of $\underline{A}$.
(6) $\underline{A} \backslash \underline{B} = \underline{A} \cap \underline{B}^c$.

Proposition 2.10 ([17])**.** *$(D(X), \cup, \cap, ^c)$ is a Morgan Algebra.*

Definition 2.11 ([17])**.** Two double sets $\underline{A}$ and $\underline{B}$ are said to be quasi-coincident, denoted by $\underline{A} q \underline{B}$, if and only if $A_1 \cap B_2 \neq \phi$ or $A_2 \cap B_1 \neq \phi$. $\underline{A}$ is not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \bar{q} \underline{B}$, if and only if $A_1 \cap B_2 = \phi$ and $A_2 \cap B_1 = \phi$.

Theorem 2.12 ([17])**.** *Let $\underline{A}, \underline{B}, \underline{C} \in D(X)$ and $x_t, y_r \in X_p$. Then:*

(1) $\underline{A} q \underline{B} \Rightarrow \underline{A} \cap \underline{B} \neq \underline{\phi}$,
(2) $\underline{A} q \underline{B} \Leftrightarrow \exists x_t \in \underline{A}$ *such that* $x_t q \underline{B}$,
(3) $\underline{A} \bar{q} \underline{B} \Leftrightarrow \underline{A} \subseteq \underline{B}^c$,
(4) $x_t \bar{q} \underline{A} \Leftrightarrow x_t \in \underline{A}^c$,
(5) $\underline{A} \subseteq \underline{B} \Leftrightarrow x_t \in \underline{A}$ *implies* $x_t \in \underline{B} \Leftrightarrow x_t q \underline{A}$ *implies* $x_t q \underline{B}$,
(6) $\underline{A} \bar{q} \underline{A}^c$,
(7) $\underline{A} = \bigcup\{x_t : x_t \in \underline{A}\} = \bigcup\{x_t : x_t \bar{q} \underline{A}^c\}$.

Definition 2.13 ([17])**.** Let X be a nonempty set. Then:

(1) $\eta \subseteq D(X)$ is called a double topology on X if the following axioms are satisfied:
(*i*) $\underline{\phi}$, $\underline{X} \in \eta$,
(*ii*) If $\underline{A}$, $\underline{B} \in \eta$, then $\underline{A} \cap \underline{B} \in \eta$ and
(*iii*) If $\{\underline{A}_\alpha : \alpha \in \Lambda\} \subseteq \eta$, then $\cup_{\alpha\in\Lambda}\underline{A}_\alpha \in \eta$.
(2) If $\underline{G} \in \eta$, then $\underline{G}$ is called an open double set and $\underline{G}^c$ is called a closed double set.
(3) The family of all closed double sets is denoted by $\eta^c = \{\underline{F} : \underline{F}^c \in \eta\}$.
(4) A double set $\underline{O}_{x_t}$ is called a neighborhood (nbd, for short) of the double point x_t if and only if $x_t \in \underline{O}_{x_t} \in \eta$. The family of all nbd of x_t denoted by $N_\eta(x_t)$. Also, $\underline{O}_{x_t}$ is called a quasi neighborhood (q-nbd, for short) of the double point x_t if and only if $x_t q \underline{O}_{x_t} \in \eta$. The family of all q-nbd of x_t denoted by $N^q_\eta(x_t)$.
(5) If $\underline{A} \in D(X)$. Then
(*i*) The closure of $\underline{A}$, denoted by $cl_\eta(\underline{A})$, is defined by $cl_\eta(\underline{A}) = \cap\{\underline{F} : \underline{A} \subseteq \underline{F} \in \eta^c\}$.
(*ii*)The interior of $\underline{A}$, denoted by $int_\eta(\underline{A})$, is defined by $int_\eta(\underline{A}) = \cup\{\underline{G} : \underline{G} \in \eta, \underline{G} \subseteq \underline{A}\}$.
(6) A double set $\underline{A}$ is called a double dense in X iff $cl_\eta(\underline{A}) = \underline{X}$.

Theorem 2.14 ([17]). *Let* (X, η) *be a double topological space and* $\underline{A} \in D(X)$. *Then* $int_\eta(\underline{A}) = (cl_\eta(\underline{A}^c))^c$.

Theorem 2.15 ([17]). *Let* (X, τ) *be a topological space, and let* $\underline{A} \in D(X)$. *Then* $\tau\hat{\times}\tau$ *is a double topology on* X *and* $cl_{\tau\hat{\times}\tau}(\underline{A}) = (cl_\tau(A_1), cl_\tau(A_2))$.

Theorem 2.16 ([17]). *Let* (X, η) *be a double topological space. Then*
(1) $\pi_1 = \{A_1 : (A_1, A_2) \in \eta\}$,
(2) $\pi_2 = \{A_2 : (A_1, A_2) \in \eta\}$ *and*
(3) $\pi_3 = \{A : (A, X) \in \eta\}$ *are topologies on* X.

Definition 2.17 ([15]). A double topological space (X, η) is said to be a double Hausdorff(DT_2, for short) if $\forall x_t, y_r \in X_p, x_t\bar{q}y_r$ there exists $\underline{O}_{x_t}, \underline{O}_{y_r} \in \eta$ such that $\underline{O}_{x_t}\bar{q}\underline{O}_{y_r}$.

Definition 2.18 ([16]). Let(X, η) be a double topological space and let $\underline{A} \in D(X)$. A collection $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq D(X)$ is said to be a double cover(D-cover, for short) of $\underline{A}$ if $\underline{A} \subseteq \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. If $\underline{\gamma} \subseteq \eta$, then $\underline{\gamma}$ is called double open cover(D-open cover, for short).

Definition 2.19 ([16]). A double topological space (X, η) is said to be a CD-compact space if for every double closed set $\underline{F}$ and for every D-open cover $\underline{\gamma}$ of $\underline{F}$ has a finite sub-cover.

Definition 2.20 ([14]). Let X be a nonempty set. A nonempty collection $\underline{\mathcal{I}} \subseteq D(X)$ is said to be a double ideal(D-ideal) on X, if it satisfies the following two conditions:
(i): $\underline{A} \in \underline{\mathcal{I}}$ and $\underline{B} \subseteq \underline{A} \Rightarrow \underline{B} \in \underline{\mathcal{I}}$ (hereditary),
(ii): $\underline{A} \in \underline{\mathcal{I}}$ and $\underline{B} \in \underline{\mathcal{I}} \Rightarrow \underline{A} \cup \underline{B} \in \underline{\mathcal{I}}$ (finite additivity).

Example 2.21 ([14]). Let X be a nonempty set:

(1) $\{\underline{\phi}\}$ and $D(X)$ are trivial examples of double ideals on X.
(2) $\underline{\mathcal{I}}_f$, the double ideal of all finite double subsets in X,
(3) $\underline{\mathcal{I}}_c$, the double ideal of all countable double subsets in X,
(4) $\underline{\mathcal{I}}_n$, the double ideal of all nowhere dense double subsets in X,
(5) $\underline{\mathcal{I}}^q_{x_t} = \{\underline{A} : \underline{A} \in D(X),\ x_t\bar{q}\underline{A}\}$ is a quasi excluded point double ideal on X,
(6) $\underline{\mathcal{I}}^+ = \{(\phi, A) : A \in P(X)\}$.

The set of all double ideals on X is denoted by $DI(X)$.

Proposition 2.22 ([13]). *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X. Then the double product $\mathcal{I}\hat{\times}\mathcal{J}(= \{(A, B) : (A, B) \in \mathcal{I} \times \mathcal{J}, A \subseteq B\}$ is a D-ideal on X.*

Proposition 2.23 ([14]). *Let X be a nonempty set and $\underline{\mathcal{I}} \in DI(X)$. Then*
(1) *$\mathcal{I}^1 = \{A_1 : (A_1, A_2) \in \underline{\mathcal{I}}\}$, and*
(2) *$\mathcal{I}^2 = \{A_2 : (A_1, A_2) \in \underline{\mathcal{I}}\}$ are ideals on X.*

Definition 2.24 ([13]). Let (X, η) be a double topological space and let $\underline{\mathcal{I}}$ be a D-ideal on X. Then η is said to be compatible with $\underline{\mathcal{I}}$, denoted by $\eta \sim \underline{\mathcal{I}}$ if $\underline{A} \cap \underline{A}^* = \underline{\phi}$, then $\underline{A} \in \underline{\mathcal{I}}$.

