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Solutions of fuzzy heat-like equations by variational iteration method

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Abstract. In present paper, the same strategy as Buckley-Feuring by the variational iteration method are used for finding the exact fuzzy solution of the fuzzy heat-like equations in one and two dimensions. Several examples are given to show the Buckley-Feuring solution and the Seikkala solution. The results show that the variational iteration method, without linearization or small perturbation, is very effective and convenient.

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1. Introduction

There are strong and efficient techniques to find approximate solutions for the linear and nonlinear equations, that most of these equations don't have exact solution such as heat-like equations. In mathematics, in order to solve the model of heat-like equations, we will introduce some imprecise parameters.

In this work, the contribution is solving heat-like equations with fuzzy parameters via the same strategy as Buckley and Feuring [3, 4] using Variational Iteration Method VIM. The VIM proposed by He in [8, 9, 10], is a method of solving linear or nonlinear problems [14, 15] and gives rapid convergent successive approximations of the exact solution if that last exists.

In comparison with the paper [1], we investigate problems with fuzzy parameters, fuzzy initial value and fuzzy forcing functions, we propose a new theorem for finding the exact fuzzy solutions, witch extended to the Buckley-Feuring for the proposed models.

We begin Section 2 by defining the notation where we will use in the paper and then in Section 3 and 4, fuzzy heat-like equations and the VIM are illustrated, respectively. In Section 5, the same strategy as in Buckley-Feuring is presented for two-dimensional fuzzy heat-like equation. Some examples in Section 6 illustrated.

2. Preliminaries

We place a bar over a capital letter to denote a fuzzy number of $\mathbb{R}^n$. So, $\overline{A}$, $\overline{K}$, $\overline{\gamma}$, $\overline{\beta}$ etc. all represent fuzzy numbers of $\mathbb{R}^n$ for some n. We write $\mu_{\overline{A}}(t)$, a number in $[0,1]$, for the membership function of $\overline{A}$ evaluated at $t \in \mathbb{R}^n$. An α-cut of $\overline{A}$ is always a closed and bounded interval that written $\overline{A}[\alpha]$, is defined as $\{t|\mu_{\overline{U}}(t) \geq \alpha\}$ for $0 < \alpha < 1$. We separately specify $\overline{A}[0]$ as the closure of the union of all the $\overline{U}[\alpha]$ for $0 < \alpha \leq 1$

Definition 2.1 ([5]). Let $\mathbb{R}_{\mathcal{F}} = \left\{\overline{A} \mid \overline{A} : \mathbb{R} \to [0,1], \text{ satisfies } (1) - (4)\right\}$:

(1) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is normal.
(2) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is a fuzzy convex set.
(3) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is upper semi-continuous on $\mathbb{R}$.
(4) $\overline{A}[0]$ is a compact set.

Then $\mathbb{R}_{\mathcal{F}}$ is called fuzzy number space and $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is called a fuzzy number.

Definition 2.2 ([5, 12]). We represent an arbitrary fuzzy number by an ordered pair of functions $\overline{A}[\alpha] = [A_1(\alpha), A_2(\alpha)]$, $\quad \alpha \in [0,1]$, which satisfy the following requirements :

(1) $A_1(\alpha)$ is a nondecreasing function over $[0,1]$,
(2) $A_2(\alpha)$ is a nonincreasing function on $[0,1]$
(3) $A_1(\alpha)$ and $A_2(\alpha)$ are bounded left continuous on $(0,1]$, and right continuous at $\alpha = 0$, and
(4) $A_1(\alpha) \leq A_2(\alpha)$, for $0 \leq \alpha \leq 1$

Definition 2.3. Let $\overline{A} = (a_1, a_2, a_3)$, $(a_1 < a_2 < a_3)$. $\overline{A}$ is called triangular fuzzy number with peak (center) a_2, left width $a_2 - a_1 > 0$ and right width $a_3 - a_2 > 0$, if its membership function has the following form :

$$\mu_{\overline{A}}(t) = \begin{cases} 1 - \dfrac{(a_2 - t)}{a_2 - a_1}, & a_1 \leq t \leq a_2 \\ 1 - \dfrac{(t - a_2)}{a_3 - a_2}, & a_2 \leq t \leq a_3 \\ 0, & \text{otherwise.} \end{cases}$$

The support of $\overline{A}$ is $[a_1, a_3]$. We can write :

(1) $\overline{A} > 0$ if $a_1 > 0$,
(2) $\overline{A} \geq 0$ if $a_1 \geq 0$,
(3) $\overline{A} < 0$ if $a_3 < 0$,
(4) $\overline{A} \leq 0$ if $a_3 \leq 0$.

Definition 2.4. For arbitrary fuzzy numbers $\overline{A}[\alpha] = \Big[a_1(\alpha), a_2(\alpha)\Big]$ and $\overline{B}[\alpha] = \Big[b_1(\alpha), b_2(\alpha)\Big]$ we have algebraic operations as follows :

(1) $(\overline{A} + \overline{B})[\alpha] = \Big[a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)\Big]$
(2) $(\overline{A} - \overline{B})[\alpha] = \Big[a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)\Big]$

(3)

$$k\overline{A}[\alpha] = \begin{cases} \Big[ka_1(\alpha), ka_2(\alpha)\Big] & k \geq 0 \\ \\ \Big[ka_2(\alpha), ka_1(\alpha)\Big] & k < 0 \end{cases}$$

(4) $(\overline{A}.\overline{B})[\alpha] = \{\min z, \max z\}$ with

$$z = \Big\{a_1(\alpha).b_1(\alpha),\ a_1(\alpha).b_2(\alpha),\ a_2(\alpha).b_1(\alpha),\ a_2(\alpha).b_2(\alpha)\Big\}$$

(5) If $0 \notin \Big[b_1(\alpha), b_2(\alpha)\Big]$

$$\frac{\overline{A}}{\overline{B}}[\alpha] = \Big[(\frac{a_1}{b_1})(\alpha), (\frac{a_2}{b_2})(\alpha)\Big]$$

where

$$\left(\frac{a_1}{b_1}\right)(\alpha) = \min\left\{\frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}\right\}$$
$$\left(\frac{a_2}{b_2}\right)(\alpha) = \max\left\{\frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}\right\}$$

We adopt the general definition of a fuzzy number given in [6].

