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Vector soft topology

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Abstract. The aim of this paper is to introduce vector soft topology and soft neighbourhood system of zero soft element and to study separation properties of vector soft topology.

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1. Introduction

The concept of soft set theory has been initiated by Molodtsov [22] in 1999 as a general mathematical tool for modeling uncertainties. He also pointed out several applications of this theory in solving many practical problems in economics, engineering, social sciences, medical sciences etc. After that, in 2003, Maji et al. [19, 20] introduced several operations on soft sets and made a theoretical study of soft set theory in more detail and applied soft sets to decision making problems. In 2007, Aktas and Cagman [1] introduced a basic version of soft group theory, which extends the notion of a group to include the algebraic structures of soft sets. Jun [12, 13] investigated soft BCK/BCI- algebras and its application in ideal theory. Feng [9] in 2008, dealt with the concept of soft semirings; Shabir and Ali [29] (2009) studied soft semigroups and soft ideals; Kharal and Ahmed [17] as well as Majumdar and Samanta [21] defined soft mappings. In 2011, Shabir and Naz [30] came up with an idea of soft topological spaces. Later Aygun et al. [2], Zorlutuna et al. [31], Cagman et al. [4], Hussain et al. [11], Hazra et al. [10], studied on soft topological spaces. As a continuation of this, it is natural to investigate the behaviour of a combination of algebraic and topological structures in soft set theoretic form. Sk. Nazmul and S. K. Samanta studied topological group structures in soft setting approaching from different perspectives in [23, 24, 25, 26, 27]. In 2012 Sujoy Das and S. K. Samanta introduced soft real sets, soft real nos [5]. Later they introduced soft complex nos, soft complex sets, soft metrics etc [6, 7]. In 2013, Sujoy Das, Pinaki Majumdar and S. K. Samanta [8] introduced the concept of soft linear spaces and soft normed linear spaces. In view of this and also considering the importance of topological vector space in developing the theory of functional analysis, we have introduced in this paper a notion of vector soft topology. In this connection it is worth mentioning that in fuzzy setting some significant works have been done on fuzzy topological vector space structure by A. K. Katsaras [14, 15, 16] and D. B. Liu [14].

The organization of the paper is as follows:

Section 2 is the preliminary part where definitions and some properties of soft sets, soft topological spaces and soft product topological spaces are given. In section 3, we introduced the notion of convex and balanced soft sets in a vector space and studied some properties of them. In section 4, for the first time, we introduced a notion of vector soft topologies and studied some of its basic properties. In section 5, some facets of the system of neighbourhoods of the zero soft elements of vector soft topology are established. Section 6, concludes the paper. The straightforward proofs of the propositions have been omitted.

2. Preliminaries

Definition 2.1 ([22])**.** Let X be a universal set and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of α - approximate elements of the soft set (F, A).

In [18] the soft sets are redefined as follows:

Let E be the set of parameters and $A \subseteq E$. Then for each soft set (F, A) over X a soft set (H, E) is constructed over X, where $\forall \alpha \in E$,

$$H(\alpha) = \begin{cases} F(\alpha) & if\ \alpha \in A \\ \phi & if\ \alpha \in E \setminus A, \end{cases}$$

Thus the soft sets (F, A) and (H, E) are equivalent to each other and the usual set operations of the soft sets $(F_i, A_i), i \in \Delta$ is the same as those of the soft sets $(H_i, E), i \in \Delta$. For this reason, in this paper, we have considered our soft sets over the same parameter set A.

Following Molodtsov and Maji et al. [19, 20, 22] definitions of soft subset, absolute soft set, null soft set, arbitrary union of soft sets etc. are presented in [25] considering the same parameter set. For arbitrary intersection of soft sets we follow [9] considering the same parameter set. For image and inverse image of a soft set we follow definitions in [24].

Unless otherwise stated, X will be assumed to be an initial universal set, A will be taken to be a set of parameters and $S(X, A)$ denote the set of all soft sets over X.

Definition 2.2 ([26])**.** A soft set (E, A) over X is said to be a soft element if $\exists \alpha \in A$ such that $E(\alpha)$ is a singleton, say, $\{x\}$ and $E(\beta) = \phi$, $\forall \beta (\neq \alpha) \in A$. Such a soft element is denoted by E_α^x. Let $\Im$ be the set of all soft elements of the universal set X.

Definition 2.3 ([26]). The soft element E_α^x is said to be in the soft set (F, A), denoted by $E_\alpha^x \tilde{\in} (F, A)$, if $x \in F(\alpha)$.

Definition 2.4 ([3]). Let (F, A) and (G, A) be two soft sets over X. The parallel product of (F, A) and (G, A) is defined as $(F, A)\tilde{\times}(G, A) = (F\tilde{\times}G, A)$ where $[F\tilde{\times}G](\alpha) = F(\alpha)\times G(\alpha), \forall\alpha \in A$. It is clear that $(F\tilde{\times}G, A)$ is a soft set over $X\times X$.

Definition 2.5 ([27]). Let $S(X, A)$ denote the set of all soft sets over X under the parameter set A. A soft set $(F, A) \in S(X, A)$ is said to be pseudo constant soft set if $F(\alpha) = X$ or ϕ, $\forall\alpha \in A$. Let $CS(X, A)$ denote the set of all pseudo constant soft sets over X under the parameter set A.

Definition 2.6 ([30]). Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if

(i) $(\tilde{\Phi}, A)$, $(\tilde{X}, A) \in \tau$, where $\tilde{\Phi}(\alpha) = \phi$ and $\tilde{X}(\alpha) = X$, $\forall\alpha \in A$.

(ii) the intersection of any two soft sets in τ belongs to τ.

(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X, A, τ) is called a soft topological space over X.

Definition 2.7 ([30]). A crisp element $x \in X$ is said to be in the soft set (F, A) over X, denoted by $x\tilde{\in}(F, A)$ iff $x \in F(\alpha)$, $\forall\alpha \in A$.

Definition 2.8 ([30]). A soft set (F, A) is said to be τ soft nbd of an element $x \in X$ if $\exists$ $(G, A) \in \tau$ such that $x\tilde{\in}\,(G, A)\,\tilde{\subseteq}(F, A)$.

Definition 2.9 ([27]). A soft topology τ on X is said to be an enriched soft topology if (i) of Definition 2.6 is replaced by $(i)'$ $(F, A) \in \tau$, $\forall(F, A) \in CS(X, A)$.
The triplet (X, A, τ) is called an enriched soft topological space over X.

Definition 2.10 ([30]). Let (X, A, τ) be a soft topological space over X. Then the collection $\tau^\alpha = \{F(\alpha) : (F, A) \in \tau\}$ for each $\alpha \in A$, defines a topology on X.

Proposition 2.11. *Let X be a non-empty set, A be the set of parameters and for each $\alpha \in A$, τ^α is a crisp topology on X. Then $\tau^* = \{(G, A) \in S(X, A) : G(\alpha) \in \tau^\alpha, \forall\alpha \in A\}$ is an enriched soft topology on X.*

Proposition 2.12 ([26]). *If (X, A, τ) be a soft topological space and if $\tau^* = \{(G, A) \in S(X, A) : G(\alpha) \in \tau^\alpha, \forall\alpha \in A\}$, then τ^* is an enriched soft topology on X such that $\tau \subseteq \tau^*$ and $[\tau^*]^\alpha = \tau^\alpha, \forall\alpha \in A$.*

Definition 2.13 ([26]). Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Then
(i) the image of a soft set $(F, A) \in (X, A)$ under the mapping f is denoted by $f\left[(F, A)\right]$ and is defined by $f\left[(F, A)\right] = (f(F), A)$, where $\left[f(F)\right](\alpha) = f\left[F(\alpha)\right], \forall\alpha \in A$.
(ii) the inverse image of a soft set $(G, A) \in (Y, A)$ under the mapping f is denoted by $f^{-1}\left[(G, A)\right]$ and is defined by $f^{-1}\left[(G, A)\right] = (f^{-1}(G), A)$, where $\left[f^{-1}(G)\right](\alpha) = f^{-1}\left[G(\alpha)\right], \forall\alpha \in A$.

Note : For simplicity of notation we use $f\,(F, A)$ instead of any different symbol like $\tilde{f}\,[(F, A)]$ etc., assuming the meaning is contextually clear.

Definition 2.14 ([26])**.** Let (X, A, τ) and (Y, A, ν) be soft topological spaces. The mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be
(i) soft continuous if $f^{-1}[(F, A)] \in \tau, \forall (F, A) \in \nu$.
(ii) soft homeomorphism if f is bijective and f, f^{-1} are soft continuous.
(iii) soft open if $(F, A) \in \tau \Rightarrow f[(F, A)] \in \nu$.
(iv) soft closed if (F, A) is soft closed in $(X, A, \tau) \Rightarrow f[(F, A)]$ is soft closed in (Y, A, ν).

Proposition 2.15 ([25])**.** *Let (X, A, τ), (Y, A, ν) and (Z, A, ω) be soft topological spaces. If $f : (X, A, \tau) \to (Y, A, \nu)$ and $g : (Y, A, \nu) \to (Z, A, \omega)$ are soft continuous and $f(X) \subseteq Y$, then the mapping $gf : (X, A, \tau) \to (Z, A, \omega)$ is soft continuous.*

Definition 2.16 ([26])**.** Let τ be a soft topology on X. Then a soft set (F, A) is said to be a $\tau-$ soft neighbourhood (shortly soft nbd) of the soft element E^x_α if there exists a soft set $(G, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$.

The soft nbd system of a soft element E^x_α in (X, A, τ) is denoted by $N_\tau(E^x_\alpha)$.