Theorem 2.25 ([13]). *Let (X, η) be a double topological space and let $\underline{\mathcal{I}}$ be a D-ideal on X. If $\eta \sim \underline{\mathcal{I}}$, then $\beta(\eta, \underline{\mathcal{I}}) = \eta^*(\underline{\mathcal{I}})$.*
For more information see [13, 14].

3. D-Compactness modulo D-ideal

In this section, we introduce and study the idea of a new type of D-compactness, defined in terms of a D-ideal in a double topological space (X, η). Calling it $\underline{\mathcal{I}}$-D-compactness, we investigate its relation with compactness, among other things.

Definition 3.1. A double topological space (X, η) is said to be D-compact space if every D-open cover $\underline{\gamma}$ of X has a finite sub-cover.

Theorem 3.2. *Let (X, η) be a D-compact space. Then every crisp double closed set is a C-set.*

Proof. Let (X, η) be a D-compact space, $\underline{A}$ be a crisp double closed set and let the collection $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta$ be a D-open cover of $\underline{A}$, i.e. $\underline{A} \subseteq \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. Then $\underline{A} \cup \underline{A}^c \subseteq \cup_{\alpha\in\Lambda}\underline{G}_\alpha \cup \underline{A}^c \Rightarrow \underline{X} = \cup_{\alpha\in\Lambda}\underline{G}_\alpha \cup \underline{A}^c$. Hence the collection $\underline{\gamma}^* = \gamma \cup \{\underline{A}^c\}$ is a D-open cover of X. Since (X, η) is D-compact, there exists a finite sub cover $\underline{\gamma}_0 = \{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\} \cup \underline{A}^c$ of X, i.e. $\underline{X} = \cup_{i=1}^n\underline{G}_{\alpha_i} \cup \underline{A}^c \Rightarrow \underline{A} = (\cup_{i=1}^n\underline{G}_{\alpha_i} \cup \underline{A}^c) \cap \underline{A} \Rightarrow \underline{A} \subseteq \cup_{i=1}^n\underline{G}_{\alpha_i}$. Hence $\underline{A}$ is a C-set. □

The converse of the above Theorem may not be true in general as shown by the following example.

Example 3.3. Let $X = \{a, b\}$ and $\eta = \{\underline{\phi}, \underline{X}\}$. Then (X, η) is a D-compact. Now, let $\underline{A} = (\{a\}, \{a\})$. Then $\underline{A}$ is a C-set, but its not D-closed as $\underline{A}^c = (\{b\}, \{b\}) \notin \eta$.

Theorem 3.4. *Let (X, η) be a double topological space. Then (X, η) is D-compact $\Leftrightarrow$ (X, π_1) is compact , where $\pi_1 = \{A_1 : (A_1, A_2) \in \eta\}$.*

Proof. Let (X, η) be a D-compact space and let $\gamma_1 = \{G_{1\alpha} : \alpha \in \Lambda\} \subseteq \pi_1$ be an open cover of X, i.e. $X = \cup_{\alpha\in\Lambda} G_{1\alpha}$. Then the family $\underline{\gamma} = \{(G_{1\alpha}, G_{2\alpha}) : G_{2\alpha} \in \pi_2, \alpha \in \Lambda\}$ is a D-open cover of X. Since (X, η) is a D-compact, then there exists a finite subcover $\underline{\gamma}^* = \{(G_{1\alpha_i}, G_{2\alpha_i}) : i = 1, 2, 3, ..., n\} \subseteq \underline{\gamma}$ of X, i.e. $(X, X) = \underline{X} = \cup_{i=1}^{n}(G_{1\alpha_i}, G_{2\alpha_i})$. So $X = \cup_{i=1}^{n} G_{1\alpha_i}$, and hence $\gamma_1^* = \{G_{1\alpha_i} : i = 1, 2, 3..., n\} \subseteq \gamma_1$ is a finite subcover of X. Therefore (X, π_1) is a π_1-compact.
Conversely, let (X, π_1) be a π_1-compact and let $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta$ be a D-open cover of X, i.e. $(X, X) = \underline{X} = \cup_{\alpha\in\Lambda} \underline{G}_\alpha$. Since for each $\underline{G}_\alpha$ in $\underline{\gamma}$ there exists $G_{1\alpha}$, $G_{2\alpha} \in P(X)$, $G_{1\alpha} \subseteq G_{2\alpha}$ such that $\underline{G}_\alpha = (G_{1\alpha}, G_{2\alpha})$. Then $X = \cup_{\alpha\in\Lambda} G_{1\alpha}$, i.e. the collection $\gamma_1 = \{G_{1\alpha} : (G_{1\alpha}, G_{2\alpha}) \in \underline{\gamma}\} \subseteq \pi_1$ is a π_1-open cover of X, but (X, π_1) is π_1-compact, then there exists a finite subcover $\gamma_1^* = \{G_{1\alpha_i} : i = 1, 2, 3, ..., n\} \subseteq \gamma_1$ such that $X = \cup_{i=1}^{n} G_{1\alpha_i}$. Now, since $G_{1\alpha_i} \subseteq G_{2\alpha_i}$, then $\cup_{i=1}^{n} G_{1\alpha_i} \subseteq \cup_{i=1}^{n} G_{2\alpha_i}$, therefore, $\underline{X} = (\cup_{i=1}^{n} G_{1\alpha_i}, \cup_{i=1}^{n} G_{2\alpha_i}) = \cup_{i=1}^{n} \underline{G}_{\alpha_i}$. Hence (X, η) is a D-compact. □

Theorem 3.5. *Let (X, η) be a double topological space. Then*
(X, η) is D-compact $\Rightarrow$ (X, π_3) is compact, where $\pi_3 = \{A : (A, X) \in \eta\}$.

Proof. Let (X, η) be a D-compact and let $\gamma = \{G_\alpha : \alpha \in \Lambda, (G_\alpha, X) \in \eta\} \subseteq \pi_3$ be an open cover of X, i.e. $X = \cup_{\alpha\in\Lambda} G_\alpha$. Then $\underline{X} = \cup_{\alpha\in\Lambda}(G_\alpha, X)$. Hence the collection $\underline{\gamma} = \{(G_\alpha, X) : G_\alpha \in \gamma\}$ is a D-open cover of X. But, (X, η) is a D-compact, then, there exists a finite subcover $\underline{\gamma}^* = \{(G_{\alpha_i}, X) : i = 1, 2, 3, ..., n\} \subseteq \underline{\gamma}$ of X,i.e. $\underline{X} = \cup_{i=1}^{n}(G_{\alpha_i}, X) \Rightarrow X = \cup_{i=1}^{n} G_{\alpha_i}$. Hence (X, π_3) is compact. □

Theorem 3.6. *Let (X, η) be a double topological space. Then*
- **(1):** $\eta_l = \{(A_1, A_1) : (A_1, A_2) \in \eta\}$
- **(2):** $\eta_r = \{(A_2, A_2) : (A_1, A_2) \in \eta\}$

are double topologies on X.

Proof. (1): Since $(X, X), (\phi, \phi) \in \eta$, then $(X, X), (\phi, \phi) \in \eta_l$. Let $(A_1, A_1), (B_1, B_1) \in \eta_l$. Then, there exists $A_2, B_2 \in P(X)$ such that $(A_1, A_2), (B_1, B_2) \in \eta \Rightarrow (A_1 \cap B_1, A_2 \cap B_2) \in \eta \Rightarrow (A_1 \cap B_1, A_1 \cap B_1) \in \eta_l$. Hence $(A_1, A_1) \cap (B_1, B_1) \in \eta_l$. Now, let $\{(A_{1\alpha}, A_{1\alpha}) : \alpha \in \Lambda\} \subseteq \eta_l$. Then, for each $A_{1\alpha}$, there exists $A_{2\alpha} \in P(X)$ such that $(A_{1\alpha}, A_{2\alpha}) \in \eta \Rightarrow \{(A_{1\alpha}, A_{2\alpha}) : \alpha \in \Lambda\} \subseteq \eta \Rightarrow \cup_{\alpha\in\Lambda}(A_{1\alpha}, A_{2\alpha}) \in \eta$. Therefore, $\cup_{\alpha\in\Lambda}(A_{1\alpha}, A_{1\alpha}) \in \eta_l$. Consequently, η_l is a double topology on X.
(2): Similarly to the proof of part (1). □