3. Fuzzy heat-like equations

We consider the heat-like equations in one and two dimensional cases which can be written in the forms

- One-dimensional [1] :

$$U_t(t,x) + P(x,\gamma)U_{xx}(t,x) = F(t,x,k) \tag{3.1}$$

- Two-dimensional [1] :

$$U_t(t,x,y) + P(x,\gamma)U_{xx}(t,x,y) + Q(y,\beta)U_{yy}(t,x,y) = F(t,x,y,k) \tag{3.2}$$

or

$$U_t(t,x,y) + Q(y,\beta)U_{xx}(t,x,y) + P(x,\gamma)U_{yy}(t,x,y) = F(t,x,y,k) \tag{3.3}$$

subject to certain initial and boundary conditions.
These initial and boundary conditions, in state two-dimensional, can come in a variety of forms such as

$$U(0,x,y) = c_1 \ \text{ or } \ U(0,x,y) = g_1(x,y,c_2) \ \text{ or } \ U(M_1,x,y) = g_2(x,y,c_3,c_4), \ \ldots$$

In this paper the method is applied for the heat-like equation (3.2). For (3.1) and (3.3), the same discussion can be made. In following lines, the components of (3.2) are enumerated :

- $I_1 = [0, M_1]$, $I_2 = [M_2, M_3]$ and $I_3 = [M_4, M_5]$ are three intervals, which M_{n_1} ($n_1 = 2,3,4,5$) is negative or positive and $M_1 > 0$.
- $F(t,x,y,k)$, $U(t,x,y)$, $P(x,\gamma)$ and $Q(y,\beta)$ will be continuous functions for $(t,x,y) \in \prod_{j=1}^{3} I_j$.
- $P(x,\gamma)$ and $Q(y,\beta)$ have a finite number of roots for each $(x,y) \in I_2 \times I_3$

- $k = \Big(k_1, \dots, k_n\Big)$, $c = \Big(c_1, \dots, c_m\Big)$, $\gamma = \Big(\gamma_1, \dots, \gamma_s\Big)$ and $\beta = \Big(\beta_1, \dots, \beta_e\Big)$ are vectors of constants with $k_j \in J_j$, $c_i \in L_i$ and $\gamma_r \in H_r$ and $\beta_l \in D_l$.

Assume that (3.2) has a solution

$$U(t,x,y) = G(t,x,y,k,c,\gamma,\beta) \tag{3.4}$$

for G and $G_t(t,x,y,k,c,\gamma,\beta) + P(x,\gamma)G_{xx}(t,x,y,k,c,\gamma,\beta) + Q(y,\beta)G_{yy}(t,x,y,k,c,\gamma,\beta)$ are continuous with $(t,x,y) \in \prod_{j=1}^{3} I_j$, $k \in J = \prod_{j=1}^{n} J_j$, $c \in L = \prod_{i=1}^{m} L_i$, $\gamma \in H = \prod_{r=1}^{s} H_r$ and $\beta \in D = \prod_{l=1}^{e} D_l$.

Suppose the constant k_j, c_i, γ_r and β_l are imprecise in their values. We will model this uncertainty by substituting triangular fuzzy numbers for the k_j, c_i , γ_r and β_l. If we fuzzify (3.2), then we obtain the fuzzy heat-like equation. Using the extension principle, we compute $\overline{F}$, $\overline{P}$ and $\overline{Q}$ from F, P and Q where $\overline{F}(t,x,y,\overline{K})$ has $\overline{K} = (\overline{k}_1, \dots, \overline{k}_n)$ and $\overline{P}(x,\overline{\gamma})$ has $\overline{\gamma} = (\overline{\gamma}_1, \dots, \overline{\gamma}_s)$ and $\overline{Q}(y,\overline{\beta})$ a $\overline{\beta} = (\overline{\beta}_1, \dots, \overline{\beta}_e)$ for $\overline{k}_j$, $\overline{\gamma}_r$ and $\overline{\beta}_l$ a triangular fuzzy numbers in J_j $(0 \le j \le n)$, H_r $(0 \le r \le s)$ and D_l $(0 \le l \le e)$.

The function U is changed to $\overline{U}$ where $\overline{U} : \prod_{j=1}^{3} I_j \to \mathcal{F}(\mathbb{R})$. That is, $\overline{U}(t,x,y)$ is a fuzzy function. The fuzzy heat-like equation is

$$\overline{U}_t(t,x,y) + \overline{P}(x,\overline{\gamma})\overline{U}_{xx}(t,x,y) + \overline{Q}(y,\overline{\beta})\overline{U}_{yy}(t,x,y) = \overline{F}(t,x,y,\overline{K}) \tag{3.5}$$

subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form

$$\overline{U}(0,x,y) = \overline{C}_1 \text{ or } \overline{U}(0,x,y) = \overline{g}_1(x,y,\overline{C}_2) \text{ or } \overline{U}(M_1,x,y) = \overline{g}_2(x,y,\overline{C}_3,\overline{C}_4)$$

The $\overline{g}_j$ is the fuzzification g_i via extension principle. Then, we will solve the problem given in (3.5). Finally, we fuzzify G in (3.4).
Let $\overline{Z}(t,x,y) = \overline{G}(t,x,y,\overline{K},\overline{C},\overline{\gamma},\overline{\beta})$ where $\overline{Z}$ is computed using the extension principale and is a fuzzy solution. In section 5, we will discuss concept of the solution with the same strategy as Buckley-Feuring for fuzzy heat-like equation.
Let $\overline{K}[\alpha] = \prod_{j=1}^{n} \overline{K}_j[\alpha]$, $\overline{\gamma}[\alpha] = \prod_{r=1}^{s} \overline{\gamma}_r[\alpha]$, $\overline{C}[\alpha] = \prod_{i=1}^{m} \overline{C}_i[\alpha]$ and $\overline{\beta}[\alpha] = \prod_{l=1}^{e} \overline{\beta}_l[\alpha]$

