Fuzzy, rough, rough fuzzy interior ideals of ternary semigroups

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Abstract. In this paper we introduce the notion of fuzzy interior ideals in ternary semigroups and investigated relations between fuzzy ideals and fuzzy interior ideals in terms of regularity. Here a characterization of fuzzy interior ideals is obtained in terms of fuzzy translation operator. The notions of rough and rough fuzzy interior ideals in a ternary semigroup are introduced. Relation between congruences and rough interior ideals are also established here.

1. Introduction

A fuzzy set theory was conceptualized by Professor L.A. Zadeh[29] in 1965 as a generalization of abstract set theory. Zadeh's initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications in different fields of mathematics has gained further ground almost round the globe. A paradigm is a concept encompassing rules and regulations which define boundaries and suggest standards as to how to successfully solve problems within these limits. For example the use of integrated circuit in place of transistors is a paradigm shift. In the late years of the decade of 1980s a variety of user-friendly tools for fuzzy control, fuzzy expert systems and fuzzy data analysis have come into force. Practical applications of fuzzy sets is divided into four main parts: engineering and natural sciences, medicine, management and behaviorial cum social sciences. Fuzzy set theory has been applied in different fields of mathematics also. Many papers

on fuzzy sets have appeared showing the importance of the concept and its application to logic, set theory, group theory, semigroup theory, real analysis, measure theory, topology etc. It was first applied to the theory of groups by A. Rosenfeld[24]. Nobuaki Kuroki[15, 16] pioneered the fuzzy ideal theory of semigroups. Sardar et al.[27] and Davvaz et al.[3] studied fuzzy interior ideals and intuitionistic fuzzy interior ideals in Γ-semigroups.

The notion of rough set was introduced by Z. Pawlak[21]. The theory of rough sets has emerged as a major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning and knowledge discovery. In connection with algebraic structures, R. Biswas and S. Nanda[2] introduced the notion of rough subgroups and N. Kuroki[17] introduced rough ideals in semigroups. Y.B. Jun[10] introduced roughness in Γ-semigroups. P. Petchkhaew and R. Chinram[22] initiated the study of rough and rough fuzzy ideals in ternary semigroups.

In 1932, D.H. Lehmer[18] investigated certain ternary algebraic structures called triplexes, but earlier such structures was studied by Kasner[14] and Prüfer[23]. Los[20] showed, by an example, that a ternary semigroup does not necessarily reduce to an ordinary semigroup. The algebraic structures of ternary semigroups were studied by several authors. In [28], Sioson studied ideal theory of ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. Santiago studied regular ternary semigroups[26], Dixit and Dewan[5] presented quasi-ideals and bi-ideals in ternary semigroups. In [6], Dutta et al. studied some properties of regular ternary semigroup, completely regular ternary semigroup, intra-regular ternary semigroup and characterized them by using various ideals of ternary semigroups and in [7], they studied prime ideals and prime radicals of ternary semigroups. Kar and Maity[12] investigated congruences of ternary semigroups and Iampan[8, 9] studied minimal and maximal lateral ideals of ternary semigroups. For further details on ternary semigroups, we refer the readers to [4, 5, 25]. Several researchers studied ternary semigroups in terms of fuzzy sets. We refer the readers to [1, 11, 13, 19, 22, 25] for the study of ternary semigroups in terms of fuzzy sets.

This paper is divided into four sections. Other than the present section and the next section on important preliminaries the third and the fourth constitute the main text of the paper. In the third section, we mainly study fuzzy interior ideals in ternary semigroups and investigate relations between fuzzy ideals and fuzzy interior ideals in terms of regularity. Also a characterization of fuzzy interior ideals is found in terms of fuzzy translation operator in this section. In the fourth section, rough and rough fuzzy interior ideals in a ternary semigroup are introduced. Relation between congruences and rough interior ideals are also established here.

2. Preliminaries

Definition 2.1 ([6])**.** A non-empty set S together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in S$.

Example 2.2 ([13]). Let Z^- be the set of all negative integers. Then Z^- forms a ternary semigroup where composition is given by the usual multiplication. But it is not a semigroup with respect to the usual multiplication.

Example 2.3 ([8]). $T = \{-i, 0, i\}$ is a ternary semigroup while T is not a semigroup under the multiplication over complex numbers.

Definition 2.4 ([28]). A non-empty subset I of a ternary semigroup S is called a subsemigroup of S if $III \subseteq I$.
A non-empty subset I of a ternary semigroup S is called a left ideal of S if $SSI \subseteq I$.
A non-empty subset I of a ternary semigroup S is called a lateral ideal of S if $SIS \subseteq I$.
A non-empty subset I of a ternary semigroup S is called a right ideal of S if $ISS \subseteq I$.
A non-empty subset I of a ternary semigroup S is called an ideal of S if I is a left ideal, a lateral ideal and a right ideal of S.
An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$.