Theorem 3.7. *Let (X, η) be a double topological space. Then (X, η) is an η-D-compact $\Leftrightarrow$ (X, η_l) is an η_l-D-compact.*

Proof. Let (X, η) be an η-D-compact and let $\underline{\gamma} = \{(G_{1\alpha}, G_{1\alpha}) : \alpha \in \Lambda\} \subseteq \eta_l$ be an η_l-D-open cover of X, i.e. $\underline{X} = \cup_{\alpha\in\Lambda}(G_{1\alpha}, G_{1\alpha})$. For each $G_{1\alpha}$ there exists $G_{2\alpha} \in P(X)$ such that $(G_{\alpha_1}, G_{\alpha_1}) \subseteq (G_{\alpha_1}, G_{\alpha_2}) \in \eta \Rightarrow \underline{X} = \cup_{\alpha\in\Lambda}(G_{1\alpha}, G_{2\alpha})$. Hence the collection $\underline{\zeta} = \{(G_{1\alpha}, G_{2\alpha}) : \alpha \in \Lambda\} \subseteq \eta$ is an η-D-open cover of X, then , by given, there exists a finite subcover $\underline{\zeta}^* = \{(G_{1\alpha_i}, G_{2\alpha_i}) : i = 1, 2, 3, ..., n\} \subseteq \underline{\zeta}$ of X, i.e. $\underline{X} = \cup_{i=1}^{n}(G_{1\alpha_i}, G_{2\alpha_i}) \Rightarrow X = \cup_{i=1}^{n} G_{1\alpha_i} \Rightarrow \underline{X} = \cup_{i=1}^{n}(G_{1\alpha_i}, G_{1\alpha_i})$. Hence (X, η_l) is an η_l-D-compact.
Conversely, let (X, η_l) be an η_l-D-compact and let $\underline{\gamma} = \{(G_{1\alpha}, G_{2\alpha}) : \alpha \in \Lambda\} \subseteq \eta$ be an η-D-compact, i.e. $\underline{X} = \cup_{\alpha\in\Lambda}(G_{1\alpha}, G_{2\alpha})$. Then $X = \cup_{\alpha\in\Lambda} G_{1\alpha} \Rightarrow \underline{X} =$

$\cup_{\alpha\in\Lambda}(G_{1\alpha}, G_{1\alpha})$. Hence the collection $\underline{\zeta} = \{(G_{1\alpha}, G_{1\alpha}) : \alpha \in \Lambda\} \subseteq \eta_l$ is an η_l-D-open cover of X, then , by given, there exists a finite subcover $\underline{\zeta}^* = \{(G_{1\alpha_i}, G_{1\alpha_i}) : i = 1, 2, 3, ..., n\} \subseteq \underline{\zeta}$ of X, i.e. $\underline{X} = \cup_{i=1}^{n}(G_{1\alpha_i}, G_{1\alpha_i}) \Rightarrow \underline{X} = \cup_{i=1}^{n}(G_{1\alpha_i}, G_{2\alpha_i})$. Hence (X, η) is an η-D-compact. □

Corollary 3.8. *Let (X, η) be a double topological space. Then (X, η_l) is an η_l-D-compact $\Leftrightarrow$ (X, π_1) is a π_1- compact.*

Proof. It follows from Theorem 3.4 and Theorem 3.7. □

Theorem 3.9. *Let (X, η) be a double topological space. Then (X, η_r) is an η_r-D-compact $\Leftrightarrow$ (X, π_2) is a π_2-compact.*

Proof. Let (X, η_r) be an η_r-D-compact and let $\gamma = \{G_{2\alpha} : \alpha \in \Lambda\} \subseteq \pi_2$ be a π_2-open cover of X, i.e. $X = \cup_{\alpha\in\Lambda}G_{2\alpha}$. Then $\underline{X} = \cup_{\alpha\in\Lambda}(G_{2\alpha}, G_{2\alpha})$. Therefore, the collection $\underline{\gamma} = \{(G_{2\alpha}, G_{2\alpha}) : G_{2\alpha} \in \gamma\} \subseteq \eta_r$ is an η_r-D-open cover of X. Hence, by given, there exists a finite subcover $\underline{\gamma}^* = \{(G_{2\alpha_i}, G_{2\alpha_i}) : i = 1, 2, 3, ..., n\} \subseteq \eta_r$ of X, i.e. $\underline{X} = \cup_{i=1}^{n}(G_{2\alpha_i}, G_{2\alpha_i}) \Rightarrow X = \cup_{i=1}^{n}G_{2\alpha_i}$. Hence (X, π_2) is a π_2-compact.
Conversely, let (X, π_2) be a π_2-compact and let $\underline{\gamma} = \{(G_{2\alpha}, G_{2\alpha}) : \alpha \in \Lambda\} \subseteq \eta_r$ be an η_r-D-open cover of X, i.e. $\underline{X} = \cup_{\alpha\in\Lambda}(G_{2\alpha}, G_{2\alpha})$. Then $X = \cup_{\alpha\in\Lambda}G_{2\alpha}$. Therefore, the collection $\gamma = \{G_{2\alpha} : (G_{2\alpha}, G_{2\alpha}) \in \underline{\gamma}\} \subseteq \pi_2$ is a π_2-open cover of X. Hence, by given, there exists a finite subcover $\gamma^* = \{G_{2\alpha_i} : i = 1, 2, 3, ..., n\} \subseteq \gamma$ of X, i.e. $X = \cup_{i=1}^{n}G_{2\alpha_i} \Rightarrow \underline{X} = \cup_{i=1}^{n}(G_{2\alpha_i}, G_{2\alpha_i})$. it follows that (X, η_r) is an η_r-D-compact. □

Theorem 3.10. *Let (X, η) be a double topological space. Then (X, η) is a D-compact $\Rightarrow$ (X, π_Δ) is a π_Δ- compact.*

Proof. Straightforward. □

Theorem 3.11. *Let (X, τ) be an ordinary topological space. Then (X, τ) is compact $\Leftrightarrow$ $(X, \tau\hat{\times}\tau)$ is D-compact.*

Proof. Suppose that (X, τ) be a compact and let $\underline{\gamma}$=$\{\underline{G}_\alpha : \alpha \in \Lambda\}$ be a D-open cover of X, i.e. $\underline{X} = \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. Then, for each $\underline{G}_\alpha$ in $\underline{\gamma}$ there exists $G_{1\alpha}$, $G_{2\alpha} \in \tau$, $G_{1\alpha} \subseteq G_{2\alpha}$ such that $\underline{G}_\alpha = (G_{1\alpha}, G_{2\alpha})$. So, the collection $\gamma_1(= \{G_{1\alpha} : \alpha \in \Lambda\} \subseteq \tau)$ is open cover of X, i.e. $X = \cup_{\alpha\in\Lambda}G_{1\alpha}$. Since (X, τ) is compact, then there exists a finite sub-collection $\gamma_1^*(= \{G_{1\alpha_i} : i = 1, 2, 3, ..., n\})$ of γ_1 such that $X = \cup_{i=1}^{n}G_{1\alpha_i}$. Now, $G_{1\alpha_i} \subseteq G_{2\alpha_i} \Rightarrow \cup_{i=1}^{n}G_{1\alpha_i} \subseteq \cup_{i=1}^{n}G_{2\alpha_i}$. Therefore, $\underline{X} = (\cup_{i=1}^{n}G_{1\alpha_i}, \cup_{i=1}^{n}G_{2\alpha_i}) = \cup_{i=1}^{n}\underline{G}_{\alpha_i}$. Hence $(X, \tau\hat{\times}\tau)$ is D-compact.
Conversely, let $(X, \tau\hat{\times}\tau)$ be a D-compact and let $\gamma(= \{G_\alpha : \alpha \in \Lambda\} \subseteq \tau)$ be an open cover of X. Then the collection $\underline{\gamma} = \{(G_\alpha, G_\alpha) : \alpha \in \Lambda\} \subseteq \tau\hat{\times}\tau$ is a D-open cover of X. Therefore, there exists a finite sub-collection $\underline{\gamma}^*(= \{(G_{\alpha_i}, G_{\alpha_i}) : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} = \cup_{i=1}^{n}(G_{\alpha_i}, G_{\alpha_i})$ which implies that $X = \cup_{i=1}^{n}G_{\alpha_i}$. Thus (X, τ) is compact. □