4. The variational iteration method

To illustrate the basic idea of the VIM we consider the following PDE model

$$L_t U + L_x U + L_y U + NU = F(t,x,y,k) \tag{4.1}$$

where L_t, L_x and L_y are linear operators of t, x and y, respectively, and N is a nonlinear operator, also $F(t,x,y,k)$ is the source non-homogeneous term. According to the VIM [14, 15], we can express the following correction function for (4.1) in t, x and y directions can be written as

$$U_{n+1}(t,x,y) = U_n(t,x,y) + \int_0^t \lambda\{L_s U_n + (L_x + L_y + N)\widetilde{U}_n - F(s,x,y,k)\}ds$$

where λ is general lagrange multiplier [11], which can be identified optimally via the variational theory [7, 15], and $\widetilde{U}_n$ is a restricted variation which means $\delta\widetilde{U}_n = 0$. First, it is required to determine the lagrange multipliers λ that will be identified optimally via integration by parts. The approximations $U_{n+1}, n \geq 0$, of the solution $U(t,x,y)$ will immediately follow upon using any selective function U_0.
The initial values $U(0,x,y)$ is usually used for the selected zeroth approximations U_0. With the Lagrange multiplies λ determined, then several approximation $u_i(t,x,y)$, $i \geq 0$ can be determined. Consequently, the solution is given as

$$U(t,x,y) = \lim_{n\to\infty} U_n(t,x,y)$$

According to the VIM, we construct a correction functional for (3.2) in t-direction as follows

$$\begin{aligned} U_{n+1}(t,x,y) &= U_n(t,x,y) \\ &+ \int_0^t \lambda(s)\Big\{(U_n)_s + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\Big\}ds \end{aligned} \tag{4.2}$$

where $n \geq 0$ and λ is a lagrange multiplier. We now determine the lagrange multiplier

$$\begin{aligned} \delta U_{n+1}(t,x,y) &= \delta U_n(t,x,y) \\ &+ \delta\int_0^t \lambda(s)\Big\{(U_n)_s + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\Big\}ds \end{aligned}$$

Therefore, the stationary conditions are:

$$\begin{gathered} \lambda'(s) = 0 \\ 1 + \lambda(s)|_{s=t} = 0 \end{gathered}$$

Thus, the lagrange multiplier is $\lambda = -1$. Submitting the results into (4.2) leads to the following iteration formula

$$\begin{aligned} U_{n+1}(t,x,y) &= U_n(t,x,y) \\ &- \int_0^t \{(U_n)_s + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\}ds \end{aligned} \tag{4.3}$$

Iteration formula start with initial approximation, for example $U_0(t,x,y) = U(0,x,y)$. Also the VIM used for system of linear and nonlinear partial differential equation [15] which handled in obtain Seikkala solution.

5. Buckley-Feuring Solution (BFS) and Seikkala solution (SS)

5.1. **Buckley-Feuring solution.**

Buckley-Feuring first proposed the BFS [2, 3]. They define for all t, x, y and $\alpha \in [0,1]$,

$$\overline{Z}(t,x,y)[\alpha] = \Big[z_1(t,x,y,\alpha), z_2(t,x,y,\alpha)\Big], \quad \overline{F}\Big(t,x,y,\overline{k}\Big)[\alpha] = \Big[F_1(t,x,y,\alpha), F_2(t,x,y,\alpha)\Big]$$

and to check (3.5) we must compute $\overline{P}\Big(x,\overline{\gamma}\Big)$ and $\overline{Q}\Big(y,\overline{\beta}\Big)$. The α-cuts of $\overline{P}\Big(x,\overline{\gamma}\Big)$ and $\overline{Q}\Big(y,\overline{\beta}\Big)$ can be found as follows :

$\forall \alpha \in [0,1]$

$$\overline{P}(x,\overline{\gamma})[\alpha] = \Big[P_1(x,\alpha), P_2(x,\alpha)\Big], \qquad \overline{Q}(y,\overline{\beta})[\alpha] = \Big[Q_1(y,\alpha), Q_2(y,\alpha)\Big]$$

Let $W = \overline{K}[\alpha] \times \overline{C}[\alpha] \times \overline{\gamma}[\alpha] \times \overline{\beta}[\alpha]$. By definition

$$z_1(t,x,y,\alpha) = \min\Big\{G(t,x,y,k,c,\gamma,\beta) \,:\, (k,c,\gamma,\beta) \in W\Big\} \tag{5.1}$$

$$z_2(t,x,y,\alpha) = \max\Big\{G(t,x,y,k,c,\gamma,\beta) \,:\, (k,c,\gamma,\beta) \in W\Big\} \tag{5.2}$$

and

$$F_1(t,x,y,\alpha) = \min\Big\{F(t,x,y,k) \,:\, k \in \overline{K}[\alpha]\Big\} \tag{5.3}$$

$$F_2(t,x,y,\alpha) = \max\Big\{F(t,x,y,k) \,:\, k \in \overline{K}[\alpha]\Big\} \tag{5.4}$$

$\forall (t,x,y) \in \prod_{j=1}^{3} I_j$ *and* $\alpha \in [0,1]$
and

$$P_1(x,\alpha) = \min\left\{P(x,\gamma) | \gamma \in \overline{\gamma}[\alpha]\right\}, \quad P_2(x,\alpha) = \max\left\{P(x,\gamma) | \gamma \in \overline{\gamma}[\alpha]\right\} \tag{5.5}$$

$\forall x \in I_2$ and $\alpha \in [0,1]$
and

$$Q_1(y,\alpha) = \min\left\{Q(y,\beta) | \beta \in \overline{\beta}[\alpha]\right\}, \quad Q_2(y,\alpha) = \max\left\{Q(y,\beta) | \beta \in \overline{\beta}[\alpha]\right\} \tag{5.6}$$