Definition 2.5 ([29]). Let S be a non-empty set. A fuzzy subset μ of S is a function $\mu : S \to [0, 1]$.

Definition 2.6. A non-empty fuzzy subset μ of a ternary semigroup S is called a fuzzy subsemigroup of S if $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in S$.

Definition 2.7 ([22]). A non-empty fuzzy subset μ of a ternary semigroup S is called a fuzzy left ideal(fuzzy lateral ideal, fuzzy right ideal) of S if $\mu(xyz) \geq \mu(z)$(resp. $\mu(xyz) \geq \mu(y)$, $\mu(xyz) \geq \mu(x)$) for all $x, y, z \in S$.

Definition 2.8 ([22]). A non-empty fuzzy subset μ of a ternary semigroup S is called a fuzzy ideal of S if μ is a fuzzy left ideal, a fuzzy lateral ideal and a fuzzy right ideal of S.

3. Fuzzy Interior Ideals

Definition 3.1. A subsemigroup I of a ternary semigroup S is called an interior ideal of S if $SSISS \subseteq I$.

Definition 3.2. A fuzzy subsemigroup μ of a ternary semigroup S is called a fuzzy interior ideal of S if it satisfies $\mu(xsaty) \geq \mu(a)$ for all $x, s, a, t, y \in S$.

Definition 3.3. Let μ be a fuzzy subset of a ternary semigroup S and let $\alpha \in [0, 1]$. Then the set $\mu_\alpha := \{x \in S : \mu(x) \geq \alpha\}$ is called the α-cut of μ.

Theorem 3.4. *Let μ be a non-empty fuzzy subset of a ternary semigroup S. Then μ is a fuzzy interior ideal of S if and only if the α-cut μ_α of μ is an interior ideal of S for every $\alpha \in [0, 1]$, provided it is non-empty.*

Proof. Let μ be a fuzzy interior ideal of S and $\alpha \in [0, 1]$ be such that μ_α is non-empty. Let $x, y, z \in \mu_\alpha$. Then $\mu(x), \mu(y), \mu(z) \geq \alpha$. Since μ is a fuzzy interior ideal of S, it is a fuzzy subsemigroup of S and hence $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq \alpha$. Thus $\mu(xyz) \geq \alpha$. Hence $xyz \in \mu_\alpha$. Consequently, μ_α is a subsemigroup of S. Now let $x, y, s, t \in$ and $a \in \mu_\alpha$. Then $\mu(a) \geq \alpha$. Since μ is a fuzzy interior ideal of S, $\mu(xsaty) \geq \mu(a) \geq \alpha$ and hence $xsaty \in \mu_\alpha$. Consequently, μ_α is an interior ideal of S.

Conversely, suppose that each μ_α, $\alpha \in [0,1]$, is an interior ideal of S, provided $\mu_\alpha \neq \phi$. Let us assume that μ is not a fuzzy subsemigroup. Then there exist $x_0, y_0, z_0 \in S$ such that $\mu(x_0 y_0 z_0) < \min\{\mu(x_0), \mu(y_0), \mu(z_0)\}$.

Taking

$$\alpha_0 := \tfrac{1}{2}[\mu(x_0 y_0 z_0) + \min\{\mu(x_0), \mu(y_0), \mu(z_0)\}],$$

we obtain

$$\mu(x_0 y_0 z_0) < \alpha_0 < \min\{\mu(x_0), \mu(y_0), \mu(z_0))\}.$$

This implies that $x_0, y_0, z_0 \in \mu_{\alpha_0}$ and $x_0 y_0 z_0 \notin \mu_{\alpha_0}$, which contradicts the fact that μ_{α_0} is an interior ideal. Hence μ is a fuzzy subsemigroup. Similarly, we can prove that μ is an fuzzy interior ideal of S. $\qquad\square$

The following corollary is a general version of the characteristic function criterion of fuzzy interior ideal of a ternary semigroup.

Theorem 3.5. *Let I be a non-empty subset of a ternary semigroup S and let μ be a fuzzy subset of S defined by*

$$\mu(x) := \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1 & \text{otherwise,} \end{cases}$$

for all $x \in S$ and $\alpha_0, \alpha_1 \in [0,1]$ such that $\alpha_0 > \alpha_1$. Then I is an interior ideal of S if and only if μ is a fuzzy interior ideal of S. Furthermore $\mu_{\alpha_0} = I$.