Theorem 3.12. *Every CD-compact space is a D-compact.*

Proof. Straightforward. □

Theorem 3.13. *Let (X, η_1), (X, η_2) be two double topological spaces such that η_2 is finer than η_1. If (X, η_2) is a D-compact, then (X, η_1) is a D-compact.*

Proof. Straightforward. □

Theorem 3.14. *The D-continuous image of a D-compact space is a C-set.*

Proof. Let (X, η_1), (Y, η_2) be two double topological spaces and let f be a D-continuous function from X into Y. Let $\underline{\gamma} = \{\underline{D}_\alpha : \alpha \in \Lambda\} \subseteq \eta_2$ be an arbitrary η_2-D-open cover of $f(\underline{X})$, i.e. $f(\underline{X}) \subseteq \cup_{\alpha\in\Lambda}\underline{D}_\alpha$. Then $\underline{X} \subseteq f^{-1}(f(\underline{X})) \subseteq f^{-1}(\cup_{\alpha\in\Lambda}\underline{D}_\alpha) \subseteq \underline{X}$, i.e. $\underline{X} = \cup_{\alpha\in\Lambda} f^{-1}(\underline{D}_\alpha)$. Since $\underline{D}_\alpha \in \eta_2$ and f is a D-continuous function, then $f^{-1}(\underline{D}_\alpha) \in \eta_1$ $\forall \alpha \in \Lambda$. Thus, the collection $\underline{\gamma} = \{f^{-1}(\underline{D}_\alpha) : \alpha \in \Lambda\}$ is an η_1-D-open cover of $\underline{X}$. But, (X, η_1) is a D-compact, then, there exists $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ such that $\underline{X} = \cup_{i=1}^n f^{-1}(\underline{D}_{\alpha_i}) \Rightarrow f(\underline{X}) \subseteq ff^{-1}(\underline{D}_{\alpha_1}) \cup ff^{-1}(\underline{D}_{\alpha_2}) \cup ... \cup ff^{-1}(\underline{D}_{\alpha_n}) \subseteq D_{\alpha_1} \cup D_{\alpha_2} \cup ... \cup D_{\alpha_n}$. Consequently, $f(\underline{X})$ is a C-set. More-over, if f is onto, then (Y, η_2) is a D-compact space. □

Definition 3.15. Let (X, η) be a double topological space . The collection $\mathcal{A} = \{\underline{H}_\alpha : \alpha \in \Lambda\} \subseteq D(X)$ is said to have the finite intersection property (FIP, for short) if for every finite sub-collection $\{\underline{H}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$, we have $\cap_{i=1}^n \underline{H}_{\alpha_i} \neq \underline{\phi}$.

Theorem 3.16. *Let (X, η) be a double topological space. Then (X, η) is a D-compact iff every collection $\{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets with FIP, we have $\cap_{\alpha\in\Lambda}\underline{F}_\alpha \neq \underline{\phi}$.*

Proof. Straightforward. □

Theorem 3.17. *Let (X, η) be a double topological space. Then (X, η) is a D-compact iff every collection $\mathcal{A} = \{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets such that $\cap_{\alpha\in\Lambda}\underline{F}_\alpha = \underline{\phi}$, there exists a finite sub-collection $\{\underline{F}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$ such that $\cap_{i=1}^n \underline{F}_{\alpha_i} = \underline{\phi}$.*

Proof. Straightforward. □

Definition 3.18. Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). A D-open cover $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ of X is said to be an $\underline{\mathcal{I}}$ - cover of X if there exists a finite sub-collection $\underline{\gamma}^*(= \{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \in \underline{\mathcal{I}}$.

Definition 3.19. Let $\underline{\mathcal{I}}$ be a D-ideal on X. A double topological space (X, η) is said to be an $\underline{\mathcal{I}}$-D-compact space if every D-open cover of X is an $\underline{\mathcal{I}}$ - cover.

Theorem 3.20. *Every D-compact space (X, η) is an $\underline{\mathcal{I}}$-D-compact for any D-ideal $\underline{\mathcal{I}}$ on X.*

Proof. Straightforward. □

Corollary 3.21. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. Then*

$$(X, \eta^*(\underline{\mathcal{I}})) \text{ is a D-compact} \Rightarrow (X, \eta^*(\underline{\mathcal{I}})) \text{ is an } \underline{\mathcal{I}} \text{ - D-compact.}$$

The converse of Theorem 3.20 may not be true in general as shown by the following example.

Example 3.22. Let $X = \mathbb{R}$ and let τ_N be the usual topology on $\mathbb{R}$. Then, by Theorem 3.11, $(\mathbb{R}, \tau_N \hat{\times} \tau_N)$ is not D-compact. Let $x_t \in \mathbb{R}_p$ and let the collection $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \tau_N \hat{\times} \tau_N$ be a D-open cover of $\mathbb{R}$, i.e. $\underline{\mathbb{R}} = \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. Since $x_t \in \mathbb{R}_p$, then $x_t \in \underline{\mathbb{R}} = \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. So there exist $\underline{G}_{\alpha_o} \in \underline{\gamma}$ such that $x_t \in \underline{G}_{\alpha_o} \Rightarrow x_t \bar{q} \underline{G}^c_{\alpha_o} \Rightarrow \underline{\mathbb{R}} \backslash \underline{G}_{\alpha_o} = \underline{G}^c_{\alpha_o} \in \underline{\mathcal{I}}^q_{x_t}$. Hence $(\mathbb{R}, \tau_N \hat{\times} \tau_N)$ is an $\underline{\mathcal{I}}^q_{x_t}$-D-compact.

Theorem 3.23. *Let X be uncountable set, τ_∞ be the co-finite topology on X and let $\underline{\mathcal{I}}_c$ be the countable D-ideal on X. Then $(\tau_\infty \hat{\times} \tau_\infty)^*(\underline{\mathcal{I}}_c) = \tau_{co} \hat{\times} \tau_{co}$, where τ_{co} is the co-countable topology on X.*

Proof. Let $\underline{G} \in \tau_{co} \hat{\times} \tau_{co}$. Then $\underline{G}^c$ is countable double set, so $\underline{G}^c \in \underline{\mathcal{I}}_c$. Since $\underline{G} = \underline{X} \backslash \underline{G}^c$, $\underline{X} \in \tau_\infty \hat{\times} \tau_\infty$ and $\underline{G}^c \in \underline{\mathcal{I}}_c$, then $\underline{G} \in \beta(\tau_\infty \hat{\times} \tau_\infty, \underline{\mathcal{I}}_c)$. It follows that $\underline{G} \in (\tau_\infty \hat{\times} \tau_\infty)^*$. Hence $\tau_{co} \hat{\times} \tau_{co} \subseteq (\tau_\infty \hat{\times} \tau_\infty)^*$ Also, let $\underline{G} \in (\tau_\infty \hat{\times} \tau_\infty)^*$. Since $\tau_\infty \hat{\times} \tau_\infty$ compatible with $\underline{\mathcal{I}}_c$[13], then $(\tau_\infty \hat{\times} \tau_\infty)^* = \beta(\tau_\infty \hat{\times} \tau_\infty, \underline{\mathcal{I}}_c)$, then there exists $\underline{H} \in \tau_\infty \hat{\times} \tau_\infty$, $\underline{A} \in \underline{\mathcal{I}}_c$ such that $\underline{G} = \underline{H} \backslash \underline{A}$, and so $\underline{G}^c$ is countable double set. It follows that $\underline{G} \in \tau_{co} \hat{\times} \tau_{co}$. Hence $(\tau_\infty \hat{\times} \tau_\infty)^* \subseteq \tau_{co} \hat{\times} \tau_{co}$. Consequently, $(\tau_\infty \hat{\times} \tau_\infty)^*(\underline{\mathcal{I}}_c) = \tau_{co} \hat{\times} \tau_{co}$. □

The converse of Corollary 3.21 may not be true in general as shown by the following example.