$\forall y \in I_3$ and $\alpha \in [0,1]$
Assume that $P(x,\gamma) > 0$, $(P_1(x,\alpha) > 0)$, $Q(y,\beta) > 0$, $(Q_1(y,\alpha) > 0)$ and the $z_i(t,x,y,\alpha)$ $i = 1,2$, has continuous partial derivatives so $(z_i)_t + P_i(z_i)_{xx} + Q_i(z_i)_{yy}$ is continuous for all $t,x,y \in \prod_{j=1}^{3} I_j$ and all $\alpha \in [0,1]$.
Define

$$\Gamma(t,x,y,\alpha) = \Big[(z_1)_t + P_1(x,\alpha)(z_1)_{xx} + Q_1(y,\beta)(z_1)_{yy}, \\ (z_2)_t + P_2(x,\alpha)(z_2)_{xx} + Q_2(y,\beta)(z_2)_{yy}\Big]$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all α.
If, for each fixed $t,x,y \in \prod_{j=1}^{3} I_j$, $\Gamma(t,x,y,\alpha)$ defines the α-cut of a fuzzy number, then will be said that $\overline{Z}(t,x,y)$ is differentiable and is written

$$\overline{Z}_t[\alpha] + \overline{P}[\alpha]\overline{Z}_{xx}[\alpha] + \overline{Q}[\alpha]\overline{Z}_{yy}[\alpha] = \Gamma(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all α

Sufficient conditions for $\Gamma(t,x,y,\alpha)$ to define α-cut of a fuzzy number are [6] :

(i) $(z_1)_t(t,x,y,\alpha) + P_1(x,\alpha)(z_1)_{xx}(t,x,y,\alpha) + Q_1(y,\alpha)(z_1)_{yy}(t,x,y,\alpha)$ is an increasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$

(ii) $(z_2)_t(t,x,y,\alpha) + P_2(x,\alpha)(z_2)_{xx}(t,x,y,\alpha) + Q_2(y,\alpha)(z_2)_{yy}(t,x,y,\alpha)$ is an decreasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$ and

(iii) for $(t,x,y) \in \prod_{j=1}^{3} I_j$

$$\begin{aligned}\big(z_1\big)_t(t,x,y,1) + P_1(x,1)\big(z_1\big)_{xx}(t,x,y,1) + Q_1(y,1)\big(z_1\big)_{yy}(t,x,y,1)\\ \le \big(z_2\big)_t(t,x,y,1) + P_2(x,1)\big(z_2\big)_{xx}(t,x,y,1) + Q_2(y,1)\big(z_2\big)_{yy}(t,x,y,1)\end{aligned}$$

Now we assume that the $z_i(t,x,y,\alpha)$ has continuous partial derivatives so $(z_i)_t + P_i(x,\alpha)(z_i)_{xx} + Q_i(y,\alpha)(z_i)_{yy}$ is continuous on $\prod_{j=1}^{3} I_j \times [0,1]$ $\;i = 1,2$. Hence, if conditions (i)-(iii) above are hold, $\overline{Z}(t,x,y)$ is differentiable.

For $\overline{Z}(t,x,y)$ to be a BFS of the fuzzy heat-like equation we need

(a) $\overline{Z}(t,x,y)$ differentiable
(b) (3.5) hold for $\overline{U}(t,x,y) = \overline{Z}(t,x,y)$,
(c) $\overline{Z}(t,x,y)$ satisfies the initial and boundary conditions.

$\overline{Z}(t,x,y)$ is a BFS (without the initial and boundary conditions) if $\overline{Z}(t,x,y)$ is differentiable and $(\overline{Z})_t + \overline{P}(x,\overline{\gamma})(\overline{Z})_{xx} + \overline{Q}(y,\overline{\beta})(\overline{Z})_{yy} = \overline{F}(t,x,y,\overline{k})$ or the following equations must hold

$$(z_1)_t + P_1(x,\alpha)(z_1)_{xx} + Q_1(y,\alpha)(z_1)_{yy} = F_1(t,x,y,\alpha) \tag{5.7}$$

$$(z_2)_t + P_2(x,\alpha)(z_2)_{xx} + Q_2(y,\alpha)(z_2)_{yy} = F_2(t,x,y,\alpha) \tag{5.8}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$.
Now, we will present a sufficient condition for the BFS to exist such as Buckley and Feuring. Since there are such a variety of possible initial and boundary conditions, so we will omit them from the following theorem. One must separately check out the initial and boundary conditions. Thus, we will omit the constants c_i, $1 \le i \le m$, from the problem. Therefore, (3.4) becomes $U(t,x,y) = G(t,x,y,k,\gamma,\beta)$, so $\overline{Z}(t,x,y) = \overline{G}\Big(t,x,y,\overline{K},\overline{\gamma},\overline{\beta}\Big)$.

Theorem 5.1. *Assume $\overline{Z}(t,x,y)$ is differentiable.*

(a)

$$\textit{if } P(x,\gamma_i) > 0 \;\; \textit{and} \;\; \frac{\partial P}{\partial \gamma_i}\frac{\partial G}{\partial \gamma_i} > 0 \quad x \in I_2 \;\; \textit{for} \;\; i = 1,2,\dots,m \tag{5.9}$$

and

$$\textit{if} \;\; Q(y,\beta_l) > 0 \; \textit{and} \;\; \frac{\partial Q}{\partial \beta_l}\frac{\partial G}{\partial \beta_l} > 0 \quad y \in I_3 \;\; \textit{for} \;\; l = 1,2,\dots,e \tag{5.10}$$

and

$$\textit{if} \;\; \frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \; \textit{for} \; j = 1,2,...,n \tag{5.11}$$

Then BFS $= \overline{Z}(t,x,y)$

(b) *If relations (5.9) does not hold for some i or relation (5.10) does not hold for some l, or relation (5.11) does not hold for some j, then $\overline{Z}(t,x,y)$ is not a BFS.*

Proof.
(a) For simplicity assume $k_j = k$, $\gamma_i = \gamma$, $\beta_l = \beta$ and $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$, $\frac{\partial P}{\partial \gamma} > 0$, $\frac{\partial G}{\partial \gamma} > 0$, $\frac{\partial Q}{\partial \beta} < 0$ and $\frac{\partial G}{\partial \beta} < 0$. The proof for $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial P}{\partial \gamma} < 0$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial Q}{\partial \beta} > 0$ and $\frac{\partial G}{\partial \beta} > 0$ is similar and omitted.
Since $\frac{\partial G}{\partial k} < 0$, $\frac{\partial G}{\partial \gamma} > 0$ and $\frac{\partial G}{\partial \beta} < 0$, then from (5.1) and (5.2) we have