Proposition 2.17 ([26])**.** *If $\{N_\tau(E^x_\alpha) : E^x_\alpha \in \Im\}$ be the system of soft nbds then*
(i) $N_\tau(E^x_\alpha) \neq \phi$, $\forall E^x_\alpha \in \Im$
(ii) $E^x_\alpha \tilde{\in} (F, A), \forall (F, A) \in N_\tau(E^x_\alpha)$
(iii) $(F, A) \in N_\tau(E^x_\alpha), (F, A) \tilde{\subseteq} (G, A) \Rightarrow (G, A) \in N_\tau(E^x_\alpha)$
(iv) $(F, A), (G, A) \in N_\tau(E^x_\alpha) \Rightarrow (F, A) \tilde{\cap} (G, A) \in N_\tau(E^x_\alpha)$
(v) $(F, A) \in N_\tau(E^x_\alpha) \Rightarrow \exists (G, A) \in N_\tau(E^x_\alpha)$ such that $(F, A) \tilde{\subseteq} (G, A)$ and $(G, A) \in N_\tau(E^x_\alpha), \forall E^x_\alpha \in (G, A)$.

Definition 2.18 ([27])**.** Let (X, A, τ) be a soft topological space. A soft element $E^x_\alpha \in \Im$ is said to be a limiting soft element of a soft set (F, A) over X if every open soft set containing E^x_α contains at least one soft element E^y_α of (F, A) other than E^x_α, i.e. if $(G, A) \in \tau$ with $E^x_\alpha \tilde{\in} (G, A)$, $F(\alpha) \cap [G(\alpha) - \{x\}] \neq \phi$.
The union of all limiting soft elements of (F, A) is a soft set over X, called the derived soft set of (F, A) and is denoted by $(F, A)'$ or (F', A).
The closure of a soft set (F, A) denoted by $\overline{(F, A)} = (\overline{F}, A)$ is defined by $\overline{(F, A)} = (F, A) \tilde{\cup} [(F', A)$.

Proposition 2.19 ([27])**.** *Let τ be a soft topology over X. A soft set (F, A) over X is a $\tau-$open soft set iff for $E^x_\alpha \tilde{\in} (F, A)$, $\exists (G, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$.*

Proposition 2.20 ([27])**.** *Let (X, A, τ) be a soft topological space. If (F, A) be a closed soft set then $(F, A) = \overline{(F, A)}$. But if (X, A, τ) is an enriched soft topological space and $(F, A) = \overline{(F, A)}$, then (F, A) is a closed soft set.*

Proposition 2.21 ([27])**.** *For any soft set (F, A), $\overline{F}(\alpha) = \overline{F(\alpha)}^\alpha$, $\forall \alpha \in A$, where $\overline{F(\alpha)}^\alpha$ is the closure of $F(\alpha)$ with respect to the topology τ^α.*

Proposition 2.22 ([25])**.** *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. A mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is soft continuous iff $\forall x \in X$ and $\forall (V, A) \in \nu$ such that $E^{f(x)}_\alpha \tilde{\in} (V, A)$, $\exists (U, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (U, A)$ and $f[(U, A)] \tilde{\subseteq} (V, A)$.*

Proposition 2.23 ([26])**.** *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. For a bijective mapping $f : (X, A, \tau) \to (Y, A, \nu)$, the following statements are equivalent:*
(i) $f : (X, A, \tau) \to (Y, A, \nu)$ is soft homeomorphism;
(ii) $f : (X, A, \tau) \to (Y, A, \nu)$ and $f^{-1} : (Y, A, \nu) \to (X, A, \tau)$ are soft continuous;
(iii) $f : (X, A, \tau) \to (Y, A, \nu)$ is both soft continuous and soft open;
(iv) $f : (X, A, \tau) \to (Y, A, \nu)$ is both soft continuous and soft closed;
If further, τ and ν are enriched soft topology, then $f : (X, A, \tau) \to (Y, A, \nu)$ is soft homeomorphism $\Leftrightarrow f[\overline{(F, A)}] = \overline{f[(F, A)]}, \forall (F, A) \in S(X, A)$.

Definition 2.24 ([27])**.** Let (X, A, τ) be a soft topological space. If for $E^x_\alpha, E^y_\beta \in \Im$ with $E^x_\alpha \neq E^y_\beta$, there exists
(i) $(F, A) \in \tau$ such that $[E^x_\alpha \tilde{\in} (F, A)$ and $E^y_\beta \tilde{\notin} (F, A)]$ or $[E^y_\beta \tilde{\in} (F, A)$ and $E^x_\alpha \tilde{\notin} (F, A)]$, then (X, A, τ) is called a soft T_0-space.
(ii) $(F, A), (G, A) \in \tau$ such that $[E^x_\alpha \tilde{\in} (F, A)$ and $E^y_\beta \tilde{\notin} (F, A)]$ and $[E^y_\beta \tilde{\in} (G, A)$ and $E^x_\alpha \tilde{\notin} (G, A)]$, then (X, A, τ) is called a soft T_1-space.
(iii) $(F, A), (G, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (F, A)$, $E^y_\beta \tilde{\in} (G, A)$ and $(F, A) \tilde{\cap} (G, A) = (\tilde{\Phi}, A)$, then (X, A, τ) is called a soft T_2-space.

Proposition 2.25 ([27])**.** *A soft topological space (X, A, τ) is soft T_1- space iff $\forall E^x_\alpha \in \Im$, $\{E^x_\alpha\}$ is soft closed.*

Definition 2.26 ([27])**.** A soft topological space (X, A, τ) is said to be soft regular space if for any soft closed set (F, A) and any soft element E^x_α such that $E^x_\alpha \tilde{\notin} (F, A)$ open soft sets $(U, A), (W, A)$ such that $E^x_\alpha \tilde{\in} (U, A)$, $(F, A) \tilde{\subseteq} (W, A)$ and $(U, A) \tilde{\cap} (W, A) = (\tilde{\Phi}, A)$.

Proposition 2.27 ([27])**.** *If a soft topological space (X, A, τ) is soft regular then $\forall E^x_\alpha \in \Im$ and $\forall (U.A) \in \tau$ such that $E^x_\alpha \in (U, A)$, $\exists (V, A) \in \tau$ such that $E^x_\alpha \in (V, A)$ and $\overline{(V, A)} \tilde{\subseteq} (U, A)$. The converse is true if τ is enriched.*

Definition 2.28 ([26])**.** Let (X, A, τ) be a soft topological space. A sub-collection $\mathscr{B}$ of τ is said to be an open base of τ if every member of τ can be expressed as the union of some members of $\mathscr{B}$.

Definition 2.29 ([25])**.** The soft topology in $X \times Y$ induced by the open base $\mathcal{F} = \{(F, A) \tilde{\times} (G, A) : (F, A) \in \tau, (G, A) \in \nu\}$ is said to be the product soft topology of the soft topologies τ and ν. It is denoted by $\tau \tilde{\times} \nu$. The soft topological space $[X \times Y, A, \tau \tilde{\times} \nu]$ is said to be the soft topological product of the soft topological spaces (X, A, τ) and (Y, A, ν).

Proposition 2.30 ([25])**.** *Let (X, A, τ) be the product space of two soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) respectively. Then the projection mappings $\pi_i : (X, A, \tau) \to (X_i, A, \tau_i)$, $i = 1, 2$ are soft continuous and soft open. Also $\tau_1 \tilde{\times} \tau_2$ is the smallest soft topology in $X \times Y$ for which the projection mappings are soft continuous. If further, (Y, A, ν) be any soft topological space then the mapping $f : (Y, A, \nu) \to (X, A, \tau)$ is soft continuous iff the mappings $\pi_i f : (Y, A, \nu) \to (X_i, A, \tau_i)$, $i = 1, 2$ are soft continuous.*

Proposition 2.31 ([25])**.** *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. Then the mapping $f : (X, A, \tau) \to (Y, A, \nu)$ defined by $f(x) = y_0$, $\forall x \in X$, where y_0 is a fixed element of Y, called a constant mapping, is soft continuous if τ contains all those soft sets (F, A) where $F(\alpha) = X$ or $F(\alpha) = \phi$, $\forall \alpha \in A$.*

Proposition 2.32 ([25])**.** *Let (X, A, τ) be the product space of two soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) respectively, where τ_2 (or τ_1) is an enriched soft topology. Let $a \in X_1$ (or X_2). Then the mapping $f : (X_2, A, \tau_2) \to (X, A, \tau)$ (or $f : (X_1, A, \tau_2) \to (X, A, \tau)$) defined by $f(x_2) = (a, x_2)$ (or $f(x_1) = (x_1, a)$) is soft continuous $\forall x_2 \in X_2$ (or $\forall x_1 \in X_1$).*

Proposition 2.33 ([25])**.** *Let (X, A, τ) be a soft topological space. Then for each $\alpha \in A, (\tau \tilde{\times} \tau)^\alpha = \tau^\alpha \times \tau^\alpha$.*

Proposition 2.34 ([25])**.** *Let (X, A, τ) be a soft topological space and define $T^* = \{(F, A) \in S(X \times X)\}$ such that $F(\alpha) \in \tau^\alpha \times \tau^\alpha$. Then T^* is a soft topology over $X \times X$ and $T^* = \tau^* \tilde{\times} \tau^*$, where τ^* is as in Proposition 2.12.*

We now state and prove the following results which will be useful in this paper.

Proposition 2.35. *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. Then for each $\alpha \in A$, $(\tau \tilde{\times} \nu)^\alpha = \tau^\alpha \times \nu^\alpha$.*

Proof. The proof is same as of Proposition 2.33. □

Proposition 2.36. *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces and define $T^* = \{(F, A) \in S(X \times Y) : F(\alpha) \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A\}$. Then T^* is a soft topology over $X \times Y$. and $T^* = \tau^* \tilde{\times} \nu^*$ where $\tau^* = \{(G, A) \in S(X) : G(\alpha) \in \tau^\alpha, \forall \alpha \in A\}$ and $\nu^* = \{(H, A) \in S(Y) : H(\alpha) \in \nu^\alpha, \forall \alpha \in A\}$.*

Proof. Since ϕ and $X \times Y \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$ we have $(\tilde{\phi}, A), (\widetilde{X \times Y}, A) \in T^*$.