Example 3.24. Let X be uncountable set, τ_∞ be the co-finite topology on X and let $\underline{\mathcal{I}}_c$ be the countable double ideal. Then, by Theorem 3.11, $(X, \tau_\infty \hat{\times} \tau_\infty)$ is a D-compact. By Theorem 3.23, $(X, (\tau_\infty \hat{\times} \tau_\infty)^*(\underline{\mathcal{I}}_c)) = (X, \tau_{co} \hat{\times} \tau_{co})$ which is not D-compact. On the other hand, let $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq (\tau_\infty \hat{\times} \tau_\infty)^*(\underline{\mathcal{I}}_c) = \beta(\tau_\infty \hat{\times} \tau_\infty, \underline{\mathcal{I}}_c)$ such that $\underline{X} = \cup_{\alpha\in\Lambda}\underline{G}_\alpha$. Then $\underline{X} = \cup_{\alpha\in\Lambda}(\underline{H}_\alpha \backslash \underline{A}_\alpha)$ where $\underline{H}_\alpha \in \tau_\infty \hat{\times} \tau_\infty$ and $\underline{A}_\alpha \in \underline{\mathcal{I}}_c$, and so the collection $\zeta = \{\underline{H}_\alpha : \alpha \in \Lambda\} \subseteq \tau_\infty \hat{\times} \tau_\infty$ is a D-open cover of $\underline{X}$, but $(X, \tau_\infty \hat{\times} \tau_\infty)$ is $\underline{\mathcal{I}}_c$-D-compact, then there exists a finite sub-collection $\zeta^* = \{\underline{H}_{\alpha_i} : i = 1, 2, 3, ..., n\} \subseteq \zeta$ such that $\underline{X} \backslash \cup^n_{i=1} \underline{H}_{\alpha_i} \in \underline{\mathcal{I}}_c$. Now, $\forall \underline{H}_{\alpha_i} \exists \underline{A}_{\alpha_i} \in \underline{\mathcal{I}}_c$ such that $\underline{G}_{\alpha_i} = \underline{H}_{\alpha_i} \backslash \underline{A}_{\alpha_i} \Rightarrow \underline{X} \backslash \cup^n_{i=1} \underline{G}_{\alpha_i} = (\underline{X} \backslash \cup^n_{i=1} \underline{H}_{\alpha_i}) \cup (\cap^n_{i=1} \underline{A}_{\alpha_i}) \in \underline{\mathcal{I}}_c$. Hence $\underline{X} \backslash \cup^n_{i=1} \underline{G}_{\alpha_i} \in \underline{\mathcal{I}}_c$. Therefore, $(X, \tau_\infty \hat{\times} \tau_\infty)^*(\underline{\mathcal{I}}_c)) = (X, \tau_{co} \hat{\times} \tau_{co})$ is an $\underline{\mathcal{I}}_c$-D-compact.

Theorem 3.25. *A double topological space (X, η) is D-compact if and only if it's $\{\underline{\phi}\}$-D-compact.*

Proof. Straightforward. □

Theorem 3.26. *Let (X, η_1), (X, η_2) be two double topological spaces such that η_2 is finer than η_1. If (X, η_2) is an $\underline{\mathcal{I}}$-D-compact, then (X, η_1) is an $\underline{\mathcal{I}}$ - D-compact.*

Proof. Straightforward. □

The converse of The above Theorem may not be true in general as shown by the following example.

Example 3.27. Let X be uncountable set. Then the co-finite double topology $\tau_\infty \hat{\times} \tau_\infty$ induced by τ_∞ on X is finer than the co-countable double topology $\tau_{co} \hat{\times} \tau_{co}$ induced by τ_{co} on X, however, $(X, \tau_\infty \hat{\times} \tau_\infty)$ is $\{\underline{\phi}\}$-D-compact, but $(X, \tau_{co} \hat{\times} \tau_{co})$ is not $\{\underline{\phi}\}$-D-compact.

Theorem 3.28. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. If $(X, \eta^*(\underline{\mathcal{I}}))$ is a D-compact, then (X, η) is $\underline{\mathcal{I}}$-D-compact.*

Proof. It follows from Theorem 3.20 and Theorem 3.26. □

Theorem 3.29. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. Then (X, η) is an $\underline{\mathcal{I}}$ - D-compact if and only if $(X, \eta^*(\underline{\mathcal{I}}))$ is an $\underline{\mathcal{I}}$ - D-compact.*

Proof. Let $\underline{\gamma}$={$\underline{G}_\alpha : \alpha \in \Lambda$} be a basic η^*-D-open cover of X. Then for each $\alpha \in \Lambda$, $\underline{G}_\alpha = \underline{H}_\alpha \backslash \underline{A}_\alpha$ where $\underline{H}_\alpha \in \eta$ and $\underline{A}_\alpha \in \underline{\mathcal{I}}$. Therefore, the collection $\underline{\Omega} = \{\underline{H}_\alpha : \alpha \in \Lambda\}$ is a η-D-open cover of X. Hence there exist a finite sub-collection $\underline{\Omega}_0 = \{\underline{H}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\underline{\Omega}$ such that $\underline{X} \backslash \cup_{i=1}^n \underline{H}_{\alpha_i} \in \underline{\mathcal{I}}$. Now, $\underline{X} \backslash \cup_{i=1}^n \underline{G}_{\alpha_i} = \underline{X} \backslash \cup_{i=1}^n (\underline{H}_{\alpha_i} \backslash \underline{A}_{\alpha_i}) = (\underline{X} \backslash \cup_{i=1}^n \underline{H}_{\alpha_i}) \cup (\cap_{i=1}^n \underline{A}_{\alpha_i}) \in \underline{\mathcal{I}}$ (for $\underline{A}_{\alpha_i} \in \underline{\mathcal{I}}$). Thus, $(X, \eta^*(\underline{\mathcal{I}}))$ is an $\underline{\mathcal{I}}$-D-compact.
Conversely, the sufficiency of the Theorem follows from Theorem 3.26. □

Theorem 3.30. *Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. Then*

$$(X, \eta) \text{ is } \underline{\mathcal{I}}\text{-D-compact} \Rightarrow (X, \pi_1) \text{ is an } \mathcal{I}^2\text{-compact.}$$

Proof. Let (X, η) be an $\underline{\mathcal{I}}$-D-compact and let $\gamma_1 (= \{G_{1\alpha} : \alpha \in \Lambda\} \subseteq \pi_1)$ be a π_1-open cover of X i.e. $X = \cup_{\alpha \in \Lambda} G_{1\alpha}$. Then the family $\underline{\gamma} = \{(G_{1\alpha}, G_{2\alpha}) : G_{2\alpha} \in \pi_2, \alpha \in \Lambda\} \subseteq \eta$ is D-open cover of X. Hence there exists a finite sub-collection $\underline{\gamma}^* = \{(G_{1\alpha_i}, G_{2\alpha_i}) : i = 1, 2, 3, ..., n\} \subseteq \underline{\gamma}$ such that $\underline{X} \backslash \cup_{i=1}^n (G_{1\alpha_i}, G_{2\alpha_i}) \in \underline{\mathcal{I}}$, and so $X \backslash \cup_{i=1}^n G_{1\alpha_i} \in \mathcal{I}^2$. Thus (X, π_1) is $\mathcal{I}^2$-compact. □

Theorem 3.31. *Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. If $(X, \eta^*(\underline{\mathcal{I}}))$ is $\underline{\mathcal{I}}$-D-compact, then (X, π_1^*) is an $\mathcal{I}^2$ - compact. Where $\pi_1^* = \{A_1 : (A_1, A_2) \in \eta^*\}$.*

Proof. Similarly to the proof of Theorem 3.30. □

Corollary 3.32. *Let (X, η) double topological space and $\underline{\mathcal{I}} \in DI(X)$. The following implication diagram holds :*

$$\begin{array}{ccc}
(X, \eta^*(\underline{\mathcal{I}})) \text{ is D-compact} & \Rightarrow & (X, \eta) \text{ is D-compact} \Leftrightarrow \text{ it's } \{\underline{\phi}\} \text{ -D-compact} \\
\Downarrow & & \Downarrow \\
(X, \eta^*(\underline{\mathcal{I}})) \text{ is } \underline{\mathcal{I}}\text{-D-compact} & \Leftrightarrow & (X, \eta) \text{ is } \underline{\mathcal{I}} \text{ - D-compact} \\
\Downarrow & & \Downarrow \\
(X, \pi_1^*) \text{ is an } \mathcal{I}^2 \text{ - compact} & \Rightarrow & (X, \pi_1) \text{ is an } \mathcal{I}^2\text{-compact}
\end{array}$$

Theorem 3.33. *Let (X, τ) be an ordinary topological space and let $\mathcal{I}$ be an ideal on X. Then*

$$(X, \tau) \text{ is an } \mathcal{I}\text{-compact} \Leftrightarrow (X, \tau \hat{\times} \tau) \text{ is an } \mathcal{I} \hat{\times} \mathcal{I}\text{-D-compact.}$$