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big),$$
$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big)$$

from (5.3), (5.4) and $\frac{\partial F}{\partial k} < 0$ we have

$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big) \qquad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big)$$

since (5.5) and $\frac{\partial P}{\partial \gamma} > 0$ we have

$$P_1(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big) \qquad P_2(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big)$$

from (5.6) and $\frac{\partial Q}{\partial \beta} < 0$ we have

$$Q_1(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big) \qquad Q_2(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big)$$

for all $\alpha \in [0,1]$ where $\overline{K}[\alpha] = \Big[k_1(\alpha),k_2(\alpha)\Big]$, $\overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha),\gamma_2(\alpha)\Big]$ and $\overline{\beta}[\alpha] = \Big[\beta_1(\alpha),\beta_2(\alpha)\Big]$.
As we know $G(t,x,y,k,\gamma,\beta)$ solves (3.2), which means

$$G_t + P(x,\gamma)G_{xx} + Q(y,\beta)G_{yy} = F(t,x,y,k)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, $k \in J$, $\gamma \in H$ and $\beta \in D$
Suppose $\overline{Z}(t,x,y)$ is differentiable and $P(x,\gamma) > 0$ and $Q(y,\beta) > 0$ so

$$\partial_t z_1(t,x,y,\alpha) + P_1(x,\alpha)\partial_{xx} z_1(t,x,y,\alpha) + Q_1(y,\alpha)\partial_{yy} z_1(t,x,y,\alpha) = F_1(t,x,y,\alpha)$$

$$\partial_t z_2(t,x,y,\alpha) + P_2(x,\alpha)\partial_{xx} z_2(t,x,y,\alpha) + Q_2(y,\alpha)\partial_{yy} z_2(t,x,y,\alpha) = F_2(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$
Hence, (5.7) and (5.8) holds and $\overline{Z}(t,x,y)$ is a BFS.

(b) Now consider the situation where (5.9) or (5.10) or (5.11) does not hold.

Let us only look at one case where $\frac{\partial Q}{\partial \beta} < 0$ $\Big($ assume $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial G}{\partial \gamma} > 0$, $\frac{\partial P}{\partial \gamma} > 0$ and $\frac{\partial G}{\partial \beta} > 0$, $P(x,\gamma) > 0$ and $Q(y,\beta) > 0\Big)$. Then we have

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big)$$
$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big)$$
$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big), \quad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big)$$

and

$$P_1(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big) \qquad P_2(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big)$$
$$Q_1(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big) \qquad Q_2(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big)$$

then we have

$$\partial_t z_1(t,x,y,\alpha) + P_1(x,\alpha)\partial_{xx} z_1(t,x,y,\alpha) + Q_1(y,\alpha)\partial_{yy} z_1(t,x,y,\alpha) = F_1(t,x,y,\alpha)$$
$$\partial_t z_2(t,x,y,\alpha) + P_2(x,\alpha)\partial_{xx} z_2(t,x,y,\alpha) + Q_2(y,\alpha)\partial_{yy} z_2(t,x,y,\alpha) = F_2(t,x,y,\alpha)$$

which is not true, because

$$\begin{aligned} G_t\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) + P\Big(x,\gamma_1(\alpha)\Big)G_{xx}\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) \\ + Q\Big(y,\beta_2(\alpha)\Big)G_{yy}\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) = F\Big(t,x,y,k_1(\alpha)\Big) \end{aligned}$$

$$\begin{aligned} G_t\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big) + P\Big(x,\gamma_1(\alpha)\Big)G_{xx}\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big) \\ + Q\Big(y,\beta_1(\alpha)\Big)G_{yy}\Big(t,x,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big) = F\Big(t,x,y,k_2(\alpha)\Big) \end{aligned}$$

□

Therefore, if $\overline{Z}(t,x,y)$ is a BFS and it satisfies the initial and boundary conditions we will say that $\overline{Z}(t,x,y)$ is a BFS satisfying the initial and boundary conditions. If $\overline{Z}(t,x,y)$ is not a BFS, then we will consider the SS.

5.2. **Seikkala solution (SS).**
Now let us define the SS [13]. Let

$$\overline{U}(t,x,y)[\alpha] = \Big[u_1(t,x,y,\alpha), u_2(t,x,y,\alpha)\Big]$$

For example suppose $P(x,\gamma) < 0$ and $Q(y,\beta) > 0$, so consider the system of heat-like equations

$$(u_1)_t + P_1(x,\alpha)(u_2)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha) \tag{5.12}$$

$$(u_2)_t + P_2(x,\alpha)(u_1)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha) \tag{5.13}$$

Or if $P(x,\gamma) > 0$, $Q(y,\beta) > 0$, $\frac{\partial P}{\partial \gamma} > 0$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial Q}{\partial \beta} > 0$, $\frac{\partial G}{\partial \beta} > 0$

$$(u_1)_t + P_1(x,\alpha)(u_1)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha) \tag{5.14}$$

(5.15) $$(u_2)_t + P_2(x,\alpha)(u_2)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$. We append to Eqs. (5.12) and (5.13) any initial and boundary conditions. For example, if it was $\overline{U}(0,x,y) = \overline{C}$ then we add

(5.16) $$u_1(0,x,y,\alpha) = c_1(\alpha)$$

(5.17) $$u_2(0,x,y,\alpha) = c_2(\alpha)$$

where $\overline{C}[\alpha] = \Big[c_1(\alpha), c_2(\alpha)\Big]$.

Let $u_i(t,x,y,\alpha)$ i=1,2 solve Eqs. (5.12) and (5.13) plus initial and boundary conditions. If

$$\Big[u_1(t,x,y,\alpha), u_2(t,x,y,\alpha)\Big],$$

defines the α-cut of a fuzzy number, for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, then $\overline{U}(t,x,y)$ is the SS.

We will say that derivative condition holds for fuzzy heat-like equation when Eqs.(5.9), (5.10) and (5.11) are true.