Again, let $(F_1, A), (F_2, A) \in T^*$. Then $F_1(\alpha), F_2(\alpha) \in \tau^\alpha \times \nu^\alpha$.

So, $(F_1 \tilde{\bigcap} F_2)(\alpha) = F_1(\alpha) \bigcap F_2(\alpha) \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$.

Thus $(F_1, A) \tilde{\bigcap} (F_2, A) \in T^*$.

Next let $(F_i, A) \in T^*$, $i \in I$. So $(\bigcup_{i \in I} F_i)(\alpha) = \bigcup_{i \in I} [F_i(\alpha)] \in \tau^\alpha \times \nu^\alpha$, $\forall \alpha \in A$. Thus $\tilde{\bigcup}_{i \in I} (F_i, A) \in T^*$.

Therefore T^* is a soft topology over $X \times Y$.

Now let $(F, A) \in T^*$ and $\alpha \in A$.

Then $F(\alpha) \in \tau^\alpha \times \nu^\alpha$ and hence $\exists\ U_i \in \tau^\alpha, V_i \in \nu^\alpha, i \in I$ such that $F(\alpha) = \bigcup_{i \in I} U_i \times V_i$. for each pair $U_i \in \tau^\alpha, V_i \in \nu^\alpha$ take soft sets (F_{U_i}, A) and (F_{V_i}, A) such that $F_{U_i}(\alpha) = U_i, F_{U_i}(\beta) = \phi$, $\forall \beta (\neq \alpha) \in A$ and $F_{V_i}(\alpha) = V_i, F_{V_i}(\beta) = \phi$, $\forall \beta (\neq \alpha) \in A$.

So, $(F_{U_i}, A) \in \tau^*$ and $(F_{V_i}, A) \in \nu^*$ and hence $(F_{U_i} \tilde{\times} F_{V_i}, A) \in \tau^* \times \nu^*$. Also $(F_{U_i} \tilde{\times} F_{V_i})(\alpha) = F_{U_i}(\alpha) \times F_{V_i}(\alpha) = U_i \times V_i$ and $(F_{U_i} \tilde{\times} F_{V_i})(\beta) = F_{U_i}(\beta) \times F_{V_i}(\beta) = \phi, \forall \beta (\neq \alpha) \in A$.

Let $(G_\alpha, A) = \tilde{\bigcup}_{i \in I} (F_{U_i} \tilde{\times} F_{V_i}, A)$. Then $(G_\alpha, A) \in \tau^* \tilde{\times} \nu^*$ and $G_\alpha(\alpha) = \bigcup_{i \in I} U_i \times V_i = F(\alpha)$, $G_\alpha(\beta) = \phi, \forall \beta (\neq \alpha) \in A$.

Again, let $(G, A) = \tilde{\bigcup}_{\alpha \in A} (G_\alpha, A)$. Then $(G, A) \in \tau^* \tilde{\times} \nu^*$ and $G(\alpha) = F(\alpha)$, $\forall \alpha \in A$.

Thus $(F,A) = (G,A) \in \tau^* \tilde{\times} \nu^*$.
$\therefore T^* \subseteq \tau^* \tilde{\times} \nu^*$.
Also, let $(F,A) \in \tau^* \tilde{\times} \nu^*$. Then $\exists \{(U_i,A) \in \tau^*, (V_i,A) \in \nu^*, i \in I\}$ such that $(F,A) = \tilde{\bigcup}_{i \in I}[(U_i,A) \tilde{\times} (V_i,A)]$. Also $F(\alpha) = \bigcup_{i\in I}[U_i(\alpha) \times V_i(\alpha)] \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$.
Hence $(F,A) \in T^*$.
$\therefore \tau^* \tilde{\times} \nu^* \subseteq T^*$. Thus $T^* = \tau^* \tilde{\times} \nu^*$. □

3. Convex and Balanced Soft Set

Definition 3.1. Let (F,A) and (G,A) be two soft sets over the vector space V over the field K, the field of real and complex numbers. Then
(i) $(F,A) + (G,A) = (F+G,A)$ where $(F+G)(\alpha) = F(\alpha) + G(\alpha)$, $\forall \alpha \in A$.
(ii) $k(F,A) = (kF,A)$ where $(kF)(\alpha) = \{k \cdot x : x \in F(\alpha)\}$, $\forall \alpha \in A, \forall k \in K$.
(iii) $x + (F,A) = (x+F,A)$ where $(x+F)(\alpha) = \{x+y : y \in F(\alpha)\}$, $\forall \alpha \in A, \forall x \in V$.
(iv) If (E,A) be any soft set over K then $(E,A) \cdot (F,A) = (E \cdot F, A)$ where $(E \cdot F)(\alpha) = E(\alpha) \cdot F(\alpha)$, $\forall \alpha \in A$.

Note : Here actually if $\oplus$ is vector addition in V and $+$, $\cdot$ are the scalar addition and scalar multiplication in K, then $\tilde{+}$, $\tilde{\cdot}$ may be used for denoting operations on soft sets (F,A), (G,A) (e.g. $(F,A) \tilde{+} (G,A)$, $x \tilde{+} (F,A)$, $k \tilde{\cdot} (F,A)$ etc.). But for the simplicity of notations we use the same symbols " $+$" and " juxtaposition " instead of " $\tilde{+}$ " and " $\tilde{\cdot}$ " respectively as the differences in their use are contextually understood.

Definition 3.2. A soft set (F,A) over a vector space V is said to be
(a) convex if $k(F,A) + (1-k)(F,A) \tilde{\subseteq} (F,A)$, $\forall k \in [0,1]$.
(b) balanced if $k(F,A) \tilde{\subseteq} (F,A)$ for all scalar k with $\mid k \mid \leq 1$.
(c) absolutely convex if it is balanced and convex.

Note : It is to be noted that
(1) (F,A) is convex (balanced) soft set iff for each $\alpha \in A$, the ordinary set $F(\alpha)$ is convex (balanced).
(2) If (F,A) and (G,A) are two convex (balanced) soft sets in a vector space V over the scalar field K, then $k_1(F,A) + k_2(G,A)$ is a convex (balanced) soft set in V for all scalars $k_1, k_2 \in K$.
(3) If $\{(F_i,A)\}_{i \in I}$ is a family of convex (balanced) soft sets in a vector space V, then $(F,A) = \tilde{\bigcap}_{i\in I}(F_i,A)$ is a convex (balanced) soft set in V.

Proposition 3.3. *Let V and W be two vector spaces over the scalar field K and let $f : V \to W$ be a linear map.*
(a) *If (F,A) is a convex (balanced) soft set in V, then $f[(F,A)]$ is a convex (balanced) soft set in W.*
(b) *$f^{-1}[(G,A)]$ is a convex (balanced) soft set in V whenever (G,A) is a convex (balanced) soft set in W.*

Proof. (a) We will prove the result for the convex case. The proof for the balanced case is similar. Let $k \in [0,1]$ and (F,A) be a convex soft set in V. Then
$[kf[(F,A)] + (1-k)f[(F,A)]](\alpha)$
$= [kf[(F,A)]](\alpha) + [(1-k)f[(F,A)]](\alpha)$

$= kf(F(\alpha)) + (1-k)f(F(\alpha))$
$= f(kF(\alpha) + (1-k)F(\alpha))$
$\subseteq f(F(\alpha)), \forall \alpha \in A.$
$\therefore kf\left[(F,A)\right] + (1-k)f\left[(F,A)\right] \tilde{\subseteq} f\left[(F,A)\right]$, which proves that $f\left[(F,A)\right]$ is a convex soft set.

(b) Assume next that (G,A) is a convex soft set in W and let $k \in [0,1]$. Set $(M,A) = kf^{-1}\left[(G,A)\right] + (1-k)f^{-1}\left[(G,A)\right]$. Then $\forall \alpha \in A$,
$\left[f(M)\right](\alpha) = kf\left[\left[f^{-1}(G)\right](\alpha)\right] + (1-k)f\left[\left[f^{-1}(G)\right](\alpha)\right] = kf\left[f^{-1}\left[G(\alpha)\right]\right] + (1-k)f\left[f^{-1}\left[G(\alpha)\right]\right] \subseteq kG(\alpha) + (1-k)G(\alpha) \subseteq G(\alpha)$,
and hence $M(\alpha) \subseteq f^{-1}\left[G(\alpha)\right]$.
Therefore $(M,A) \tilde{\subseteq} f^{-1}\left[(G,A)\right]$. Thus $f^{-1}\left[(G,A)\right]$ is a convex soft set in V. □

4. Vector Soft Topology

Throughout the rest of the paper we use the notation V for the vector space $(V,+,\cdot)$ over the scalar field K, where K is the field of real or complex numbers, A is the parameter set. Also, we use the notation xy instead of $x \cdot y$.

Definition 4.1. Let K be the field of real or complex numbers, A be the parameter set and ν^α be the usual topology on K, $\forall \alpha \in A$. Then the soft topology ν defined as in Proposition 2.11 is called the soft usual topology on K.

Definition 4.2. Let V be a vector space over the scalar field K endowed with the soft usual topology ν, A be the parameter set and τ be a soft topology on V. Then τ is said to be a vector soft topology on V if the mappings:
(1) $f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x,y) = x + y$ and
(2) $g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k,x) = kx$
are soft continuous $\forall x, y \in V$ and $\forall k \in K$.

Note : When we consider τ as a vector soft topology on V we always consider the scalar field K with the soft usual topology.

