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On L-fuzzy soft ideals of nearrings

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Abstract. The soft set theory offers a general mathematical tool for dealing with uncertain fuzzy objects.Our object in this paper is to introduce and study the new concept of L-fuzzy soft sets in a nearring and investigate some basic properties. We obtained an exact analogue of L-fuzzy soft ideals for nearrings.

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1. Introduction

Molodtsov [9] initiated the concept of soft set as mathematical tool for dealing with uncertainties. Maji et al. [7] defined some operations on soft sets. Ali et al. [1] introduced several new operations of soft sets. The theory has also seen a wide-ranging applications in the algebraic structures such as groups, semirings, rings, BCK/BCI-algebras, nearrings and soft substructures and union soft substructures (see [2, 3, 11, 12]).

The fundamental concept of fuzzy set was introduced by Zadeh [14] in 1965. Rosenfeld inspired the fuzzification of algebraic structures and introduced the notions of fuzzy subgroups.

The concept of fuzzy subnearring and fuzzy ideal was discussed further by many researchers among whom Davvaz, Dutta, Jun, Kim, Narayanan, Kandasmy, Saikia and Barthakur. In [4] Bashir and Kanwal studied prime and strongly prime bi-ideals of nearrings.

This paper is devoted to the discussion of algebraic structures of L- fuzzy soft sets, L-fuzzy soft subnearring, L-fuzzy soft left (right) ideal, over a universe U. We study some of their properties.

2. Preliminaries

A nearring is a non-empty set N together with two binary operations "+" and "." such that

(i) $(N,+)$ is a group (not necessarily abelian).

(ii) $(N,.)$ is a semigroup.

(iii) For all $n_1, n_2, n_3 \in N : n_1(n_2+n_3) = n_1n_2 + n_1n_3$ (left distributive law).

In view of (iii), one speaks more precisely of a "left nearring". Postulating $(iii)^{'}$ For all $n_1, n_2, n_3 \in N : (n_1+n_2)n_3 = n_1n_3 + n_2n_3$ instead of (iii), one gets " right nearring " (see [8, 10]). In this paper we consider left nearrings.

Let N be a nearring. A subgroup $(M,+)$ of $(N,+)$ is called a subnearring of N if $m_1m_2 \in M$ for all $m_1, m_2 \in M$. Let N be a nearring. A normal subgroup I of $(N,+)$ is called an ideal of N if

(i) $NI \subseteq I$, that is $ni \in I$ for all $i \in I$ and $n \in N$.

(ii) For all $n_1, n_2 \in N$ and $i \in I$, $(n_1+i)n_2 - nn_2 \in I$.

Normal subgroup I of $(N,+)$ with (i) is called a left ideal of N, while normal subgroup I of $(N,+)$ with (ii) is called a right ideal of N.

A partially ordered set (poset) $(L,\leq)$ is called

1) a lattice, if $a \vee b \in L$, $a \wedge b \in L$ for any $a, b \in L$.

2) a complete lattice, if $\vee A \in L$, $\wedge A \in L$ for any $A \subseteq L$.

3) distributive, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$; $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Let L be a lattice with top element 1_L and bottom element 0_L and let $a, b \in L$. Then b is called a complement of a, if $a \vee b = 1_L$ and $a \wedge b = 0_L$. If $a \in L$ has a complement element, then it is unique. It is denoted by a'.

A lattice L is called a Boolean lattice, if

(i) L is distributive,

(ii) L has 0_L and 1_L

(iii) each $a \in L$ has the complement $a' \in L$.

Let X be a non empty set. By a fuzzy subset f of X we mean a membership function $f : X \to [0,1]$ which associates with each element in X a real number from the unit interval $[0,1]$, the value $f(x)$ represents the "grade of membership" of x in f.

Goguen [5] generalized the concept of fuzzy set, by replacing $[0,1]$ by a complete distributive lattice L, and called it an L-fuzzy set. An L- fuzzy set A in a nonempty set X is a function $A : X \to L$, where L is a complete distributive lattice with 1_L and 0_L. We denote by L^X the set of all L-fuzzy sets in X. Let $A, B \in L^X$. Then their union and intersection are L-fuzzy sets in X, defined as

$(A \cup B)(x) = A(x) \vee B(x)$ and $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in X$.