Theorem 5.2.

(1) *If BFS* $= \overline{Z}(t,x,y)$*, then SS* $= \overline{Z}(t,x,y)$.

(2) *If SS* $= \overline{Z}(t,x,y)$ *and the derivative condition holds, then BFS* $= \overline{U}(t,x,y)$.

Proof.

(1) Follows from the definition of BFS and SS.

(2) If SS $= \overline{U}(t,x,y)$ then the Seikkala derivative [3] exists and since the derivative condition holds, therefore, equation following holds

$$(u_1)_t + P_1(x,\alpha)(u_1)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha)$$
$$(u_2)_t + P_2(x,\alpha)(u_2)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha)$$

Also suppose one $k_j = k$, $\gamma_i = \gamma$, $\beta_l = \beta$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial P}{\partial \gamma} < 0$, $\frac{\partial G}{\partial k} < 0$ and $\frac{\partial F}{\partial k} < 0$, $\frac{\partial G}{\partial \beta} > 0$, $\frac{\partial Q}{\partial \beta} > 0$ (the other cases are similar and are omitted). We see

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big)$$
$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big)$$
$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big), \quad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big)$$
$$P_1(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big), \quad P_2(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big)$$
$$Q_1(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big), \quad Q_2(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big)$$

Now look at Eqs. (5.7), (5.8) also Eqs. (5.1) and (5.2), implies that

$$u_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big) = z_1(t,x,y,\alpha)$$
$$u_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big) = z_2(t,x,y,\alpha)$$

Therefore $BFS = \overline{U}(t,x,y)$ □

Lemma 5.3. *Consider (3.1) suppose $\overline{Z}(t,x)$ is differentiable.*

(a)

$$\text{if } P(x,\gamma_i) > 0 \text{ and } \frac{\partial P}{\partial \gamma_i}\frac{\partial G}{\partial \gamma_i} > 0 \quad x \in I_2 \quad \text{for} \quad i = 1,2,\dots,m \tag{5.18}$$

and

$$\text{if } \frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \text{ for } j = 1,2,\dots,n \tag{5.19}$$

Then BFS=$\overline{Z}(t,x)$

(b) *If relations (5.18) does not hold for some i or relation (5.19) does not hold for some j, then $\overline{Z}(t,x)$ is not a BFS.*

Proof. It is similar to theorem 5.1 □

6. Examples

We consider the following examples ([1],[14]) and we added fuzzy parameters to these references.

Example 6.1. We first consider the one-dimensional heat-like equation with variable coefficients as

$$U_t + \frac{\gamma}{2}x^2 U_{xx} = k \tag{6.1}$$

with the initial conditions

$$U(0,x) = cx^2$$

where $t \in (0,M_1], x \in (0,M_2]$, $k \in [0,J]$, $\gamma \in [0,H]$ and $c \in [L,0]$ are constants.
According to the VIM, a correct functional for (6.1) from (4.3) can be constructed as follows

$$U_{n+1}(t,x) = U_n(t,x) - \int_0^t \{(U_n)_s(s,x) + \frac{\gamma}{2}x^2(\widetilde{U}_n)_{xx}(s,x) - F(s,x,k)\}ds$$

Beginning with an initial approximation $U_0(t,x) = U(0,x) = cx^2$, we can obtain the following successive approximations

$U_1(t,x) = cx^2(1-\gamma t) + kt$

$U_2(t,x) = cx^2(1-\gamma t + \gamma^2\frac{t^2}{2!}) + kt$

$U_3(t,x) = cx^2(1-\gamma t + \gamma^2\frac{t^2}{2!} - \gamma^3\frac{t^3}{3!}) + kt$

and $U_n(t,x) = cx^2(1-\gamma t + \gamma^2\frac{t^2}{2!} - \gamma^3\frac{t^3}{3!} + \cdots + (-1)^n\gamma^n\frac{t^n}{n!}) + kt, \quad n \geq 1$

The VIM admits the use of $U(t,x) = \lim_{n\to\infty} U_n(t,x)$, which gives the exact solution

$$U(t,x) = cx^2\exp(-\gamma t) + kt$$

Now we fuzzify $F(t,x,k)$, $P(x,\gamma)$ and

$$G(t,x,k,c,\gamma) = cx^2\exp(-\gamma t) + kt.$$

Clearly

$$\begin{aligned} \overline{F}(t,x,\overline{K}) &= \overline{K} \\ \overline{P}(x,\overline{\gamma}) &= \frac{\overline{\gamma}}{2}x^2 \end{aligned}$$

so that

$$F_1(t,x,\alpha) = k_1(\alpha), \quad F_2(t,x,\alpha) = k_2(\alpha)$$
$$P_1(x,\alpha) = \frac{\gamma_1}{2}x^2, \quad P_2(x,\alpha) = \frac{\gamma_2}{2}x^2$$

Also $\overline{G}(t,x,\overline{K},\overline{C},\overline{\gamma}) = \overline{C}x^2\exp(-\overline{\gamma}t) + \overline{K}t$, therefore

$$z_i(t,x,\alpha) = c_i(\alpha)x^2\exp(-\gamma_i(\alpha)t) + k_i(\alpha)t$$

for $i = 1,2$ and $\overline{C} < 0$ ($\overline{C} = (c_1,c_2,c_3)$ also with $c_3 < 0$), $\overline{K}[\alpha] = [k_1(\alpha),k_2(\alpha)]$, $\overline{C}[\alpha] = [c_1(\alpha),c_2(\alpha)]$, and $\overline{\gamma}[\alpha] = [\gamma_1(\alpha),\gamma_2(\alpha)]$.
$\overline{Z}(t,x)$ is differentiable because $(z_i(t,x,\alpha))_t + \frac{\gamma_i(\alpha)}{2}x^2(z_i(t,x,\alpha))_{xx} = k_i(\alpha)$ for $i = 1,2$ are α-cuts of $\overline{k}$ i.e. α-cuts of a fuzzy number. Due to

$$P(x,\gamma) > 0$$
$$\frac{\partial G}{\partial k} > 0, \quad \frac{\partial F}{\partial k} > 0$$
$$\frac{\partial P}{\partial \gamma} > 0, \quad \frac{\partial G}{\partial \gamma} = -cx^2t\exp(-\gamma t) > 0$$