Proposition 4.3. *Let τ be a vector soft topology on a vector space V over the field K, A be the parameter set and ν be the soft usual topology on K. Then τ^α is a vector topology on V, $\forall \alpha \in A$.*

Proof. Let $U \in \tau^\alpha$. Then $\exists\, (F,A) \in \tau$ such that $F(\alpha) = U$. Since τ is a vector soft topology, we have the mappings
$f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x,y) = x + y$ and
$g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k,x) = kx$
are soft continuous and hence $f^{-1}((F,A)) \in \tau \tilde{\times} \tau$ and $g^{-1}((F,A)) \in \nu \tilde{\times} \tau$.
So, $f^{-1}(F(\alpha)) \in (\tau \tilde{\times} \tau)^\alpha = \tau^\alpha \times \tau^\alpha, \forall \alpha \in A$ and $g^{-1}(F(\alpha)) \in (\nu \tilde{\times} \tau)^\alpha = \nu^\alpha \times \tau^\alpha, \forall \alpha \in A$.
$\therefore f : (V \times V, \tau^\alpha \times \tau^\alpha) \to (V, \tau^\alpha)$ and $g : (K \times V, \nu^\alpha \times \tau^\alpha) \to (V, \tau^\alpha)$ are continuous, $\forall \alpha \in A$. Hence, τ^α is a vector topology on V, $\forall \alpha \in A$. □

Proposition 4.4. *Let V be a vector space over the scalar field K endowed with the soft usual topology ν, A be the parameter set and $\forall \alpha \in A$, τ^α be a vector topology on V. Then τ^* is a vector soft topology on V, where τ^* is defined as in Proposition 2.11.*

Proof. Let $(F, A) \in \tau^*$. Then $F(\alpha) \in \tau^\alpha$, $\forall \alpha \in A$.
Since $\forall \alpha \in A$, τ^α is a vector topology on V whereas ν^α is a usual topology on K, we have the mappings $f : (V \times V, \tau^\alpha \times \tau^\alpha) \to (V, \tau^\alpha)$, defined by $f(x, y) = x + y$ and $g : (K \times V, \nu^\alpha \times \tau^\alpha) \to (V, \tau^\alpha)$, defined by $g(k, x) = kx$ are continuous, $\forall x, y \in V$ and $\forall k \in K$, $\forall \alpha \in A$.
So, $f^{-1}(F(\alpha)) \in \tau^\alpha \times \tau^\alpha, \forall \alpha \in A$ and $g^{-1}(F(\alpha)) \in \nu^\alpha \times \tau^\alpha, \forall \alpha \in A$ and hence, by Proposition 2.34 and Proposition 2.36 $f^{-1}((F, A)) \in T^* = \tau^* \tilde{\times} \tau^*$ and $g^{-1}((F, A)) \in S^* = \nu^* \tilde{\times} \tau^*$.

Thus, the mappings $f : (V \times V, A, \tau^* \tilde{\times} \tau^*) \to (V, A, \tau^*)$, defined by $f(x, y) = x + y$ and $g : (K \times V, A, \nu^* \tilde{\times} \tau^*) \to (V, A, \tau^*)$, defined by $g(k, x) = kx$ are soft continuous $\forall x, y \in V$ and $\forall k \in K$. Therefore, τ^* is a vector soft topology on V. □

Example 4.5. (i) Let V be a vector space over the scalar field K where K is equipped with the soft usual topology; $\tau_1 = \left\{\left(\tilde{\Phi}, A\right), \left(\tilde{V}, A\right)\right\}$ and $\tau_2 = \{$all soft sets over $V\}$ are soft topologies on V.
Then $\tau_1 \tilde{\times} \tau_1 = \left\{\left(\tilde{\Phi}, A\right), \left(\widetilde{V \times V}, A\right)\right\}$ and $\tau_2 \tilde{\times} \tau_2 = \{$all soft sets over $V \times V\}$. It can be easily shown that τ_1 and τ_2 are vector soft topologies.

(ii) Consider the vector space $\mathbb{R}$ over the scalar field $\mathbb{R}$ where the scalar field $\mathbb{R}$ is equipped with the soft usual topology. Let $A = \{e_1, e_2, e_3\}$ and $\tau^{e_1} =$ Indiscrete topology on $\mathbb{R}$, $\tau^{e_2} =$ Discrete topology on $\mathbb{R}$, $\tau^{e_3} =$ The usual topology on $\mathbb{R}$. Then by Proposition 4.4, $\tau^* = \{(F, A) : F(e_i) \in \tau^{e_i}; i = 1, 2, 3\}$ is a vector soft topology on the vector space $\mathbb{R}$.

From Proposition 2.31, it is seen that if τ is enriched, then any constant mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is soft continuous.
But the converse is not true. i.e. for continuity of constant mapping enrichedness of τ is sufficient but not necessary, which follows from the following two *Examples*.

Example 4.6. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{\alpha, \beta, \gamma\}$ and $\tau = \left\{\left(\tilde{X}, A\right), \left(\tilde{\Phi}, A\right), (F_1, A) = \{\{x_1\}, \phi, \phi\}, (F_2, A) = \{X, \phi, \phi\}\right\}$. Then τ is a soft topology on X and any constant map $f : (X, A, \tau) \to (X, A, \tau)$ is soft continuous, though τ is not enriched.

Example 4.7. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3\}$, $A = \{\alpha, \beta, \gamma\}$,
$\tau = \left\{\left(\tilde{X}, A\right), \left(\tilde{\Phi}, A\right), (F_1, A) = \{X, \phi, \phi\}, (F_2, A) = \{\phi, X, \phi\}, (F_3, A) = \{X, X, \phi\}\right\}$ and $\nu = \left\{\left(\tilde{X}, A\right), \left(\tilde{\Phi}, A\right), (G_1, A) = \{\{y_2, y_3\}, \{y_1\}, \phi\}\right\}$. Then (X, A, τ) and (Y, A, ν) are two soft topological spaces and every constant map $f : (X, A, \tau) \to (Y, A, \nu)$ is soft continuous though τ is not enriched.

So, we introduce the following definition:

Definition 4.8. Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. Then τ is said to be weak enriched iff any constant mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is soft continuous.

Thus the Proposition 2.32 can be modified as:

Proposition 4.9. *Let (X, A, τ) be the product space of two soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) respectively, where τ_2 (or τ_1) is a weak enriched soft topology. Let $a \in X_1$ (or X_2). Then the mapping $f : (X_2, A, \tau_2) \to (X, A, \tau)$ (or $f : (X_1, A, \tau_2) \to (X, A, \tau)$) defined by $f(x_2) = (a, x_2)$ (or $f(x_1) = (x_1, a)$) is soft continuous $\forall x_2 \in X_2$(or $\forall x_1 \in X_1$).*

Proposition 4.10. *Let τ be a vector soft topology on a vector space V over the field K endowed with the soft usual topology ν. If further, τ is a weak enriched soft topology, then the map $M_k : (V, A, \tau) \to (V, A, \tau)$, defined by $M_k(x) = kx$ is soft continuous, $\forall k \in K$ and M_k is a soft homeomorphism for $k \neq 0$.*

Proof. Since τ be a vector soft topology on a vector space V, the map $g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k, x) = kx$ is soft continuous. Also since τ is weak enriched, by Proposition 4.9, the map $h : (V, A, \tau) \to (K \times V, A, \nu \tilde{\times} \tau)$, defined by $h(x) = (k, x)$ is soft continuous for a fixed $k \in K$. Hence $M_k = g \circ h$ is soft continuous. In case $k \neq 0$, $M_k^{-1}(x) = x/k$ is soft continuous. Therefore, M_k is a soft homeomorphism. □

Proposition 4.11. *Let τ be a vector soft topology on a vector space V over the field K endowed with the soft usual topology ν. If further, τ is a weak enriched soft topology, then the map $T_a : (V, A, \tau) \to (V, A, \tau)$, defined by $T_a(x) = a + x$ is soft homeomorphism for any $a \in V$.*

Proof. The proof is similar as above. □

Proposition 4.12. *Let τ be a vector soft topology on a vector space V over the field K endowed with the soft usual topology ν and τ be a weak enriched soft topology. Then the mapping $h : (V \times V, A, \tau \tilde{\times} \tau) \to (V \times V, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is soft continuous for all scalars $a, b \in K$ and $x, y \in V$.*

Proof. We know that the mappings $\pi_i : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau), i = 1, 2$; defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are soft continuous. Also, $M_k : (V, A, \tau) \to (V, A, \tau)$, defined by $M_k(x) = kx$ is soft continuous, $\forall k \in K$.
Now, $\pi_1 h : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $\pi_1 h(x, y) = \pi_1(ax, by) = ax = M_a \pi_1(x, y)$.
$\therefore \pi_1 h \, (= M_a \pi_1)$ is soft continuous.
Similarly, $\pi_2 h \, (= M_b \pi_2)$ is soft continuous.
Then from Proposition 2.30, we get that the mapping $h : (V \times V, A, \tau \tilde{\times} \tau) \to (V \times V, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is soft continuous for all scalars $a, b \in K$ and $x, y \in V$. □

Proposition 4.13. *A weak enriched soft topology τ on a vector space V over the field K, where K endowed with the soft usual topology ν, is a vector soft topology iff the mapping $L_{(a,b)} : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is soft continuous $\forall a, b \in K$ and $\forall x, y \in V$.*

Proof. Let τ be a vector soft topology. Therefore $f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x, y) = x + y$ is continuous, $\forall x, y \in V$. Also, from Proposition 4.12, the mapping $h : (V \times V, A, \tau \tilde{\times} \tau) \to (V \times V, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is soft continuous for all scalars $a, b \in K$ and $x, y \in V$.

Therefore, $L_{(a,b)} = f \circ h : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by

$L_{(a,b)}(x, y) = f(h(x, y)) = f(ax, by) = ax + by$, is soft continuous $\forall a, b \in K$ and $\forall x, y \in V$.

Conversely, let the mapping $L_{(a,b)} : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is soft continuous $\forall a, b \in K$ and $\forall x, y \in V$.

We know that the mappings $\pi_1 : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $\pi_1(x, y) = x$ and $\pi_2 : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $\pi_2(k, y) = y$ are soft continuous $\forall k \in K$ and $\forall x, y \in V$. Also, since τ is weak enriched by Proposotion 4.9, $h_0 : (V, A, \tau) \to (V \times V, A, \tau \tilde{\times} \tau)$, defined by $h_0(x) = (x, \theta)$ is soft continuous $\forall x \in V$, where θ is the zero element of V.