Proof. Let I be an interior ideal of S and $x, y, z \in S$. Then following cases may arise: (1) $x \in I, y \in I, z \in I$; (2) $x \in I, y \notin I, z \in I$; (3) $x \in I, y \in I, z \notin I$; (4) $x \in I, y \notin I, z \notin I$; (5) $x \notin I, y \in I, z \in I$; (6) $x \notin I, y \notin I, z \in I$; (7) $x \notin I, y \in I, z \notin I$; (8) $x \notin I, y \notin I, z \notin I$. When $x \notin I, y \notin I, z \notin I$, we have $\mu(x) = \mu(y) = \mu(z) = \alpha_1$ and so $\min\{\mu(x), \mu(y), \mu(z)\} = \alpha_1$. Now, $\mu(xyz) = \alpha_0$ or α_1 according as $xyz \in I$ or $xyz \notin I$. Since $\alpha_0 > \alpha_1$, $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. For the other cases, by using similar argument, we obtain $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S. Now let $x, s, a, t, y \in S$. If $a \notin I$, then $\mu(a) = \alpha_1$ and $\mu(xsaty) = \alpha_0$ or $\mu(xsaty) = \alpha_1$ according as $xsaty \in I$ or $xsaty \notin I$. Again, if $a \in I$ then $xsaty \in I$ and so $\mu(xsaty) = \alpha_0 = \mu(a)$. Thus $\mu(xsaty) \geq \mu(a)$. Hence μ is a fuzzy interior ideal of S.

In order to prove the converse, we first observe that by definition of μ, $\mu_{\alpha_0} = I$. Then the proof follows from Theorem 3.4. $\qquad\square$

Following result is the characteristic function criterion of fuzzy interior ideal of a ternary semigroup which follows as an easy consequence of the above result.

Corollary 3.6. *Let μ_I be the characteristic function of a subset $I \neq \phi$ of a ternary semigroup S. Then μ_I is a fuzzy interior ideal of S if and only if I is an interior ideal of S.*

Proposition 3.7. *In a ternary semigroup every fuzzy ideal is a fuzzy interior ideal.*

Proof. Let μ be a fuzzy ideal of a ternary semigroup S. Then μ is a fuzzy left, fuzzy lateral and fuzzy right ideal of S. Then $\mu(xyz) \geq \mu(z)$, $\mu(xyz) \geq \mu(y)$, $\mu(xyz) \geq \mu(x)$. Let $x, y, z \in S$. Then $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S. Now let $x, s, a, t, y \in S$. Then $\mu(xsaty) = \mu(xs(aty)) \geq \mu(aty)$ (as μ is a fuzzy left ideal of S) $\geq \mu(a)$ (as μ is a fuzzy right ideal of S). Hence μ is a fuzzy interior ideal of S. $\qquad\square$

Definition 3.8. [6, 13] Let S be a ternary semigroup. An element $x \in S$ is called regular if there exists an element $a \in S$ such that $x = xax$. A ternary semigroup is called regular if all its elements are regular.

Proposition 3.9. *In a regular ternary semigroup every fuzzy interior ideal is a fuzzy ideal.*

Proof. Let μ be a fuzzy interior ideal of S and $x, s, y \in S$. Since S is regular, there exist $u, z, v \in S$ such that $x = xuzvx$. Then $\mu(xsy) = \mu((xuzvx)sy) = \mu((xuz)vxsy) \geq \mu(x)$ (as μ is a fuzzy interior ideal). So μ is a fuzzy right ideal of S. Similarly, we can prove that μ is a fuzzy left ideal and fuzzy lateral ideal of S. Hence μ is a fuzzy ideal of S. $\qquad\square$

From Proposition 3.7 and 3.9 it is clear that in regular ternary semigroups the notions of fuzzy ideals and fuzzy interior ideals coincide.

Proposition 3.10. *Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy interior ideals of a ternary semigroup S, then $\cap_{i \in I}\mu_i(x) = \inf\{\mu_i(x) : i \in I, x \in S\}$ is a fuzzy interior ideal of S, provided it is non-empty.*

Proof. We omit the proof since it is a matter of routine verification. $\qquad\square$

In what follows $Aut(S)$ denote the set of all automorphisms of the ternary semigroup S.

Definition 3.11. An interior ideal I of a ternary semigroup S is called a characteristic interior ideal of S if $f(I) = I$ for all $f \in Aut(S)$.

Definition 3.12. A fuzzy interior ideal μ of a ternary semigroup S is called a fuzzy characteristic interior ideal of S if $\mu(f(x)) = \mu(x)$ for all $x \in S$ and $\forall f \in Aut(S)$.

Now by routine calculation we can prove following two theorems.