That is, $(\overline{Z})_t + \frac{\gamma}{2}x^2(\overline{Z})_{xx} = \overline{k}$, a fuzzy number.
So Lemma 5.3 implies the result that $\overline{Z}(t,x)$ is a BFS. We easily see that

$$z_i(0,x,\alpha) = c_i(\alpha)x^2$$

for $i = 1,2$, so $\overline{Z}(t,x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\overline{Z}(t,x) = \overline{C}x^2\exp(-\overline{\gamma}t) + \overline{K}t$$

for all $(t,x) \in (0,M_1] \times (0,M_2]$

Example 6.2. Consider the two-dimensional heat-like equation with variable coefficients as

$$\left\{ \begin{array}{l} U_t + \frac{\gamma}{2}x^2U_{xx} + \frac{\beta}{2}y^2U_{yy} = kxy \\ U(0,x,y) = c_1y^2 - c_2x^2 \end{array} \right. \tag{6.2}$$

which $t \in (0,M_1], x \in (0,M_2), y \in (0,M_3)$,$k \in [0,J]$,$\gamma \in [0,H]$, $c_1 \in [L,0[$, $c_2 \in [0,E]$ and $\beta \in [0,D]$
Similarly we can establish an iteration formula in the form

$$\begin{aligned} U_{n+1}(t,x,y) = U_n(t,x,y) - \int_0^t \Big\{ &(U_n(s,x,y))_s \\ &+ \frac{\gamma}{2}x^2(\widetilde{U}_n)_{xx}(s,x,y) + \frac{\beta}{2}y^2(\widetilde{U}_n)_{yy}(s,x,y) - F(s,x,y,k)\Big\} ds \end{aligned} \tag{6.3}$$

We begin with an initial arbitrary approximation:

$$U_0(t,x,y) = U(0,x,y) = c_1y^2 - c_2x^2$$

and using the iteration formula (6.3), we obtain the following successive approximations
$U_1(t,x,y) = c_1y^2(1-\beta t) - c_2x^2(1-\gamma t) + kxyt$
$U_2(t,x,y) = c_1y^2(1-\beta t + \frac{\beta^2t^2}{2!}) - c_2x^2(1-\gamma t + \frac{\gamma^2t^2}{2!}) + kxyt$

$U_3(t,x,y) = c_1y^2(1-\beta t+\frac{\beta^2t^2}{2!}-\frac{\beta^3t^3}{3!}) - c_2x^2(1-\gamma t+\frac{\gamma^2t^2}{2!}-\frac{\gamma^3t^3}{3!}) + kxyt$
and

$$U_n(t,x,y) = c_1y^2(1-\beta t+\frac{\beta^2t^2}{2!}+...+(-1)^n\frac{\beta^nt^n}{n!}) - c_2x^2(1-\gamma t+\frac{\gamma^2t^2}{2!}+...+(-1)^n\frac{\gamma^nt^n}{n!}) + kxyt$$

Then, the exact solution is given by

$$U(t,x,y) = c_1y^2\exp(-\beta t) - c_2x^2\exp(-\gamma t) + kxyt$$

Fuzzify $F(t,x,k)$, $P(x,\gamma)$, $Q(y,\beta)$ and

$$G(t,x,k,c,\gamma,\beta) = c_1y^2\exp(-\beta t) - c_2x^2\exp(-\gamma t) + kxyt$$

producing their α-cuts

$$\begin{aligned} z_1(t,x,y,\alpha) &= c_{11}y^2\exp(-\beta_1 t) - c_{22}x^2\exp(-\gamma_1 t) + k_1xyt \\ z_2(t,x,y,\alpha) &= c_{12}y^2\exp(-\beta_2 t) - c_{21}x^2\exp(-\gamma_2 t) + k_2xyt \end{aligned}$$

$$F_1(t,x,y,\alpha) = k_1(\alpha)xy, \quad F_2(t,x,y,\alpha) = k_2(\alpha)xy$$
$$P_1(x,\alpha) = \frac{\gamma_1(\alpha)}{2}x^2, \quad P_2(x,\alpha) = \frac{\gamma_2(\alpha)}{2}x^2$$
$$Q_1(x,\alpha) = \frac{\beta_1(\alpha)}{2}y^2, \quad Q_2(x,\alpha) = \frac{\beta_2(\alpha)}{2}y^2$$

where $\overline{C}_1 < 0$ ($\overline{C}_1 = (c_{11}, c_{12}, c_{13})$ with $c_{13} < 0$) and $\overline{K}[\alpha] = \Big[k_1(\alpha), k_2(\alpha)\Big]$, $\overline{C}_1[\alpha] = \Big[c_{11}(\alpha), c_{12}(\alpha)\Big]$, $\overline{C}_2[\alpha] = \Big[c_{21}(\alpha), c_{22}(\alpha)\Big]$, $\overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha), \gamma_2(\alpha)\Big]$, $\overline{\beta}[\alpha] = \Big[\beta_1(\alpha), \beta_2(\alpha)\Big]$.
We first check to see if $\overline{Z}(t,x,y)$ is differentiable. We compute

$$\Big[(z_1)_t + \frac{\gamma_1}{2}x^2(z_1)_{xx} + \frac{\beta_1}{2}y^2(z_1)_{yy}, (z_2)_t + \frac{\gamma_2}{2}x^2(z_2)_{xx} + \frac{\beta_2}{2}y^2(z_2)_{yy}\Big]$$

which are α-cuts of $\overline{K}xy$ i.e. α-cuts of a fuzzy number. Hence, $\overline{Z}(t,x,y)$ is differentiable.
Since

$$P(x,\gamma) > 0, \quad Q(y,\beta) > 0$$
$$\frac{\partial F}{\partial k} > 0, \quad \frac{\partial G}{\partial k} > 0$$
$$\frac{\partial P}{\partial \gamma} > 0, \quad \frac{\partial G}{\partial \gamma} > 0$$
$$\frac{\partial Q}{\partial \beta} > 0, \quad \frac{\partial G}{\partial \beta} > 0$$