Proof. Suppose that (X, τ) be an $\mathcal{I}$ - compact and let $\underline{\gamma}$={$\underline{G}_\alpha : \alpha \in \Lambda$} be an $\tau \hat{\times} \tau$-D-open cover of X, i.e. $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Then for each $\underline{G}_\alpha$ in $\underline{\gamma}$ there exists $G_{1\alpha}, G_{2\alpha} \in \tau$, $G_{1\alpha} \subseteq G_{2\alpha}$ such that $\underline{G}_\alpha = (G_{1\alpha}, G_{2\alpha})$. So, the collection $\gamma_1 = \{G_{1\alpha} : \alpha \in \Lambda\} \subseteq \tau$ is a τ - open cover of X, i.e. $X = \cup_{\alpha \in \Lambda} G_{1\alpha}$. Since (X, τ) is an $\mathcal{I}$ - compact, then there exists a finite sub-collection $\gamma_1^* = \{G_{1\alpha_i} : i = 1, 2, 3, ..., n\}$ of γ_1 such that $X \backslash \cup_{i=1}^n G_{1\alpha_i} \in \mathcal{I}$. Now, $G_{1\alpha_i} \subseteq G_{2\alpha_i} \Rightarrow X \backslash \cup_{i=1}^n G_{2\alpha_i} \subseteq X \backslash \cup_{i=1}^n G_{1\alpha_i} \in$

$\mathcal{I} \Rightarrow X \setminus \cup_{i=1}^{n} G_{2\alpha_i} \in \mathcal{I}$. Therefore, $(X \setminus \cup_{i=1}^{n} G_{2\alpha_i}, X \setminus \cup_{i=1}^{n} G_{1\alpha_i}) \in \mathcal{I} \hat{\times} \mathcal{I}$. Hence $\underline{X} \setminus \cup_{i=1}^{n} \underline{G}_{\alpha_i} \in \mathcal{I} \hat{\times} \mathcal{I}$. Consequently, $(X, \tau \hat{\times} \tau)$ is an $\mathcal{I} \hat{\times} \mathcal{I}$ - D-compact.
Conversely, let $(X, \tau \hat{\times} \tau)$ be an $\mathcal{I} \hat{\times} \mathcal{I}$-D-compact and let $\gamma = \{G_\alpha : \alpha \in \Lambda\} \subseteq \tau$ be an open cover of X. Then the collection $\underline{\gamma} = \{(G_\alpha, G_\alpha) : \alpha \in \Lambda\} \subseteq \tau \hat{\times} \tau$ is D-open cover of X, therefore there exists a finite sub-collection $\underline{\gamma}^*(= \{(G_{\alpha_i}, G_{\alpha_i}) : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} \setminus \cup_{i=1}^{n} \underline{G}_{\alpha_i} \in \mathcal{I} \hat{\times} \mathcal{I}$ which implies that $X \setminus \cup_{i=1}^{n} G_{\alpha_i} \in \mathcal{I}$. Thus (X, τ) is an $\mathcal{I}$ - compact. □

Definition 3.34. Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). The collection $\mathcal{A} = \{\underline{H}_\alpha : \alpha \in \Lambda\} \subseteq D(X)$ is said to have the finite intersection property modulo D-ideal $\underline{\mathcal{I}}$, denoted by $\underline{\mathcal{I}} - FIP$, if for every finite sub-collection $\{\underline{H}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$, we have $\cap_{i=1}^{n} \underline{H}_{\alpha_i} \notin \underline{\mathcal{I}}$.

Theorem 3.35. *Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. Then (X, η) is an $\underline{\mathcal{I}}$-D-compact iff every collection $\{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets with $\underline{\mathcal{I}} - FIP$, we have $\cap_{\alpha \in \Lambda} \underline{F}_\alpha \neq \underline{\phi}$.*

Proof. Let (X, η) be an $\underline{\mathcal{I}}$-D-compact, and let $\mathcal{A} = \{\underline{F}_\alpha : \alpha \in \Lambda\} \subseteq \eta^c$ having $\underline{\mathcal{I}} - FIP$ i.e. $\cap_{i=1}^{n} \underline{F}_{\alpha_i} \notin \underline{\mathcal{I}}\ \forall n \in N$. Assume that $\cap_{\alpha \in \Lambda} \underline{F}_\alpha = \underline{\phi}$. Then $\underline{X} = \cup_{\alpha \in \Lambda} \underline{F}_\alpha^c \Rightarrow$ the collection $\mathcal{A}^* = \{\underline{F}_\alpha^c : \underline{F}_\alpha \in \mathcal{A}\}$ is D-open cover of X. But, (X, η) is an $\underline{\mathcal{I}}$-D-compact, then there exists a finite sub-collection $\{\underline{F}_{\alpha_i}^c : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}^*$ such that $\underline{X} \setminus \cup_{i=1}^{n} \underline{F}_{\alpha_i}^c \in \underline{\mathcal{I}}$ which implies that $\cap_{i=1}^{n} \underline{F}_{\alpha_i} \in \underline{\mathcal{I}}$ a contradiction.
Conversely, suppose that for every collection $\{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets with $\underline{\mathcal{I}} - FIP$, we have $\cap_{\alpha \in \Lambda} \underline{F}_\alpha \neq \underline{\phi}$. Assume that (X, η) is not $\underline{\mathcal{I}}$-D-compact. Then there exists a D-open cover $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\}$ of X such that for any finite sub-collection $\{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\underline{\gamma}$, $\underline{X} \setminus \cup_{i=1}^{n} \underline{G}_{\alpha_i} \notin \underline{\mathcal{I}}$ which implies that $\cap_{i=1}^{n} \underline{G}_{\alpha_i}^c \notin \underline{\mathcal{I}}$. Thus, the collection $\{\underline{G}_\alpha^c : \alpha \in \Lambda\} \subseteq \eta^c$ and has $\underline{\mathcal{I}} - FIP$, and so $\cap_{\alpha \in \Lambda} \underline{G}_\alpha^c \neq \underline{\phi}$ contradicts with $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Hence (X, η) is an $\underline{\mathcal{I}}$-D-compact. □

Theorem 3.36. *Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. Then (X, η) is an $\underline{\mathcal{I}}$-D-compact iff every collection $\mathcal{A} = \{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets such that $\cap_{\alpha \in \Lambda} \underline{F}_\alpha = \underline{\phi}$, there exists a finite sub-collection $\{\underline{F}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$ such that $\cap_{i=1}^{n} \underline{F}_{\alpha_i} \in \underline{\mathcal{I}}$.*

Proof. Straightforward. □

4. Double Quasi H-Closed (DQHC) modulo double ideal

In this section, we introduce and study the idea of double quasi H-closed, defined in terms of a D-ideal in a double topological space (X, η). Calling it $\underline{\mathcal{I}}$-DQHC, we investigate its relation with compactness, among other things.

Definition 4.1. A double topological space (X, η) is said to be a double quasi H - closed (DQHC, for short) if every D-open cover $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ of X there exists a finite sub-collection $\underline{\gamma}^*(= \{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} = \cup_{i=1}^{n} cl_\eta \underline{G}_{\alpha_i}$. In this case the collection $\underline{\gamma}^*(= \{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\})$ is called a D-proximate cover of X. A double Hausdorff quasi H - closed space is called double H - closed (DHC, for short).