Theorem 3.13. *A non-empty fuzzy subset μ of a ternary semigroup S is a fuzzy characteristic interior ideal of S if and only if the α-cut of μ is a characteristic interior ideal of S for all $\alpha \in [0, 1]$, provided μ_α is non-empty.*

Theorem 3.14. *Let I be a non-empty subset of S. Then I is a characteristic interior ideal of a ternary semigroup S if and only if its characteristic function χ_I is a fuzzy characteristic interior ideal of S.*

Definition 3.15. Let X be a non-empty set, $\alpha \in [0, 1]$ and μ be a fuzzy subset of X. We define

$$T_{\alpha^+}(\mu)(x) = \min\{\mu(x) + \alpha, 1\} \quad \text{and} \quad T_{\alpha^-}(\mu)(x) = \min\{\mu(x) - \alpha, 0\},$$

for all $x \in X$. T_{α^+} and T_{α^-} are respectively called the α-up fuzzy translation operator and α-down fuzzy translation operator of μ.

Example 3.16. Let $X = \{a, b, c\}$ and $\mu : X \to [0, 1]$ be a fuzzy subset of X defined as

$$\mu(x) = \begin{cases} 1 \text{ if } x = a \\ \frac{1}{2} \text{ if } x = b \\ 0 \text{ if } x = c \end{cases}$$

Let $\alpha = \frac{1}{4}$. Then $T_{\alpha^+}(\mu)$ and $T_{\alpha^-}(\mu)$ are given by

$$T_{\alpha^+}(\mu)(x) = \begin{cases} 1 \text{ if } x = a \\ \frac{3}{4} \text{ if } x = b \\ \frac{1}{4} \text{ if } x = c \end{cases}, \quad T_{\alpha^-}(\mu)(x) = \begin{cases} \frac{3}{4} \text{ if } x = a \\ \frac{1}{4} \text{ if } x = b \\ 0 \text{ if } x = c \end{cases}.$$

Proposition 3.17. *Let μ be a fuzzy interior ideal of a ternary semigroup S. Then $T_{\alpha^+}(\mu)$ $(T_{\alpha^-}(\mu))$ is a fuzzy interior ideal of S for all $\alpha \in [0, 1]$.*

Proof. Let μ be a fuzzy interior ideal of S and $\alpha \in [0, 1]$. Then μ is a fuzzy subsemigroup of S. Let $x, y, z \in S$. Then $T_{\alpha^+}(\mu)(xyz) = \min\{\mu(xyz) + \alpha, 1\} \geq \min [\min \{ \mu(x), \mu(y), \mu(z) \} + \alpha, 1] = \min [\min \{ \mu(x) + \alpha, 1\}, \min\{\mu(y) + \alpha, 1\}, \min\{\mu(z) + \alpha, 1\}] = \min\{T_{\alpha^+}(\mu)(x), T_{\alpha^+}(\mu)(y), T_{\alpha^+}(\mu)(z)\}$. Hence $T_{\alpha^+}(\mu)$ is a fuzzy subsemigroup of S. Now let $x, s, a, t, y \in S$. Then $T_{\alpha^+}(\mu)(xsaty) = \min\{\mu(xsaty) + \alpha, 1\} \geq \min\{\mu(a) + \alpha, 1\} = T_{\alpha^+}(\mu)(a)$. Hence $T_{\alpha^+}(\mu)$ is a fuzzy interior ideal of S. Similarly, we can prove the other case also. $\square$

Example 3.18. Let $S = \{a, b, c\}$ with the following cayley table:

.	a	b	c
a	a	b	c
b	b	a	c
c	c	c	a

Then $(S, .)$ is a semigroup. We define a fuzzy subset $\mu : S \to [0, 1]$ as

$$\mu(x) = \begin{cases} \frac{1}{2} \text{ if } x = a \\ \frac{3}{4} \text{ if } x = b \\ \frac{1}{4} \text{ if } x = c. \end{cases}$$

Since $\mu(cabac) = \mu(a) = \frac{1}{2} \ngtr \mu(b) = \frac{3}{4}$, μ is not a fuzzy interior ideal of S.
Now let $\alpha = \frac{3}{5}$. Then $T_{\alpha^+}(\mu)$ is given by

$$T_{\alpha^+}(\mu)(x) = \begin{cases} 1 \text{ if } x = a, b \\ \frac{17}{20} \text{ if } x = c \end{cases}$$

which is a fuzzy interior ideal of S.

Definition 3.19. Let μ be a fuzzy interior ideal of a ternary semigroup S. Then we define $S_\mu := \{x \in S : \mu(x) = 1\}$. Clearly S_μ is an interior ideal of S.

Proposition 3.20. *If $T_{\alpha+}$ is a fuzzy interior ideal of S for some $\alpha \in [0,1]$ with $\alpha < 1 - \max\{\mu(x) : x \in S - S_\mu\}$, then μ is a fuzzy interior ideal of S.*

Proof. Let $T_{\alpha+}$ be a fuzzy interior ideal of S. Then $T_{\alpha+}$ is a fuzzy subsemigroup of S. Let $x, y, z \in S$. Then $T_{\alpha+}(\mu)(xyz) \geq \min \{ T_{\alpha+}(\mu)(x), T_{\alpha+}(\mu)(y), T_{\alpha+}(\mu)(z) \}$...(1).