Therefore, $g = \pi_1 \circ L_{(a,0)} \circ h_0 \circ \pi_2 : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$,defined by

$$\begin{aligned} g(a, x) &= \pi_1 \circ L_{(a,0)} \circ h_0 \circ \pi_2(a, x) \\ &= \pi_1(L_{(a,0)}(h_0(\pi_2(a, x)))) \\ &= \pi_1(L_{(a,0)}(h_0(x))) \\ &= \pi_1(L_{(a,0)}(x, \theta)) \\ &= \pi_1(ax, \theta) \\ &= ax \end{aligned}$$

is soft continuous, $\forall a \in K$ and $\forall x \in V$.

Since, $L_{(a,b)} : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is soft continuous $\forall a, b \in K$ and $\forall x, y \in V$, taking $a = 1, b = 1$; we can define $f = L_{(1,1)} : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, such that $f(x, y) = L_{(1,1)}(x, y) = x + y$.

Then f is soft continuous. Then τ is a vector soft topology on V. □

Proposition 4.14. *Let τ be a weak enriched vector soft topology on a vector space V over the field K. If (F, A) is a soft open set and t is a non-zero scalar, then (tF, A) is also a soft open set.*

Proof. Since for $t \neq 0$, $M_t : (V, A, \tau) \to (V, A, \tau)$ is a soft homeomorphism, $(tF, A) = M_t(F, A)$ is soft open whenever (F, A) is so. □

Corollary 4.15. *Let τ be a weak enriched vector soft topology on a vector space V over the field K. If (F, A) is a soft neighborhood of an element $x \in V$ and t is a non-zero scalar, then (tF, A) is soft neighborhood of tx.*

Proposition 4.16. *Let τ be a weak enriched vector soft topology on a vector space V over the field K. If (F, A) is soft open, then $[x_0 + (F, A)]$ is soft open for all $x_0 \in V$.*

Proof. Since the map $T_{x_0} : (V, A, \tau) \to (V, A, \tau)$, defined by $T_{x_0}(x) = x + x_0$ is a soft homeomorphism and $T_{x_0}(F, A) = [x_0 + (F, A)]$, $[x_0 + (F, A)]$ is soft open for all $x_0 \in V$ and $\forall (F, A) \in \tau$. □

Proposition 4.17. *Let τ be a weak enriched vector soft topology on a vector space V over the field K. If (F, A) is a soft neighborhood of an element $x_0 \in V$ then $-x_0 + (F, A)$ is a neighborhood of zero of V.*

Proposition 4.18. *Let (V, A, τ) be a soft topological space over a vector space V over the field K, where K is equipped with the soft usual topology ν. Then τ is a vector soft topology if and only if*

(i) $\forall x, y \in V$, $\forall \alpha \in A$ *and* $\forall (W, A) \in \tau$ *with* $E_\alpha^{x+y} \tilde{\in} (W, A)$, $\exists (F, A), (G, A) \in \tau$ *such that* $E_\alpha^x \tilde{\in} (F, A), E_\alpha^y \tilde{\in} (G, A)$ *and* $(F + G, A) \tilde{\subseteq} (W, A)$.
(ii) $\forall x \in V$, $\forall k \in K$, $\forall \alpha \in A$ *and* $\forall (W, A) \in \tau$ *with* $E_\alpha^{kx} \tilde{\in} (W, A)$, $\exists (G, A) \in \nu, (F, A) \in \tau$ *such that* $E_\alpha^x \tilde{\in} (F, A), E_\alpha^k \tilde{\in} (G, A)$ *and* $(G \cdot F, A) \tilde{\subseteq} (W, A)$.

Proof. Let τ be a vector soft topology on the vector space V over the scalar field K. Therefore $f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x, y) = x + y$, $g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k, x) = kx$ is continuous, $\forall x, y \in V, \forall k \in K$.
Let $x, y \in V$, $\alpha \in A$ and $(W, A) \in \tau$ with $E_\alpha^{f(x,y)} = E_\alpha^{x+y} \tilde{\in} (W, A)$. Then by Proposition 2.22, $\exists (U, A) \in (\tau \tilde{\times} \tau)$ such that $E_\alpha^{(x.y)} \tilde{\in} (U, A)$and $f(U, A) \tilde{\subseteq} (W, A)$.
Since $(U, A) \in (\tau \tilde{\times} \tau)$ $\exists \{(U_i, A), (G_i, A) \in \tau, i \in I\}$ such that
$(U, A) = \tilde{\cup}_{i \in I} [(U_i, A) \tilde{\times} (G_i, A)]$. So, $\exists i \in I$ such that $E_\alpha^x \tilde{\in} (U_i, A)$, $E_\alpha^y \tilde{\in} (G_i, A)$.
Now $(U_i, A) \tilde{\times} (G_i, A) \in (\tau \tilde{\times} \tau)$ and $(U_i, A) \tilde{\times} (G_i, A) \tilde{\subseteq} (U, A)$ and hence
$(U_i + V_i, A) = f[(U_i, A) \tilde{\times} (G_i, A)] \tilde{\subseteq} f[(U, A)] \tilde{\subseteq} (W, A)$.
Thus the condition (i) is satisfied.
Condition (ii) can be proved similarly.
Conversely, let the given conditions be satisfied.
Let $(W, A) \in \tau$ and $E_\alpha^{(x.y)} \tilde{\in} f^{-1}[(W, A)]$.
Then $f[E_\alpha^{(x.y)}] = E_\alpha^{f(x,y)} = E_\alpha^{x+y} \tilde{\in} (W, A)$ and hence by the given condition (i),
$\exists (F_x, A), (G_y, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (F_x, A), E_\alpha^y \tilde{\in} (G_y, A)$ and $(F_x + G_y, A) \tilde{\subseteq} (W, A)$.
So, $(F_x, A) \tilde{\times} (G_y, A) \in (\tau \tilde{\times} \tau)$ and $f[(F_x, A) \tilde{\times} (G_y, A)] = (F_x + G_y, A) \tilde{\subseteq} (W, A)$.
So, $(F_x, A) \tilde{\times} (G_y, A) \tilde{\subseteq} f^{-1}[(W, A)]$ and
$f^{-1}[(W, A)] = \tilde{\cup} \{E_\alpha^{(x.y)} : E_\alpha^{(x.y)} \tilde{\in} f^{-1}[(W, A)]\}$
$\tilde{\subseteq} \tilde{\cup} \{(F_x, A) \tilde{\times} (G_y, A) : E_\alpha^{(x.y)} \tilde{\in} f^{-1}[(W, A)]\}$
$\tilde{\subseteq} f^{-1}[(W, A)]$.
Thus $f^{-1}[(W, A)] = \tilde{\cup} \{(F_x, A) \tilde{\times} (G_y, A) : E_\alpha^{(x.y)} \tilde{\in} f^{-1}[(W, A)]\} \in (\tau \tilde{\times} \tau)$ and hence, the mapping $f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x, y) = x + y$ is soft continuous.
Continuity of the mapping $g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k, x) = kx$ can be proved similarly.
Therefore, (V, A, τ) is a vector soft topology on the vector space V over the field K. □

5. Neighbourhood system of zero

In this section we study some basic properties of the system of neighbourhoods of the zero vector. In this section, τ is a vector soft topology on a vector space V over the scalar field K, A be the parameter set, the soft topology on K is the soft usual topology ν and θ is the zero vector of V.

Definition 5.1. A collection $\mathscr{B}$ of soft neighbourhoods of a soft element E_α^x is said to be a fundamental soft nbd system or soft nbd base of E_α^x if for any soft nbd (F, A) of E_α^x, $\exists (H, A) \in \mathscr{B}$ such that $(H, A) \tilde{\subseteq} (F, A)$.

Proposition 5.2. *Let* $\mathscr{B}$ *be a fundamental soft nbd system of the soft element* E_α^θ *in* (V, A, τ). *Then* $\forall (F, A) \in \mathscr{B}$, $(-F, A)$ *is also a soft nbd of* E_α^θ, *where* $(-F)(\alpha) = \{-x : x \in F(\alpha)\}$.

Proof. Proof follows from Proposition 4.15. □

Definition 5.3. A soft set (F, A) over a vector space V over the field K is said to be symmetric iff $(-F, A) = (F, A)$. A soft nbd system is said to be symmetric soft nbd system if all the members of that system are symmetric.

Proposition 5.4. *Let τ be a vector soft topology over a vector space V over the field K. Then there exists a fundamental symmetric soft nbd system of the soft element E_α^θ in (V, A, τ).*

Proposition 5.5. *Let τ be a vector soft topology on a vector space V over the scalar field K. Then for a soft nbd (W, A) of E_α^θ $\exists$ a balanced soft nbd (F, A) of E_α^θ such that $F(\alpha) \subseteq W(\alpha)$. If further, τ is enriched soft topology, then*
(i) every soft nbd of E_α^θ contains a balanced soft nbd of E_α^θ.
(ii) every convex soft nbd of E_α^θ contains an absolutely convex soft nbd of E_α^θ.