$A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

The L-fuzzy sets $\widetilde{0}$ and $\widetilde{1}$ of X are defined as $\widetilde{0}(x) = 0$ and $\widetilde{1}(x) = 1$ for all $x \in X$. Obviously $\widetilde{0} \subseteq A \subseteq \widetilde{1}$ for all $A \in L^X$.

3. L-fuzzy soft sets

In 1999, Molodtsov defined the concept of soft set [9].

Let U be a universe, E be a set of parameters and $A \subseteq E$. A pair (F, A) is called a soft set (over U) if F is a mapping from A into the power set of U, that is $F : A \longrightarrow P(U)$. In other words, the soft set is a parametrized family of subsets of the set U.

In 2001, Maji, Biswas and Roy [7] generalized the concept of soft set and defined fuzzy soft set. They replace the power set of the universe U by the fuzzy power set of U. In 2012, Li, Zheng and Hao [6] further generalized the concept and defined L-fuzzy soft set by replacing power set of U by L-fuzzy power set of U, that is L^U. In [13], Shabir and Ghafoor worked on L-fuzzy soft semigroups.

In this section we define sum and product of L-fuzzy soft subsets of a nearring over a universe U and study some properties of these operations.

Definition 3.1 ([6])**.** Let E be a set of parameters, U be an initial universe, L be a complete distributive bounded lattice and $A \subseteq E$. An L-fuzzy soft set f_A over U is a mapping defined by $f_A : E \longrightarrow L^U$, such that $f_A(x) = \widetilde{0}$ if $x \notin A$.

The following operations on L-fuzzy soft sets are defined in [6]

(1) Let f_A and g_B be two L-fuzzy soft sets over U. Then f_A is contained in g_B denoted by $f_A \widetilde{\subseteq} g_B$ if $f_A(e) \subseteq g_B(e)$ for all $e \in E$, that is $(f_A(e))(u) \leq (g_B(e))(u)$ for all $u \in U$.

Two L-fuzzy soft sets f_A and g_B over U are said to be equal, denoted by $f_A \widetilde{=} g_B$ if $f_A \widetilde{\subseteq} g_B$ and $g_B \widetilde{\subseteq} f_A$.

(2) Let f_A and g_B be two L-fuzzy soft sets over U. Then their union $f_A \widetilde{\cup} g_B \widetilde{=} h_{A\cup B}$, where $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$ for all $e \in E$.

(3) Let f_A and g_B be two L-fuzzy soft sets over U. Then their intersection $f_A \widetilde{\cap} g_B \widetilde{=} h_{A\cap B}$, where $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$ for all $e \in E$.

(4) f_A is called null, if $f(e) = \widetilde{0}$ for any $e \in A$. We denote it by $\varnothing_A$.

(5) f_A is called absolute, if $f_A(e) = \widetilde{1}$ for any $e \in A$. We denote it by U_A.

Proposition 3.2 ([6])**.** *Let $A, B, C \subseteq E$ and f_A, g_B, h_C be three L-fuzzy soft sets over U. Then*

(1) $f_A \widetilde{\cup} f_A \widetilde{=} f_A$ *and* $f_A \widetilde{\cap} f_A \widetilde{=} f_A$.

(2) $f_A \widetilde{\cup} g_B \widetilde{=} g_B \widetilde{\cup} f_A$ *and* $f_A \widetilde{\cap} g_B \widetilde{=} g_B \widetilde{\cap} f_A$.

(3) $\left(f_A \widetilde{\cup} g_B\right) \widetilde{\cup} h_C \widetilde{=} f_A \widetilde{\cup} \left(g_B \widetilde{\cup} h_C\right)$ *and* $\left(f_A \widetilde{\cap} g_B\right) \widetilde{\cap} h_C \widetilde{=} f_A \widetilde{\cap} \left(g_B \widetilde{\cap} h_C\right)$.