Then following cases may arise: (1) $T_{\alpha+}(\mu)(x) = 1, T_{\alpha+}(\mu)(y) < 1, T_{\alpha+}(\mu)(z) = 1$; (2) $T_{\alpha+}(\mu)(x) = 1, T_{\alpha+}(\mu)(y) < 1, T_{\alpha+}(\mu)(z) < 1$; (3) $T_{\alpha+}(\mu)(x) = 1, T_{\alpha+}(\mu)(y) = 1, T_{\alpha+}(\mu)(z) < 1$; (4) $T_{\alpha+}(\mu)(x) = 1, T_{\alpha+}(\mu)(y) = 1, T_{\alpha+}(\mu)(z) = 1$; (5) $T_{\alpha+}(\mu)(x) < 1, T_{\alpha+}(\mu)(y) < 1, T_{\alpha+}(\mu)(z) = 1$; (6) $T_{\alpha+}(\mu)(x) < 1, T_{\alpha+}(\mu)(y) < 1, T_{\alpha+}(\mu)(z) < 1$; (7) $T_{\alpha+}(\mu)(x) < 1, T_{\alpha+}(\mu)(y) = 1, T_{\alpha+}(\mu)(z) < 1$; (8) $T_{\alpha+}(\mu)(x) < 1, T_{\alpha+}(\mu)(y) = 1, T_{\alpha+}(\mu)(z) = 1$.

$Case(1)$: In this case $\mu(x) + \alpha, \mu(z) + \alpha \geq 1$ and $\mu(y) + \alpha < 1$. From (1) we have $\min\{\mu(xyz) + \alpha, 1\} \geq \mu(y) + \alpha$, i.e., $\mu(xyz) + \alpha \geq \mu(y) + \alpha$, i.e., $\mu(xyz) \geq \mu(y) = \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S.

$Case(2)$: In this case we have $\mu(x) + \alpha \geq 1$, $\mu(y) + \alpha, \mu(z) + \alpha < 1$. From (1) we have $\min\{\mu(xyz) + \alpha, 1\} \geq \min[\mu(y) + \alpha, \mu(z) + \alpha]$, i.e., $\min\{\mu(xyz) + \alpha, 1\} \geq \min\{\mu(y), \mu(z)\} + \alpha$, i.e., $\mu(xyz) + \alpha \geq \min\{\mu(y), \mu(z)\} + \alpha$, i.e., $\mu(xyz) \geq \min\{\mu(y), \mu(z)\} = \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S.

$Case(3)$: Similar as Case (1).

$Case(4)$: In this case $\mu(x) + \alpha, \mu(y) + \alpha, \mu(z) + \alpha \geq 1$. Since $\alpha < 1 - \max\{\mu(x) : x \in S - S_\mu\}$, $x, y, z \in S_\mu$. As S_μ is a subsemigroup of S, $xyz \in S_\mu$. Consequently, $\mu(xyz) = 1 = \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S.

$Case(5)$: Similar as Case (2).

$Case(6)$: In this case $\mu(x) + \alpha, \mu(y) + \alpha, \mu(z) + \alpha < 1$. From (1) we have $\min\{\mu(xyz) + \alpha, 1\} \geq \min[\mu(x) + \alpha, \mu(y) + \alpha, \mu(z) + \alpha]$, i.e., $\min\{\mu(xyz) + \alpha, 1\} \geq \min\{\mu(x), \mu(y), \mu(z)\} + \alpha$, i.e., $\mu(xyz) + \alpha \geq \min\{\mu(x), \mu(y), \mu(z)\} + \alpha$, i.e., $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Hence μ is a fuzzy subsemigroup of S.

$Case(7)$: Similar as Case (2).

$Case(8)$: Similar as Case (1).

Consequently, μ is a fuzzy subsemigroup of S.

Now let $x, s, a, t, y \in S$. Since $T_{\alpha+}$ is a fuzzy interior ideal of S, $T_{\alpha+}(\mu)(xsaty) \geq T_{\alpha+}(\mu)(a) \ldots (2)$. If $T_{\alpha+}(\mu)(a) = 1$, then $\mu(a) + \alpha \geq 1$. Since $\alpha < 1 - \max\{\mu(x) :$

$x \in S - S_\mu\}$, $a \in S_\mu$ and so $xsaty \in S_\mu$ as S_μ is an interior ideal of S. Consequently, $\mu(xsaty) = 1 = \mu(a)$. Hence μ is a fuzzy interior ideal of S.