Proof. Let $(W, A) \in \tau$ and $E_\alpha^\theta \in (W, A)$. By the continuity of $g : (k, x) \to kx$ and $g\left(E_\alpha^{(0,\theta)}\right) = E_\alpha^{g(0,\theta)} = E_\alpha^\theta$, $\exists\,(H, A) \in \nu$ and $(G, A) \in \tau$ such that $E_\alpha^{(0,\theta)} \tilde{\in} \left(H \tilde{\times} G, A\right)$ and $g\left(H \tilde{\times} G, A\right) \tilde{\subseteq} (W, A)$. So $\exists\ \delta > 0$ such that $\gamma G(\alpha) \subseteq W(\alpha)$ for all γ with $\mid \gamma \mid < \delta$. Consider $(F, A) = \cup \{(\rho G, A) :\mid \rho \mid < \delta\}$. Then (F, A) is a balanced nbd of E_α^θ and $F(\alpha) \subseteq W(\alpha)$.
(i) If further, τ is an enriched then taking $F(\alpha) = \cup \{\rho H(\alpha) :\mid \rho \mid < \delta\}$ and $F(\beta) = \phi$, $\forall \beta \neq \alpha$ we see that (F, A) is a balanced soft nbd of E_α^θ and $(F, A) \subseteq (W, A)$.
(ii) Let (F, A) be a convex soft nbd of E_α^θ. Then $\exists\ (H, A) \in \tau$ such that $E_\alpha^\theta \tilde{\in} (H, A)$ $\tilde{\subseteq} (F, A)$. Then $\theta \in H(\alpha) \subseteq F(\alpha)$. Again since τ^α is a vector topology on V, $H(\alpha) \in \tau^\alpha$. Then $F(\alpha)$ is a convex nbd of θ in (V, τ^α). Then by topological vector space theory, $\exists$ a balanced and convex nbd W of θ in (V, τ^α) such that $\theta \subseteq W \subseteq F(\alpha)$.
Now construct a soft set (G, A) such that $G(\alpha) = W$ and $G(\beta) = \phi$,$\forall \beta \neq \alpha$. Then $(G, A) \in \tau$, as τ is enriched and $E_\alpha^\theta \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$. Also (G, A) is balanced and convex soft set.
$\therefore$ (G, A) is an absolutely convex soft nbd of E_α^θ such that $(G, A) \tilde{\subseteq} (F, A)$. □

Corollary 5.6. *Let τ be an enriched vector soft topology over a vector space V over the scalar field K, K is equipped with the soft usual topology. Then $\exists$ a balanced soft nbd base of the soft element E_α^θ in (V, A, τ).*

Proof. Follows from Proposition 5.5. □

Proposition 5.7. *For each soft nbd (W, A) of E_α^θ in (V, A, τ) where τ is a weak enriched soft topology and for each finite set $\{k_1, k_2, ..., k_n\}$ with $k_i = 1$ or -1, $i = 1, 2, ..., n$, $\exists$ a symmetric soft nbd (U, A) of E_α^θ such that*

$$(k_1 U, A) + (k_2 U, A) + ... + (k_n U, A) \tilde{\subseteq} (W, A).$$

Proof. Let (W, A) be a soft nbd of E_α^θ. Since $(k_1 E_\alpha^\theta) + (k_2 E_\alpha^\theta) + ... + (k_n E_\alpha^\theta) = E_\alpha^\theta$, by soft continuity of addition and scalar multiplication in (V, A, τ), it follows that there exist soft nbds (F_1, A), (F_2, A),........,(F_n, A) of E_α^θ such that $(k_1 F_1, A) + (k_2 F_2, A) + ... + (k_n F_n, A) \tilde{\subseteq} (W, A)$.
Let $(F, A) = (F_1, A) \tilde{\cap} (F_2, A) \tilde{\cap} .... \tilde{\cap} (F_n, A)$. Then (F, A) is a soft nbd of E_α^θ and since τ is a weak enriched soft topology, $(-F, A)$ is a soft nbd of E_α^θ. Let $(U, A) = (F, A) \tilde{\cap} (-F, A)$. Then (U, A) is a symmetric soft nbd of E_α^θ.

Thus $(k_1U, A) + (k_2U, A) + \ldots\ldots\ldots + (k_nU, A) \,\tilde{\subseteq}(k_1F, A) + (k_2F, A) + \ldots\ldots\ldots + (k_nF, A) \,\tilde{\subseteq}(k_1F_1, A) + (k_2F_2, A) + \ldots\ldots\ldots + (k_nF_n, A) \,\tilde{\subseteq}(W, A)$. □

Proposition 5.8. *Let $\mathscr{B}$ be a soft nbd base of the soft element E_α^θ in (V, A, τ), where τ is a weak enriched soft topology. Then $\mathscr{B}' = \{x + (F, A) : (F, A) \in \mathscr{B}\}$ is a soft nbd base of E_α^x.*

Proof. Let (G, A) be any soft nbd of E_α^x. Then $T_{-x}[(G, A)]$ is a soft nbd of $T_{-x}(E_\alpha^x)$. i.e. $-x + (G, A)$ is a soft nbd of $-x + E_\alpha^x = E_\alpha^\theta$. So, $\exists (F, A) \in \mathscr{B}$ such that $E_\alpha^\theta \tilde{\in} (F, A) \tilde{\subseteq} -x + (G, A)$. Hence $T_x[(F, A)]$ is a soft nbd of $T_x(E_\alpha^\theta)$ such that $T_x(E_\alpha^\theta)$ $\tilde{\in} T_x\,[(F, A)] \,\tilde{\subseteq} T_x\,[-x + (G, A])$. Thus $E_\alpha^x \,\tilde{\in}\, [x + (F, A)] \,\tilde{\subseteq}\, [x - x + (G, A)] = (G, A)$. Thus $\mathscr{B}'$ is a soft nbd base of E_α^x. □

Proposition 5.9. *If $\mathscr{B}$ be a soft nbd base of the soft element E_α^x in (V, A, τ), where τ is a weak enriched soft topology. Then*
(i) $\mathscr{B}' = \{-x + (F, A) : (F, A) \in \mathscr{B}\}$ is a soft nbd base of E_α^θ.
(ii) $\mathscr{B}' = \{(tF, A) : (F, A) \in \mathscr{B}\}$ is a soft nbd base of E_α^{tx}, where t is a non zero scalar.

Proposition 5.10. *If $\mathscr{B}$ be the system of all nbds of the soft element E_α^θ in (V, A, τ), then $\mathscr{B}_\alpha = \{U(\alpha) : (U, A) \in \mathscr{B}\}$ is the system of all nbds of θ in (V, τ^α).*

Proposition 5.11. *Let $\mathscr{B}$ be the system of all nbds of the soft element E_α^θ in (V, A, τ) and (F, A) be any soft set over V. Then $\overline{F(\alpha)} = \cap_{(U,A)\in\mathscr{B}}\,[F(\alpha) + U(\alpha)]$ $= \cap_{(U,A)\in\mathscr{B}}\,[U(\alpha) + F(\alpha)]$.*

Proof. Since $\mathscr{B}$ be the system of all nbds of the soft element E_α^θ, $\mathscr{B}_\alpha = \{U(\alpha) : (U, A) \in \mathscr{B}\}$ is system of all nbds of θ in (V, τ^α). So, using Proposition 2.21 we have, $\overline{F(\alpha)} = \overline{F(\alpha)}^{\alpha} = \cap_{(U,A)\in\mathscr{B}}\,[F(\alpha) + U(\alpha)] = \cap_{(U,A)\in\mathscr{B}}\,[U(\alpha) + F(\alpha)]$. □

Proposition 5.12. *For any soft nbd (U, A) of E_α^θ in (V, A, τ), where τ is a weak enriched soft topology, $\exists$ a soft nbd (W, A) of E_α^θ such that $\overline{W(\alpha)} \subseteq U(\alpha)$. If further, τ is enriched, then $\overline{(W, A)} \tilde{\subseteq} (U, A)$.*

Proof. Let (U, A) any soft nbd of E_α^θ. Then by Proposition 5.7, $\exists$ a soft nbd (F, A) of E_α^θ such that$(F, A) + (F, A) \tilde{\subseteq} (U, A)$. Also, by Proposition 2.21, $\overline{F(\alpha)} = \overline{F(\alpha)}^{\alpha}$. Since (F, A) is a nbd of E_α^θ, by Proposition 5.11, $\overline{F(\alpha)} \subseteq [F(\alpha) + F(\alpha)] \subseteq U(\alpha)$. So, taking $(W, A) = (F, A)$we get $\overline{W(\alpha)} \subseteq U(\alpha)$.
If further, τ is enriched, then taking $W(\alpha) = F(\alpha)$ and $W(\beta) = \phi$, $\forall \beta \neq \alpha$, we get $\overline{(W, A)} \tilde{\subseteq} (U, A)$. □

Definition 5.13. A soft topological space (X, A, τ) is said to be level regular space if for any soft closed set (F, A) and any soft element E_α^x such that $E_\alpha^x \tilde{\notin} (F, A)$, $\exists$ (U, A), $(V, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (U, A)$, $F(\alpha) \subseteq V(\alpha)$ and $U(\alpha) \cap V(\alpha) = \phi$.

Proposition 5.14. *A soft topological space (X, A, τ) is level regular space iff $\forall E_\alpha^x \in \Im$ and $\forall (U, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (U, A)$, $\exists (V, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (V, A)$ and $\overline{V(\alpha)} \subseteq U(\alpha)$.*

Proof. Let (X, A, τ) be a level regular space and $E^x_\alpha \in \Im$, $(U, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (U, A)$. Let $(F, A) = (\tilde{X}, A) \setminus (U, A)$. Then (F, A) is soft closed and $E^x_\alpha \tilde{\notin} (F, A)$. Since (X, A, τ) is level regular, $\exists$ (V, A), $(W, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (V, A)$, $F(\alpha) \subseteq W(\alpha)$ and $V(\alpha) \cap W(\alpha) = \phi$. Now, $V(\alpha)$, $W(\alpha) \in \tau^\alpha$ and $V(\alpha) \subseteq X \setminus W(\alpha)$. Thus $\overline{V(\alpha)} = \overline{V(\alpha)}^\alpha = \overline{X \setminus W(\alpha)}^\alpha = X \setminus W(\alpha) \subseteq X \setminus F(\alpha) = U(\alpha)$.