Then Theorem (5.1) tells us that $\overline{Z}(t,x,y)$ is a BFS. The initial condition

$$z_1(0,x,y) = c_{11}(\alpha)y^2 - c_{22}x^2$$
$$z_2(0,x,y) = c_{12}(\alpha)y^2 - c_{21}x^2$$

Therefore $\overline{Z}(t,x,y)$ is a BFS which also satisfies the initial condition. This BFS may be written

$$\overline{Z}(t,x,y) = \overline{C}_1 y^2 \exp(-\overline{\beta}t) - \overline{C}_2 x^2 \exp(-\overline{\gamma}t) + \overline{K}xyt$$

for all $(x,y) \in (0,M_2) \times (0,M_3)$, $t \in (0,M_1]$

Example 6.3. Assume $c \in]0,L]$. $\overline{K}$, $\overline{\gamma}$ and $\overline{C}$ are triangular fuzzy numbers as in Example 6.1 with $\overline{C} = (c_1,c_2,c_3)$ and $c_1 > 0$. Then there is no BFS (Lemme 5.3). We proceed to look for a SS. We must solve

$$\begin{aligned} (u_1)_t + \frac{\gamma_1(\alpha)}{2}x^2(u_1)_{xx} &= k_1(\alpha) \\ (u_2)_t + \frac{\gamma_1(\alpha)}{2}x^2(u_2)_{xx} &= k_2(\alpha) \end{aligned}$$

subject to

$$u_1(0,x,\alpha) = c_1(\alpha)x^2, \quad u_2(0,x,\alpha) = c_2(\alpha)x^2$$

If the intervals $[u_1(t,x,\alpha), u_2(t,x,\alpha)]$ define α-cuts of a fuzzy number $\overline{U}(t,x)$; then SS $=\overline{U}(t,x)$. By VIM, the solution is

$$\begin{aligned} u_1(t,x,\alpha) = c_1(\alpha)x^2 \exp(-\gamma_1(\alpha)t) + k_1(\alpha)t \\ u_2(t,x,\alpha) = c_2(\alpha)x^2 \exp(-\gamma_2(\alpha)t) + k_2(\alpha)t \end{aligned}$$

Now we show $[u_1(t,x,\alpha), u_2(t,x,\alpha)]$ defines α-cut of a fuzzy number.
Thus we only need to check if $\frac{\partial u_1}{\partial \alpha} > 0$ and $\frac{\partial u_2}{\partial \alpha} < 0$. Since $u_i(t,x,\alpha)$ are continuous and $u_1(t,x,1) = u_2(t,x,1)$. There is a region $\mathfrak{R}$ contained in $[0,1] \times [0,1]$ for which the SS exists and $[0,1] \times [0,1] - \mathfrak{R}$ there may be no SS.
Since $\overline{K}$, $\overline{C}$ and $\overline{\gamma}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k_1^{'}(\alpha)$, $c_1^{'}(\alpha)$ and $\gamma_1^{'}(\alpha)$ are all positive numbers while $k_2^{'}(\alpha)$,$c_2^{'}(\alpha)$ and $\gamma_2^{'}(\alpha)$ are negative numbers. The "prime" denotes differentiation with respect to α. Then there is a $\lambda > 0$ so that $k_1^{'}(\alpha) = c_1^{'}(\alpha) = \gamma_1^{'}(\alpha) = \lambda$ and $k_2^{'}(\alpha) = c_2^{'}(\alpha) = \gamma_2^{'}(\alpha) = -\lambda$. Then, for the SS exist we need

$$\begin{aligned} \frac{\partial u_1}{\partial \alpha} &= \lambda(x^2 \exp(-\gamma_1(\alpha)t) - c_1(\alpha)tx^2 \exp(-\gamma_1(\alpha)t) + t) > 0 \\ \frac{\partial u_2}{\partial \alpha} &= -\lambda(x^2 \exp(-\gamma_2(\alpha)t) - c_2(\alpha)tx^2 \exp(-\gamma_2(\alpha)t) + t) < 0 \end{aligned}$$

Therefore inequalities hold if $1 - c_2(\alpha)t \geq 0$ for all $\alpha \in [0,1]$. So under the above assumptions we may choose

$$\mathfrak{R} = \left\{(t,x) | 0 < t \leq \frac{1}{c_3} \quad for\ all \quad 0 < x \leq M_2 \right\}$$

and the SS exists on $\mathfrak{R}$ in form Eqs

$$\overline{U}(t,x) = \overline{C}x^2 \exp(-\overline{\gamma}t) + \overline{K}t$$

Remark 6.4. We note that if we consider heat-like equation homogenous (i.e $k = 0$) as in example 6.1, we need to solve the following system (6.1) using $\overline{0}[\alpha] = [\alpha - 1, 1 - \alpha]$, it is clear that in this new procedure, the solution is

$$u_1(t, x, \alpha) = c_1(\alpha)x^2 \exp(-\gamma_i(\alpha)t) + (\alpha - 1)t$$
$$u_2(t, x, \alpha) = c_2(\alpha)x^2 \exp(-\gamma_i(\alpha)t) + (1 - \alpha)t$$

So Theorem 5.2 implies the result that $\overline{U}(t, x) = \overline{Z}(t, x)$ is a BFS. We easily see that

$$z_i(0, x, \alpha) = c_i(\alpha)x^2$$

for i=1,2 , So $\overline{Z}(t, x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\overline{Z}(t, x) = \overline{C}x^2 \exp(-\overline{\gamma}t) + \overline{0}t$$

for all $(t, x) \in (0, M_1] \times (0, M_2]$

7. Conclusion

In this paper, we give sufficient condition for the Buckley-Feuring solution to exist by the VIM for the proposed models, we obtain the exact solution of various kinds of fuzzy heat-like equations. Application of this method is easy and calculation of successive approximations is direct and straightforward. We using the VIM and strategy based on [4] introduced two type of solutions, the Buckley-Feuring solution and the Seikkala solution. If the Buckley-Feuring solution fails to exist and when the Seikkala solution fails to exist we offer no solution to the fuzzy heat-like equations.

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