If $T_{\alpha^+}(\mu)(a) < 1$, then $\mu(a) + \alpha < 1$. Then from (2), we have $\min\{\mu(xsaty) + \alpha, 1\} \geq \mu(a) + \alpha$, i.e., $\mu(xsaty) + \alpha \geq \mu(a) + \alpha$, i.e., $\mu(xsaty) \geq \mu(a)$. Hence μ is a fuzzy interior ideal of S. □

From Propositions 3.17 and 3.20, we deduce the following characterization theorem for a fuzzy interior ideal of a ternary semigroup.

Theorem 3.21. *Let S be a ternary semigroup and μ be a fuzzy subset of S. Then μ is a fuzzy interior ideal of S if and only if T_{α^+} is a fuzzy interior ideal of S for some $\alpha \in [0,1]$ with $\alpha < 1 - \max\{\mu(x) : x \in S - S_\mu\}$.*

4. Rough and Rough Fuzzy Interior Ideals

In this section, we study rough ternary interior ideals of ternary semigroups. We recall the following concepts from Kar and Maity[12] and Petchkhaew and Chinram[22].

• Let S be a ternary semigroup. A congruence ρ on S is an equivalence relation on S such that for all $a, b, x, y \in S$, $(a, b) \in \rho$ implies $(xya, xyb) \in \rho$, $(xay, xby) \in \rho$ and $(axy, bxy) \in \rho$. For $a \in S$, the ρ-congruence class containing a is denoted by $[a]_\rho$.

• A congruence ρ of S is called complete if $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in S$.

• Let ρ be a congruence on S and I be a non-empty subset of S. Then the sets

$$\rho_-(I) = \{x \in S : [x]_\rho \subseteq I\} \quad \text{and} \quad \rho^-(I) = \{x \in S : [x]_\rho \cap I \neq \phi\}$$

are called the ρ-lower and ρ-upper approximations of I, respectively.

Proposition 4.1. [22] *Let ρ and λ be congruences on a ternary semigroup S and I and J non-empty subsets of S. Then following statements are true:*
(1) $\rho_-(I) \subseteq I \subseteq \rho^-(I)$;
(2) $\rho^-(I \cup J) = \rho^-(I) \cup \rho^-(J)$;
(3) $\rho_-(I \cap J) = \rho_-(I) \cap \rho_-(J)$;
(4) $I \subseteq J \Rightarrow \rho_-(I) \subseteq \rho_-(J)$;
(5) $I \subseteq J \Rightarrow \rho^-(I) \subseteq \rho^-(J)$;
(6) $\rho_-(I) \cup \rho_-(J) \subseteq \rho_-(I \cup J)$;
(7) $\rho^-(I \cap J) \subseteq \rho^-(I) \cap \rho^-(J)$;
(8) $\rho \subseteq \lambda \Rightarrow \lambda_-(I) \subseteq \rho_-(I)$;
(9) $\rho \subseteq \lambda \Rightarrow \lambda^-(I) \subseteq \rho^-(I)$.

Theorem 4.2. [22] *Let ρ be a complete congruence on a ternary semigroup S and I, J and K non-empty subsets of S. Then* (1) $\rho^-(I)\rho^-(J)\rho^-(K) \subseteq \rho^-(IJK)$, (2) $\rho_-(I)\rho_-(J)\rho_-(K) \subseteq \rho_-(IJK)$.

• A non-empty subset I of a ternary semigroup S is called a ρ-upper rough interior ideal of S if $\rho^-(I)$ is an interior ideal of S and I is called a ρ-lower rough interior ideal of S if $\rho_-(I)$ is an interior ideal of S.

Theorem 4.3. *Let ρ be a complete congruence on a ternary semigroup S. If a non-empty subset I of S is an interior ideal of S, then I is a ρ-upper rough interior ideal of S.*

Proof. Let I be an interior ideal of S. By Proposition 4.1(1), $\rho^-(I) \neq \phi$. Then by Theorem 4.2 and Proposition 4.1(5), $\rho^-(I)\rho^-(I)\rho^-(I) \subseteq \rho^-(III) \subseteq \rho^-(I)$. Hence I is a ρ-upper rough subsemigroup of S. Now by Theorem 4.2 and Proposition 4.1(5), $SS\rho^-(I)SS = \rho^-(S)\rho^-(S)\ \rho^-(I)\rho^-(S)\rho^-(S) \subseteq \rho^-(SSISS) \subseteq \rho^-(I)$. Hence I is a ρ-upper rough interior ideal of S. $\qquad\square$

Similarly we can prove the following theorem.