Conversely, let $E^x_\alpha \in \Im$ and (F, A) be soft closed such that $E^x_\alpha \tilde{\notin} (F, A)$. Consider $(U, A) = (\tilde{X}, A) \setminus (F, A)$. Then $(U, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (U, A)$. So, by the given condition $\exists (V, A) \in \tau$ such that $E^x_\alpha \tilde{\in} (V, A)$ and $\overline{V(\alpha)} \subseteq U(\alpha)$. Now $\overline{V(\alpha)} = \overline{V(\alpha)}^\alpha$ is closed in τ^α. Let $W(\alpha) = X \setminus \overline{V(\alpha)}$. Then $W(\alpha) \in \tau^\alpha$. Then $\exists (G, A) \in \tau$ such that $G(\alpha) = W(\alpha)$ and $F(\alpha) = X \setminus U(\alpha) \subseteq X \setminus \overline{V(\alpha)} = G(\alpha)$. Also $V(\alpha) \cap G(\alpha) = \phi$. Therefore, (X, A, τ) is a level regular space. □

Proposition 5.15. *Let τ be a vector soft topology on a vector space V over the field K, where τ is a weak enriched soft topology. Then the soft topological space (V, A, τ) is level regular. If further, τ is enriched, then (V, A, τ) is soft regular.*

Proof. Let $E^x_\alpha \in \Im$ and $(U, A) \in \tau$ be such that $E^x_\alpha \tilde{\in} (U, A)$. Then (U, A) is a soft nbd of E^x_α and hence by Proposition 5.9, $[-x + (U, A)] = (W, A)$ (say) is a soft nbd of E^θ_α. Then by Proposition 5.12, $\exists$ a soft nbd (F, A) of E^θ_α such that $\overline{F(\alpha)} \subseteq W(\alpha)$. Since τ is weak enriched, $[x + (F, A)]$ is a soft nbd of E^x_α and hence $\exists (P, A) \in \tau$ such that $E^x_\alpha \; \tilde{\in} (P, A) \; \tilde{\subseteq} [x + (F, A)]$.

Then $\overline{P(\alpha)} \subseteq \overline{x + F(\alpha)}$
$= \overline{x + F(\alpha)}^\alpha$
$= x + \overline{F(\alpha)}^\alpha$ [$\because$ (V, τ^α) is a topological vector space]
$= x + \overline{F(\alpha)}$
$\subseteq x + [-x + U(\alpha)] = U(\alpha)$.
$\therefore$ By Proposition 5.14, the soft topological space (V, A, τ) is level regular.
If further, τ is enriched, then by Proposition 5.12, $\overline{(F, A)} \tilde{\subseteq} (W, A)$.
Thus, $\overline{(P, A)} \; \tilde{\subseteq} \overline{[x + (F, A)]}$
$= \overline{T_x[(F, A)]}$
$= T_x[\overline{(F, A)}]$, [since T_x is a soft homeomorphism and τ is enriched]
$= x + \overline{(F, A)}$
$\tilde{\subseteq} x + [-x + (U, A)] = (U, A)$.
Therefore, by Proposition 2.27, the soft topological space (V, A, τ) is soft regular. □

Lemma 5.16. *Let τ be a vector soft topology on a vector space V over the scalar field K and τ be an enriched soft topology. Then*
(i) If (W, A) be any open nbd of E^θ_α, then $E^x_\alpha + (W, A)$ is an open nbd of E^x_α.
(ii) If (W, A) be any open nbd of E^x_α, then $E^t_\alpha \cdot (W, A)$ is an open nbd of E^{tx}_α, for any non zero scalar t.

Proof. (i) Let (W, A) be any open nbd of $E^\theta_\alpha, \alpha \in A$. Then $E(\alpha) = \theta \in W(\alpha) \in \tau^\alpha$ i.e. $W(\alpha)$ is an open nbd of θ in τ^α. Since τ^α is a vector topology on V and $W(\alpha)$ is an open nbd of θ in τ^α, $[x + W(\alpha)]$ is an open nbd of x in τ^α. Now, $[E^x_\alpha + (W, A)](\alpha)$ $= \{x + y : y \in W(\alpha)\}$ and $[E^x_\alpha + (W, A)](\beta) = \phi \; \forall \beta \neq \alpha$. $\therefore$ $E^x_\alpha + (W, A) \in \tau$, since τ is enriched. Again $E^x_\alpha \tilde{\in} [E^x_\alpha + (W, A)]$.

Therefore, $[E_\alpha^x + (W, A)]$ is an open nbd of E_α^x.
Proof of (ii) is similar as above. □

We now introduce the following definitions:

Definition 5.17. Let (X, A, τ) be a soft topological space. If for $E_\alpha^x, E_\alpha^y \in \Im$ with $x \neq y$,
(i) $\exists$ $(F, A) \in \tau$ such that $[E_\alpha^x \tilde{\in} (F, A)$ and $E_\alpha^y \tilde{\notin} (F, A)]$ or $[E_\alpha^y \tilde{\in} (F, A)$ and $E_\alpha^x \tilde{\notin} (F, A)]$, then (X, A, τ) is called level T_0-space.
(ii) $\exists$ $(F, A), (G, A) \in \tau$ such that $[E_\alpha^x \tilde{\in} (F, A)$ and $E_\alpha^y \tilde{\notin} (F, A)]$ and $[E_\alpha^y \tilde{\in} (G, A)$ and $E_\alpha^x \tilde{\notin} (G, A)]$, then (X, A, τ) is called level T_1-space.
(iii) $\exists$ $(F, A), (G, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (F, A)$, $E_\alpha^y \tilde{\in} (G, A)$ and $F(\alpha) \cap G(\alpha) = \phi$, then (X, A, τ) is called level T_2-space.

Proposition 5.18. *Let τ be a weak enriched vector soft topology on a vector space V over the scalar field K, K is equipped with the soft usual topology and θ is the zero vector of V. Then the following statements are related as follows:* $(i) \Leftrightarrow (ii)$ *;* $(iii) \Rightarrow (iv) \Rightarrow (i)$.
(i) *(V, A, τ) is a level T_0- space.*
(ii) *(V, A, τ) is a level T_1- space.*
(iii) *(V, A, τ) is a level T_2- space .*
(iv) $\tilde{\cap}\{(U, A) : (U, A) \in \mathscr{B}\} = \{E_\alpha^\theta\}$, *where $\mathscr{B}$ is a fundamental system of nbds of $E_\alpha^\theta, \alpha \in A$.*

Proof. $(i) \Leftrightarrow (ii)$:
Let (V, A, τ) be a level T_0- space and E_α^x, $E_\alpha^y \in \Im$ such that $E_\alpha^x \neq E_\alpha^y$. Then $\exists$ an soft open set (F, A) such that one of E_α^x and E_α^y belong to (F, A) but the other does not belong to (F, A).
Let $E_\alpha^x \tilde{\in} (F, A)$ but $E_\alpha^y \tilde{\notin} (F, A)$. Consider $(W, A) = [-x + (F, A)]$. Then (W, A) is a soft open set containing E_α^θ and $(U, A) = (W, A) \tilde{\cap} (-W, A)$ is a symmetric soft open set containing E_α^θ.
So, $[y + (U, A)]$ is an soft open set containing E_α^y.
We shall now show that $E_\alpha^x \tilde{\notin} [y + (U, A)]$. If possible let $E_\alpha^x \tilde{\in} [y + (U, A)]$. Then $E_\alpha^{-x} \tilde{\in} [(-U, A) + (-y)] = (U, A) + (-y) \tilde{\subseteq} (W, A) + (-y) = [(-x) + (F, A) + (-y)]$. Thus $E_\alpha^\theta = E_\alpha^x + E_\alpha^{-x} \tilde{\in} E_\alpha^x + [(-x) + (F, A) + (-y)] = E_\alpha^\theta + (F, A) + (-y)$. Thus, $y + E_\alpha^\theta = E_\alpha^y \tilde{\in} y + \left[E_\alpha^\theta + (F, A) + (-y)\right]$. Therefore, $E_\alpha^y \tilde{\in} (F, A)$, a contradiction.
$\therefore$ (V, A, τ) is a level T_1- space. Conversely, if (V, A, τ) is a level T_1- space then obviously (V, A, τ) is a level T_0- space.

$(iii) \Rightarrow (iv)$:
Let (V, A, τ) is a level T_2- space and $\mathscr{B}$ be a fundamental system of nbds of E_α^θ.
Let $E_\alpha^x \tilde{\in} \tilde{\cap}\{(U, A) : (U, A) \in \mathscr{B}\}$.
If possible let $E_\alpha^x \neq E_\alpha^\theta$. Since (V, A, τ) is a level T_2- space, there exist a soft open set (F, A) such that $E_\alpha^\theta \tilde{\in} (F, A)$ but $E_\alpha^x \tilde{\notin} (F, A)$. Since $\mathscr{B}$ be the fundamental system of nbds of E_α^θ, $\exists$ $(U, A) \in \tau$ such that $(U, A) \tilde{\subseteq} (F, A)$.
So, $E_\alpha^x \tilde{\notin} (U, A)$, which contradicts our assumption $E_\alpha^x \tilde{\in} (U, A)$, $\forall (U, A) \in \mathscr{B}$.
Thus $E_\alpha^x = E_\alpha^\theta$ and $\tilde{\cap}\{(U, A) : (U, A) \in \mathscr{B}\} = \{E_\alpha^\theta\}$.

$(iv) \Rightarrow (i)$:
Let $\tilde{\cap}\{(U,A) : (U,A) \in \mathscr{B}\} = \{E^{\theta}_{\alpha}\}$ and E^x_{α}, $E^y_{\alpha} \in \Im$ such that $E^x_{\alpha} \neq E^y_{\alpha}$. Then $E^{-y}_{\alpha} + E^x_{\alpha} \neq E^{\theta}_{\alpha}$ and $\exists\, (U,A) \in \mathscr{B}$ such that $E^{-y}_{\alpha} + E^x_{\alpha} \tilde{\notin} (U,A)$. So, $E^x_{\alpha} \tilde{\notin} [y + (U,A)]$.
Thus, $[y + (U,A)]$ is a soft open set containing E^y_{α} but $E^x_{\alpha} \tilde{\notin} [y + (U,A)]$.
$\therefore (V,A,\tau)$ is a level T_0- space. □

Proposition 5.19. *Let τ be an enriched vector soft topology on a vector space V over the scalar field K, K is equipped with the soft usual topology and θ is the zero vector of V. Then the following statements are equivalent:*
(i) (V,A,τ) is a soft T_0- space.
(ii) (V,A,τ) is a soft T_1- space.
(iii) (V,A,τ) is a soft T_2- space.
(iv) $\tilde{\cap}\{(U,A) : (U,A) \in \mathscr{B}\} = \{E^{\theta}_{\alpha}\}$, where $\mathscr{B}$ is a fundamental system of nbds of $E^{\theta}_{\alpha}, \alpha \in A$.