Theorem 4.2. *Let (X, τ) be an ordinary topological space. Then (X, τ) is QHC if and only if $(X, \tau \hat{\times} \tau)$ is DQHC.*

Proof. Let (X, τ) be a QHC and let $\underline{\gamma}=\{\underline{G}_\alpha : \alpha \in \Lambda\}$ be an $\tau \hat{\times} \tau$ - double open cover of X, i.e. $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Then, for each $\underline{G}_\alpha$ in $\underline{\gamma}$ there exists $G_{1\alpha}$, $G_{2\alpha} \in \tau$, $G_{1\alpha} \subseteq G_{2\alpha}$ such that $\underline{G}_\alpha = (G_{1\alpha}, G_{2\alpha})$. So, the collection $\gamma_1(= \{G_{1\alpha} : \alpha \in \Lambda\} \subseteq \tau)$ is a τ - open cover of X, i.e. $X = \cup_{\alpha \in \Lambda} G_{1\alpha}$. Since (X, τ) is QHC, there exists a finite sub-collection $\gamma_1^*(= \{G_{1i} : i = 1, 2, 3, ..., n\})$ of γ_1 such that $X = \cup_{i=1}^n cl_\tau G_{1\alpha_i}$. Now, since $G_{1\alpha_i} \subseteq G_{2\alpha_i}$, then $\cup_{i=1}^n cl_\tau G_{1\alpha_i} \subseteq \cup_{i=1}^n cl_\tau G_{2\alpha_i}$. Therefore, $\underline{X} = \cup_{i=1}^n (cl_\tau G_{1\alpha_i}, cl_\tau G_{2\alpha_i}) = \cup_{i=1}^n cl_{\tau \hat{\times} \tau} \underline{G}_{\alpha_i}$ (by Theorem **??**). Consequently, $(X, \tau \hat{\times} \tau)$ is DQHC.
Conversely, let $(X, \tau \hat{\times} \tau)$ be a DQHC and let $\gamma(= \{G_\alpha : \alpha \in \Lambda\} \subseteq \tau)$ be an open cover of X. Then the collection $\underline{\gamma} = \{(G_\alpha, G_\alpha) : \alpha \in \Lambda\} \subseteq \tau \hat{\times} \tau$ is a D-open cove of X. So, there exists a finite sub-collection $\underline{\gamma}^*(= \{(G_{\alpha_i}, G_{\alpha_i}) : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} = \cup_{i=1}^n cl_{\tau \hat{\times} \tau}(G_{\alpha_i}, G_{\alpha_i}) = \cup_{i=1}^n (cl_\tau G_{\alpha_i}, cl_\tau G_{\alpha_i})$ which implies that $X = \cup_{i=1}^n cl_\tau G_{\alpha_i}$. Thus (X, τ) is QHC. □

Theorem 4.3. *Every D-compact space (X, η) is DQHC.*

Proof. Straightforward. □

Theorem 4.4. *Let (X, η_1), (X, η_2) be two double topological spaces such that η_2 is finer than η_1. If (X, η_2) is a DQHC, then (X, η_1) is a DQHC.*

Proof. Straightforward. □

Definition 4.5. Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. A D-open cover $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ of X is said to be an $\underline{\mathcal{I}}$ - proximate cover of X ($\underline{\mathcal{I}}$ - pcover, for short) if there exists a finite sub-collection $\underline{\gamma}^*(= \{\underline{G}_{\alpha_i} : i = 1, 2, 3, ..., n\})$ of $\underline{\gamma}$ such that $\underline{X} \setminus \cup_{i=1}^n cl_\eta(\underline{G}_{\alpha_i}) \in \underline{\mathcal{I}}$.

Definition 4.6. Let $\underline{\mathcal{I}}$ be a D-ideal on X. A double topological space (X, η) is said to be an $\underline{\mathcal{I}}$ - DQHC if every D-open cover of X is an $\underline{\mathcal{I}}$ - pcover.

Theorem 4.7. *Every DQHC space (X, η) is an $\underline{\mathcal{I}}$ - DQHC for any D-ideal $\underline{\mathcal{I}}$ on X.*

Proof. Straightforward. □

Corollary 4.8. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. Then*

$$(X, \eta^*(\underline{\mathcal{I}})) \text{ is a DQHC} \Rightarrow (X, \eta^*(\underline{\mathcal{I}})) \text{ is an } \underline{\mathcal{I}}\text{-DQHC}.$$

Theorem 4.9. *Every $\underline{\mathcal{I}}$-D-compact space (X, η) is $\underline{\mathcal{I}}$-DQHC.*

Proof. Straightforward. □

Theorem 4.10. *A double topological space (X, η) is DQHC if and only if it's $\{\underline{\phi}\}$ - DQHC.*

Proof. Straightforward. □

Theorem 4.11. *Let (X, η_1), (X, η_2) be two double topological spaces such that η_2 is finer than η_1. If (X, η_2) is an $\underline{\mathcal{I}}$-DQHC, then (X, η_1) is an $\underline{\mathcal{I}}$-DQHC.*

Proof. Straightforward. □

Theorem 4.12. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. If $(X, \eta^*(\underline{\mathcal{I}}))$ is a DQHC, then (X, η) is $\underline{\mathcal{I}}$ - DQHC.*

Proof. It follows from Theorem 4.7 and Theorem 4.11. □

On account to Theorems 4.4, 4.7, 4.10, 4.11 and Theorem 4.12 we have the following corollary.

Corollary 4.13. *Let (X, η) double topological space and $\underline{\mathcal{I}} \in DI(X)$. The following implication diagram holds :*

$$\begin{array}{ccc} (X, \eta^*(\underline{\mathcal{I}})) \text{ is DQHC} \Rightarrow & (X, \eta) \text{ is DQHC} \Leftrightarrow & \text{it is}\{\underline{\phi}\} \text{ it is-DQHC} \\ \Downarrow & \Downarrow & \\ (X, \eta^*(\underline{\mathcal{I}})) \text{ is } \underline{\mathcal{I}}\text{-DQHC} \Rightarrow & (X, \eta) \text{ is } \underline{\mathcal{I}}\text{-DQHC} & \end{array}$$

Theorem 4.14. *Let (X, η) be a double topological space and $\underline{\mathcal{I}} \in DI(X)$. The following statements are equivalent:*

1. *(X, η) is an $\underline{\mathcal{I}}$-DQHC.*
2. *For every collection $\mathcal{A} = \{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets such that $\cap_{\alpha \in \Lambda} \underline{F}_\alpha = \underline{\phi}$, there exists a finite sub-collection $\{\underline{F}_{\alpha_i} : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$ such that $\cap_{i=1}^n int_\eta(\underline{F}_{\alpha_i}) \in \underline{\mathcal{I}}$.*
3. *every collection $\mathcal{A} = \{\underline{F}_\alpha : \alpha \in \Lambda\}$ of double closed sets such that $\{int_\eta(\underline{F}_\alpha) : \underline{F}_\alpha \in \mathcal{A}\}$ with $\underline{\mathcal{I}} - FIP$, we have $\cap_{\alpha \in \Lambda} \underline{F}_\alpha \neq \underline{\phi}$.*

Proof. (1) $\Rightarrow$ (2): Let (X, η) be an $\underline{\mathcal{I}}$-DQHC and let $\mathcal{A}$ be a collection of double closed sets with $\cap_{\alpha \in \Lambda} \{\underline{F}_\alpha : \underline{F}_\alpha \in \mathcal{A}\} = \underline{\phi}$. Then the collection $\{\underline{F}_\alpha^c : \underline{F}_\alpha \in \mathcal{A}\}$ is a D-open cover of X, and hence there exists a finite sub-collection $\{F_i^c : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$ such that $\underline{X} \setminus \cup_{i=1}^n cl_\eta(\underline{F}_i^c) \in \underline{\mathcal{I}}$ which implies that $\cap_{i=1}^n int_\eta(\underline{F}_i) \in \underline{\mathcal{I}}$.
(2) $\Leftrightarrow$ (3): It is obvious.
(2) $\Rightarrow$ (1): Let $\underline{\gamma} = \{\underline{G}_\alpha : \alpha \in \Lambda\}$ be a D-open cover of X i.e. $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Then the collection $\mathcal{A} = \{\underline{G}_\alpha^c : \alpha \in \Lambda\} \subseteq \eta^c$ with $\cap_{\alpha \in \Lambda} \underline{G}_\alpha^c = \underline{\phi}$, and hence there exist a finite sub-collection $\{\underline{G}_{\alpha_i}^c : i = 1, 2, 3, ..., n\}$ of $\mathcal{A}$ such that $\cap_{i=1}^n int_\eta(\underline{G}_{\alpha_i}^c) \in \underline{\mathcal{I}}$ which implies that $\underline{X} \setminus \cup_{i=1}^n cl_\eta(\underline{G}_{\alpha_i}) \in \underline{\mathcal{I}}$. Hence (X, η) is an $\underline{\mathcal{I}}$-DQHC. □

5. The relation between the compactness modulo double ideal and the DQHC

In this section, we try to associate the notion of double quasi H - closedness with that of $\underline{\mathcal{I}}$ - double compactness.

Definition 5.1. A double ideal $\underline{\mathcal{I}}$ on a double topological space (X, η) is said to be a codense with respect to η if the complement of each of its member is a double dense.