Theorem 4.4. *Let ρ be a complete congruence on a ternary semigroup S and I be a non-empty subset of S such that $\rho_-(I) \neq \phi$. If I is an interior ideal of S, then I is a ρ-lower rough interior ideal of S.*

Definition 4.5. [22] Let ρ be a congruence on a ternary semigroup S and I be a non-empty subset of S. Then $\rho_-(I)/\rho$ and $\rho^-(I)/\rho$ are defined as

$$\rho_-(I)/\rho = \{[x]_\rho \in S/\rho : [x]_\rho \subseteq I\} \quad \text{and} \quad \rho^-(I)/\rho = \{[x]_\rho \in S/\rho : [x]_\rho \cap I \neq \phi\},$$

respectively.

Theorem 4.6. *Let ρ be a complete congruence on a ternary semigroup S. If I is an interior ideal of S, then $\rho^-(I)/\rho$ is an interior ideal of S/ρ.*

Proof. Let I be an interior ideal of S. Then I is a subsemigroup of S. Let us suppose that $[x]_\rho, [y]_\rho, [z]_\rho \in \rho^-(I)/\rho$. Then there exist $p \in [x]_\rho \cap I$, $q \in [y]_\rho \cap I$, $r \in [z]_\rho \cap I$. Since ρ is complete, $pqr \in [x]_\rho[y]_\rho[z]_\rho = [xyz]_\rho$. As I is a subsemigroup of S, $pqr \in I$. Consequently, $pqr \in [xyz]_\rho \cap I$. So $[xyz]_\rho \cap I \neq \phi$. Then $[x]_\rho[y]_\rho[z]_\rho \in \rho^-(I)/\rho$. Hence $\rho^-(I)/\rho$ is a subsemigroup of S/ρ. Now let $[x]_\rho, [s]_\rho, [t]_\rho, [y]_\rho \in S/\rho$ and $[a]_\rho \in \rho^-(I)/\rho$. Then there exist $d \in [a]_\rho \cap I$. Since ρ is complete, $xsdty \in [x]_\rho[s]_\rho[a]_\rho[t]_\rho[y]_\rho = [xsaty]_\rho$. As I is an interior ideal of S, $xsdty \in I$. Hence $xsdty \in [xsaty]_\rho \cap I$. So $[xsaty]_\rho \cap I \neq \phi$. Then $[x]_\rho[s]_\rho[a]_\rho[t]_\rho[y]_\rho \in \rho^-(I)/\rho$ and hence $(S/\rho) (S/\rho)\ (\rho^-(I)/\rho)\ (S/\rho)\ (S/\rho) \subseteq \rho^-(I)/\rho$. Consequently, $\rho^-(I)/\rho$ is an interior ideal of S/ρ. $\qquad\square$

Theorem 4.7. *Let ρ be a complete congruence on a ternary semigroup S and I be a non-empty subset of S such that $\rho_-(I)/\rho \neq \phi$. If I is an interior ideal of S, then $\rho_-(I)/\rho$ is an interior ideal of S/ρ.*

Proof. Let I be an interior ideal of S. Then I is a subsemigroup of S. Let us suppose that $[x]_\rho, [y]_\rho, [z]_\rho \in \rho_-(I)/\rho$. Then $[x]_\rho \subseteq I$, $[y]_\rho \subseteq I$, $[z]_\rho \subseteq I$. Since I is a subsemigroup S, $[x]_\rho[y]_\rho[z]_\rho \subseteq I$. Hence $[x]_\rho[y]_\rho[z]_\rho \in \rho_-(I)/\rho$. Consequently, $\rho_-(I)/\rho$ is a subsemigroup of S/ρ. Now let $[x]_\rho, [s]_\rho, [t]_\rho, [y]_\rho \in S/\rho$ and $[a]_\rho \in \rho_-(I)/\rho$. Then $[a]_\rho \subseteq I$. Since I is an interior ideal of S, $[x]_\rho[s]_\rho[a]_\rho[t]_\rho[y]_\rho \subseteq I$ whence $[x]_\rho[s]_\rho[a]_\rho[t]_\rho[y]_\rho \in \rho_-(I)/\rho$. Hence $\rho_-(I)/\rho$ is an interior ideal of S/ρ. $\qquad\square$

Definition 4.8. [22] Let μ be a fuzzy subset of a ternary semigroup. Then the fuzzy subsets

$$\rho^-(\mu)(x) = \sup_{a \in [x]_\rho} \mu(a) \quad \text{and} \quad \rho_-(\mu)(x) = \inf_{a \in [x]_\rho} \mu(a)$$

are called the ρ-upper and ρ-lower approximations of a fuzzy set μ respectively.