Proof. $(i) \Rightarrow (ii)$:
Let (V,A,τ) be a soft T_0- space and E^x_{α}, $E^y_{\beta} \in \Im$ such that $E^x_{\alpha} \neq E^y_{\beta}$. If $\alpha \neq \beta$ and since τ is enriched, $\exists$ soft open sets (F,A), (G,A) where $F(\alpha) = V$, $F(\beta) = \phi, \forall \beta \neq \alpha$ and $G(\beta) = V$, $G(\alpha) = \phi$, $\forall \alpha \neq \beta$. So, $E^x_{\alpha} \tilde{\in} (F,A)$, $E^y_{\beta} \tilde{\notin} (F,A)$ and $E^y_{\beta} \tilde{\in} (G,A)$, $E^x_{\alpha} \tilde{\notin} (G,A)$. For the case when $\alpha = \beta$ and $x \neq y$, the proof is similar as of Proposition 5.18.

$(ii) \Rightarrow (iii)$:
Next assume that (V,A,τ) is a soft T_1- space. we only consider the case when $\alpha = \beta$ and $x \neq y$. Since τ is a soft T_1- space, it follows that $\{E^x_{\alpha}\}$ is soft closed set and hence $(P,A) = (\tilde{V},A) \setminus \{E^x_{\alpha}\}$ is soft open set containing $\{E^y_{\alpha}\}$.
So, $[(-y) + (P,A)]$ is a soft nbd of E^{θ}_{α}. Then by Proposition 5.7, $\exists$ a soft open nbd (W,A) of E^{θ}_{α} such that $(W,A) + (-W,A) \tilde{\subseteq} [(-y) + (P,A)]$. So, $[y + (W,A)]$ is a soft open nbd of E^y_{α} and $(Q,A) = (\tilde{V},A) \setminus \overline{[y + (W,A)]}$ is a soft open set.
If $E^x_{\alpha} \tilde{\in} \overline{[y + (W,A)]}$, then since $E^x_{\alpha} + (W,A)$ is a soft open nbd of E^x_{α},
$[E^x_{\alpha} + (W,A)] \,\tilde{\cap}\, [y + (W,A)] \neq (\tilde{\Phi},A)$
$\Rightarrow ([E^x_{\alpha} + (W,A)] \,\tilde{\cap}\, [y + (W,A)])\,(\alpha) \neq \phi$.
Then $\exists\, z \in ([E^x_{\alpha} + (W,A)] \,\tilde{\cap}\, [y + (W,A)])\,(\alpha)$.
$\therefore z = x + t$, for some $t \in W(\alpha)$ and $z = y + s$, for some $s \in W(\alpha)$.
So, $x + t = y + s$, $t, s \in W(\alpha)$. And this implies that $x = y + s - t \in [y + W + (-W)]\,(\alpha)$.
i.e. $E^x_{\alpha} \tilde{\in} [y + (W,A) + (-W,A)] \tilde{\subseteq} y + (-y) + (P,A) = (P,A)$, which is a contradiction, because $E^x_{\alpha} \tilde{\notin} (P,A)$.
Thus (Q,A) is a soft open set containing E^x_{α}. Then $E^x_{\alpha} \tilde{\in} (Q,A)$, $E^y_{\alpha} \tilde{\in} [y + (W,A)]$ and $(Q,A) \tilde{\cap} [y + (W,A)] = (\tilde{\Phi},A)$.
Therefore, (V,A,τ) is a soft T_2- space.

Proof of $(iii) \Rightarrow (iv) \Rightarrow (i)$ is similar as above.
Therefore, the statements (i), (ii), (iii) and (iv) are equivalent. □

Proposition 5.20. *Let τ be a vector soft topology on a vector space V over the scalar field K, where τ is an enriched soft topology. Then there exists a fundamental soft nbd system $\mathscr{B}$ of closed nbds of E_α^θ such that*
(i) each $(U, A) \in \mathscr{B}$ is symmetric;
(ii) $\forall (U, A) \in \mathscr{B}$, $\exists (W, A) \in \mathscr{B}$ such that $(W, A) + (W, A) \tilde{\subseteq} (U, A)$;
(iii) $\forall (U, A) \in \mathscr{B}$ and $\forall a \in V$, $\exists (W, A) \in \mathscr{B}$ such that $(W, A) \tilde{\subseteq} [E_\alpha^{-a} + (U, A) + E_\alpha^a]$ i.e. $[E_\alpha^a + (W, A) + E_\alpha^{-a}] \tilde{\subseteq} (U, A)$;
(iv) $\forall (W, A) \in \mathscr{B}$ and $\forall E_\alpha^a \tilde{\in} (W, A)$, $\exists (U, A) \in \mathscr{B}$ such that $[E_\alpha^a + (U, A)] \tilde{\subseteq} (W, A)$.

Proof. (i) We know that $\exists$ a fundamental system $\mathscr{B}'$ of symmetric nbds of E_α^θ.

Let $\mathscr{B} = \{\overline{(U, A)} : (U, A) \in \mathscr{B}'\}$.

Since $(U, A) \in \mathscr{B}' \Rightarrow (U, A)$ is symmetric and hence $(-U, A) = (U, A)$.

Again since the scalar multiplication is soft homeomorphism and hence $\forall \alpha \in A$, $\overline{[-U]}(\alpha) = \overline{[-U]\,(\alpha)}^{\alpha} = \overline{-\,[U(\alpha)]}^{\alpha} = -\left[\overline{U(\alpha)}^{\alpha}\right] = -\left[\overline{U}(\alpha)\right] = \left[-\overline{U}\right](\alpha)$.

So, $\overline{(U, A)} = \overline{-\,(U, A)} = -\left[\overline{(U, A)}\right]$. Thus $\overline{(U, A)}$ is a symmetric closed nbd of E_α^θ.

We shall now show that $\mathscr{B}$ is a fundamental system of nbds of E_α^θ.

Let (W, A) be any nbd of E_α^θ. Then by Proposition 5.12, there exists a nbd (F, A) of E_α^θ such that $\overline{(F, A)} \tilde{\subseteq} (W, A)$.

Since $\mathscr{B}'$ is a fundamental system of nbds of E_α^θ, $\exists$ $(U, A) \in \mathscr{B}'$ such that $(U, A) \tilde{\subseteq} (F, A)$.

Then $\overline{(U, A)} \in \mathscr{B}$ and $\overline{(U, A)} \tilde{\subseteq} \overline{(F, A)} \tilde{\subseteq} (W, A)$.

Therefore $\mathscr{B}$ is a fundamental system of closed nbds of E_α^θ such that each member of $\mathscr{B}$ is symmetric.

(ii) Let $(U, A) \in \mathscr{B}$. Then from Proposition 5.7, $\exists$ a symmetric nbd (F, A) of E_α^θ such that $(F, A) + (F, A) \tilde{\subseteq} (U, A)$. Since $\mathscr{B}$ is a fundamental system of nbds of E_α^θ $\exists (W, A) \in \mathscr{B}$ such that $(W, A) \tilde{\subseteq} (F, A)$. Then $(W, A) + (W, A) \tilde{\subseteq} (F, A) + (F, A) \tilde{\subseteq} (U, A)$.

(iii) Let $(U, A) \in \mathscr{B}$. Since $\left[a + E_\alpha^\theta + (-a)\right] = E_\alpha^\theta$, (U, A) is a soft nbd of E_α^θ and the translation mapping is soft homeomorphism, $\exists$ a soft nbd (V_1, A) of E_α^θ such that $[a + (V_1, A) + (-a)] \tilde{\subseteq} (U, A)$.

Since $\mathscr{B}$ is a fundamental system of nbds of E_α^θ, $\exists (W, A) \in \mathscr{B}$ such that $(W, A) \tilde{\subseteq} (V_1, A)$.
Therefore $E_\alpha^a + (W, A) + E_\alpha^{-a} \tilde{\subseteq} [a + (W, A) + (-a)] \tilde{\subseteq} [a + (V_1, A) + (-a)] \tilde{\subseteq} (U, A)$.

(iv) Let $(W, A) \in \mathscr{B}$ and $E_\alpha^a \tilde{\in} (W, A)$.

Since $a + E_\alpha^\theta = E_\alpha^a$ i.e. $T_a(E_\alpha^\theta) = E_\alpha^a$, (W, A) is a soft nbd of E_α^a and translation mapping is soft homeomorphism, there exist a soft nbd (U_1, A) of E_α^θ such that $T_a\left[(U_1, A)\right] \tilde{\subseteq} (W, A)$ i.e. $[a + (U_1, A)] \tilde{\subseteq} (W, A)$.

Since $\mathscr{B}$ is a fundamental system of nbds of E_α^θ, $\exists (U, A) \in \mathscr{B}$ such that $(U, A) \tilde{\subseteq} (U_1, A)$.
Therefore $E_\alpha^a + (U, A) \tilde{\subseteq} [a + (U, A)] \tilde{\subseteq} [a + (U_1, A)] \tilde{\subseteq} (W, A)$. □

6. Conclusion

In this paper, we have introduced vector soft topology and studied its separation properties. In this context the neighbourhood systems of soft elements play important role because of the homeomorphism property of the translation operator. This is just a beginning of studying soft topological vector spaces. There is a huge scope of further study in extending the results of classical topological vector spaces [28] in soft setting.

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