Theorem 5.2. *Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). Then $\underline{\mathcal{I}}$ is a codense $\Leftrightarrow$ $\eta \cap \underline{\mathcal{I}} = \{\underline{\phi}\}$.*

Proof. Let $\underline{\mathcal{I}}$ be a codence with respect to η. Let $\underline{A} \in \eta \cap \underline{\mathcal{I}}$. Then $\underline{A}^c \in \eta^c$ and $cl_\eta(\underline{A}^c) = \underline{X}$. But, $cl_\eta(\underline{A}^c) = \underline{A}^c$, hence $\underline{A}^c = \underline{X}$ which implies that $\underline{A} = \underline{\phi}$. Consequently, $\eta \cap \underline{\mathcal{I}} = \{\underline{\phi}\}$.
Conversely, let $\eta \cap \underline{\mathcal{I}} = \{\underline{\phi}\}$ and assume that there exists $\underline{A} \in \underline{\mathcal{I}}$ such that $\underline{A}^c$ is not dense. Then $cl_\eta(\underline{A}^c) \neq \underline{X}$. So, $(cl_\eta \underline{A}^c)^c \neq \underline{\phi} \Rightarrow int_\eta(\underline{A}) \neq \underline{\phi}$. Since, $int_\eta \underline{A} \subseteq \underline{A} \in \underline{\mathcal{I}}$, then $int_\eta(\underline{A}) \in \underline{\mathcal{I}}, int_\eta(\underline{A}) \in \eta \Rightarrow int_\eta(\underline{A}) = \underline{\phi}$ which is a contradiction. □

Theorem 5.3. *Let (X, η) be a double topological space. Then $\eta \cap \underline{\mathcal{I}}_n = \{\underline{\phi}\}$, where $\underline{\mathcal{I}}_n$ is the double ideal of all nowhere dense double sets in (X, η).*

Proof. Let $\underline{G} \in \eta \cap \underline{\mathcal{I}}_n$. Then, $int_\eta(\underline{G}) = \underline{G}$ and $int_\eta cl_\eta(\underline{G}) = \underline{\phi}$. Since, $int_\eta(\underline{G}) \subseteq int_\eta cl_\eta(\underline{G})$, then $int_\eta(\underline{G}) = \underline{\phi}$, and so $\underline{G} = \underline{\phi}$. Hence $\eta \cap \underline{\mathcal{I}}_n = \{\underline{\phi}\}$. □

Theorem 5.4. *Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). If (X, η) is an $\underline{\mathcal{I}}$ - D-compact and $\eta \cap \underline{\mathcal{I}} = \{\underline{\phi}\}$, then (X, η) is DQHC.*

Proof. Let (X, η) be an $\underline{\mathcal{I}}$ - D-compact and let $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ be a D-open cover of X i.e. $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Then there exists $\underline{G}_{\alpha_1}, \underline{G}_{\alpha_2}, ..., \underline{G}_{\alpha_n} \in \underline{\gamma}$ such that $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \in \underline{\mathcal{I}}$. Thus, $int_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}) \in \underline{\mathcal{I}} \cap \eta = \{\underline{\phi}\} \Rightarrow int_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}) = \underline{\phi} \Rightarrow (int_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}))^c = \underline{X} \Rightarrow \underline{X} = \cup_{i=1}^n cl_\eta \underline{G}_{\alpha_i}$. Hence (X, η) is a DQHC. □

Theorem 5.5. *Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). If (X, η) is a DQHC, $\underline{\mathcal{I}}_n \subseteq \underline{\mathcal{I}}$, then (X, η) is an $\underline{\mathcal{I}}$ - D-compact.*

Proof. Let (X, η) be a DQHC, $\underline{\mathcal{I}}_n \subseteq \underline{\mathcal{I}}$ and let $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ be a D-open cover of X i.e. $\underline{X} = \cup_{\alpha \in \Lambda} \underline{G}_\alpha$. Then, there exists $\underline{G}_{\alpha_1}, \underline{G}_{\alpha_2}, ..., \underline{G}_{\alpha_n} \in \underline{\gamma}$ such that $\underline{X} \setminus \cup_{i=1}^n cl_\eta \underline{G}_{\alpha_i} = \underline{\phi}$. We claim that $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \in \underline{\mathcal{I}}_n$. In fact, $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \notin \underline{\mathcal{I}}_n \Rightarrow int_\eta cl_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}) \neq \underline{\phi}$. But, $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \in \eta^c$, then $cl_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}) = \underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}$ and so $int_\eta(\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i}) \neq \underline{\phi}$. Hence $\underline{X} \setminus \cup_{i=1}^n cl_\eta \underline{G}_{\alpha_i} \neq \underline{\phi}$, a contradiction. Therefore $\underline{X} \setminus \cup_{i=1}^n \underline{G}_{\alpha_i} \in \underline{\mathcal{I}}_n \subseteq \underline{\mathcal{I}}$. Consequently, (X, η) is an $\underline{\mathcal{I}}$ - D-compact. □

Corollary 5.6. *Let (X, η) be any double topological space. Then*
(1) *(X, η) is an $\underline{\mathcal{I}}_n$ - D-compact $\Leftrightarrow$ it's DQHC.*
(2) *If (X, η) is a double Hausdorff space, then (X, η) is an $\underline{\mathcal{I}}_n$ - D-compact $\Leftrightarrow$ it's DHC.*

Proof. It follows from Theorem 5.3 and Theorem 5.5. □

Definition 5.7. Let (X, η) be a double topological space. A double set $\underline{A}$ is said to be a DC-set if for every D-open cover $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ of $\underline{A}$, there exists a finite sub-collection $\{\underline{G}_{\alpha_1}, \underline{G}_{\alpha_2}, \underline{G}_{\alpha_3}, ..., \underline{G}_{\alpha_n}\}$ of γ such that $\underline{A} \subseteq \cup_{i=1}^n cl_\eta(\underline{G}_{\alpha_i})$.

Definition 5.8. A double topological space (X, η) is said to be a DC-compact if every double closed set is a DC-set.

Theorem 5.9. *Every DC-compact space is a DQHC.*

Proof. Straightforward. □

Theorem 5.10. *Every CD-compact space is a DC-compact.*

Proof. Straightforward. □

Theorem 5.11. *Let (X, τ) be an ordinary topological space. Then (X, τ) is a C-compact $\Leftrightarrow$ $(X, \tau\hat{\times}\tau)$ is a DC-compact.*

Proof. Straightforward. □

Definition 5.12. Let $\underline{\mathcal{I}}$ be a D-ideal on a double topological space (X, η). A double topological space (X, η) is said to be $\underline{\mathcal{I}}$-DC-compact if for every crisp double closed set $\underline{F}$ and every D-open cover $\underline{\gamma}(= \{\underline{G}_\alpha : \alpha \in \Lambda\} \subseteq \eta)$ of $\underline{F}$, there exists a finite sub-collection $\{\underline{G}_{\alpha_1}, \underline{G}_{\alpha_2}, \underline{G}_{\alpha_3}, ..., \underline{G}_{\alpha_n}\}$ of $\underline{\gamma}$ such that $\underline{F} \setminus \cup_{i=1}^{n} cl_\eta(\underline{G}_{\alpha_i}) \in \underline{\mathcal{I}}$.

Theorem 5.13. *Every DC-compact space is an $\underline{\mathcal{I}}^+$-DC-compact.*

Proof. Straightforward. □

On account of Theorems 3.12, 3.20, 4.3, 4.9, 4.7, 5.10, 5.9 and Theorem 5.13 , we have the following Corollary.

Corollary 5.14. *Let (X, η) be a double topological space and let $\underline{\mathcal{I}} \in DI(X)$. Then the following diagram is hold:*

CD-compact $\Rightarrow$ D-compact $\Rightarrow$ $\underline{\mathcal{I}}$ - D-compact

$\Downarrow$ $\Downarrow$ $\Downarrow$

DC-compact $\Rightarrow$ DQHC $\Rightarrow$ $\underline{\mathcal{I}}$-DQHC

$\Downarrow$

$\underline{\mathcal{I}}^+$-DC-compact

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A. Kandil (*dr.ali_kandil@yahoo.com*)
Department of Mathematics, Faculty of Science, Helwan University, Helwan, Egypt.

S. A. El-Sheikh (sobhyelsheikh@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

M. M. Yakout (mmyakout@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

Shawqi A. Hazza (*shawqialalimi@yahoo.com*)
Department of Mathematics, Faculty of Education, Taiz University, Taiz, Yemen.