Lemma 4.9. [22] *Let ρ be a congruence on a ternary semigroup S. If μ is a fuzzy subset of S and $\alpha \in [0,1]$, then (1) $(\rho_-(\mu))_\alpha = \rho_-(\mu_\alpha)$, (2) $(\rho^-(\mu))_\alpha = \rho^-(\mu_\alpha)$.*

Theorem 4.10. *Let ρ be a complete congruence on a ternary semigroup S. If μ is a fuzzy interior ideal of S, then $\rho^-(\mu)$ and $\rho_-(\mu)$ are fuzzy interior ideals of S.*

Proof. Let μ be a fuzzy interior ideal of S. Then by Theorem 3.4, μ_α is an interior ideal of S for each $\alpha \in [0,1]$, provided $\mu_\alpha \neq \phi$. By Theorem 4.3, $\rho^-(\mu_\alpha)$ is an interior ideal of S. So by Lemma 4.9, $(\rho^-(\mu))_\alpha$ is an interior ideal of S. Hence by Theorem 3.4, $\rho^-(\mu)$ is a fuzzy interior ideal of S. Similarly we can prove the other case also. □

Definition 4.11. [22] Let S and T be two ternary semigroups. A mapping $\varphi : S \to T$ is called a homomorphism from S to T if $\varphi(xyz) = \varphi(x)\varphi(y)\varphi(z)$ for all $x, y, z \in S$. The set $\Omega = \{(x,y) \in S \times T : \varphi(x) = \varphi(y)\}$ is called the kernel of φ. Ω is a congruence on S.

Lemma 4.12. [22] *Let φ be an onto homomorphism from a ternary semigroup S to a ternary semigroup T, ρ_2 be a congruence on T, $\rho_1 = \{(x,y) \in S \times T : (\varphi(x), \varphi(y)) \in \rho_2\}$ and I be a non-empty subset of S. Then the following statements are true:* (1) ρ_1 *is a congruence on S;* (2) *if ρ_2 is complete and φ is one-one, then ρ_1 is complete;* (3) $\varphi(\rho_1^-(I)) = \rho_2^-(\varphi(I))$; (4) $\varphi(\rho_{1_-}(I)) \subseteq \rho_{2_-}(\varphi(I))$; (5) *if φ is one-one, then* $\varphi(\rho_{1_-}(I)) = \rho_{2_-}(\varphi(I))$.

Theorem 4.13. *Let φ be an onto homomorphism from a ternary semigroup S to a ternary semigroup T, ρ_2 be a congruence on T, $\rho_1 = \{(x,y) \in S \times T : (\varphi(x), \varphi(y)) \in \rho_2\}$ and I be a non-empty subset of S. Then $\rho_1^-(I)$ is an interior ideal of S if and only if $\rho_2^-(\varphi(I))$ is an interior ideal of T.*

Proof. Let $\rho_1^-(I)$ be an interior ideal of S. Then $\rho_1^-(I)$ is a subsemigroup of S. Then $\varphi(\rho_1^-(I))\ \varphi(\rho_1^-(I))\ \varphi(\rho_1^-(I)) = \varphi(\rho_1^-(I)\rho_1^-(I)\rho_1^-(I)) \subseteq \varphi(\rho_1^-(I))$. Then by Lemma 4.12(3), $\rho_2^-(\varphi(I))\ \rho_2^-(\varphi(I))\ \rho_2^-(\varphi(I)) \subseteq \rho_2^-(\varphi(I))$. Hence $\rho_2^-(\varphi(I))$ is a subsemigroup of T. In a similar fashion as above, we see that $\rho_2^-(\varphi(I))$ is an interior ideal of T.

We omit the proof of the converse part for its similarity to the direct part.

□

Theorem 4.14. *Let φ be an onto isomorphism from a ternary semigroup S to a ternary semigroup T, ρ_2 be a congruence on T, $\rho_1 = \{(x,y) \in S \times T : (\varphi(x), \varphi(y)) \in \rho_2\}$ and I be a non-empty subset of S. Then $\rho_{1_-}(I)$ is an interior ideal of S if and only if $\rho_{2_-}(\varphi(I))$ is an interior ideal of T.*

Proof. From Lemma 4.12(5), we have $\varphi(\rho_{1_-}(I)) = \rho_{2_-}(\varphi(I))$. Now by routine calculation the proof follows. □

5. Conclusions

Fuzzy set theory, rough set theory, soft set theory are all mathematical tools for dealing with uncertainties. In this paper interior ideal of ternary semigroup has been studied via fuzzy set, rough set and rough fuzzy set. We used these concepts to introduce the notions of fuzzy interior ideal, rough interior ideal and rough fuzzy interior ideal in ternary semigroups. In further research we shall study (1) Soft and fuzzy soft interior ideals of ternary semigroups/ordered ternary semigroups, (2) Soft rough fuzzy interior ideals of ternary semigroups/ordered ternary semigroups.

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