In the remaining of this paper $E = N$, a nearring and L is a complete bounded distributive lattice and U is the universe set. We shall call an L-fuzzy soft set over U as an L-fuzzy soft set of N over U.

Definition 3.3. Let f_A and g_B be two L-fuzzy soft sets of a nearring N over the common universe U. Then the soft product $f_A \odot g_B$ is an L-fuzzy soft set of N over U defined by

$$(f_A \odot g_B)(x) = \begin{cases} \bigcup\limits_{x=yz} f_A(y) \cap g_B(z) & \text{if } \exists\ y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases}$$

for all $x \in N$.

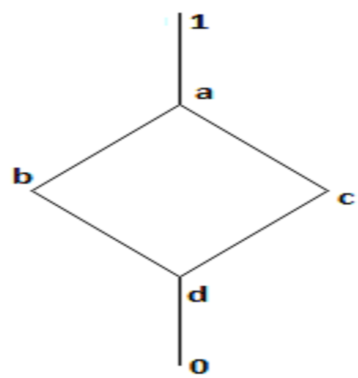
We next show that if f_A, g_B are L-fuzzy soft sets of N over U, then $f_A \odot g_B \neq g_B \odot f_A$.

Example 3.4. Let $N = \{0, x, y, z\}$ be a nearring with the binary operations as defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Let $L = \{1, a, b, c, d, 0\}$ be the complete bounded distributive lattice, $U = \{p, q\}$ and $A = B = \{x, y, z\}$.



Let f_A and g_B be two L-fuzzy soft sets of N over U as follows:

	p	q
$f_A(x)$	b	d
$f_A(y)$	a	c
$f_A(z)$	a	b

	p	q
$g_B(x)$	1	0
$g_B(y)$	1	c
$g_B(z)$	0	b

Now, for $x \in N$

$$\begin{aligned}
(f_A \odot g_B)(x) &= \bigcup_{x=yz} \{f_A(y) \cap g_B(z)\} \\
&= \bigcup_{x=yz} \{f_A(x) \cap g_B(x), f_A(x) \cap g_B(z)\} \\
&= \{(b, d) \cap (1, 0)\} \cup \{(b, d) \cap (0, b)\} \\
&= (b, 0) \cup (0, d) = (b, d).
\end{aligned}$$

where $(b, 0)$ means that image of p is b and image of q is 0. As

$$\begin{aligned}
(g_B \odot f_A)(x) &= \bigcup_{x=yz} g_B(y) \cap f_A(z) \\
&= \bigcup_{x=yz} \{g_B(x) \cap f_A(x), g_B(x) \cap f_A(z)\} \\
&= \{(1, 0) \cap (b, d)\} \cup \{(1, 0) \cap (a, b)\} \\
&= (b, 0) \cup (a, 0) = (a, 0).
\end{aligned}$$

Hence,

$$f_A \odot g_B \neq g_B \odot f_A.$$

Proposition 3.5. *Let $S(U)$ be the collection of all L-fuzzy soft sets of a nearring N over U and $f_A, g_B, h_C \in S(U)$. Then*

(i) $(f_A \odot g_B) \odot h_C \widetilde{=} f_A \odot (g_B \odot h_C)$.

(ii) $f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \odot h_C) \widetilde{\subseteq} (g_B \odot h_C)$ *and* $(h_C \odot f_A) \widetilde{\subseteq} (h_C \odot g_B)$.

(iii) $f_A \odot (g_B \widetilde{\cup} h_C) \widetilde{=} (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C)$ *and*

$$(f_A \widetilde{\cup} g_B) \odot h_C \widetilde{=} (f_A \odot h_C) \widetilde{\cup} (g_B \odot h_C).$$

(iv) $f_A \odot (g_B \widetilde{\cap} h_C) \widetilde{\subseteq} (f_A \odot g_B) \widetilde{\cap} (f_A \odot h_C)$ *and*

$$(f_A \widetilde{\cap} g_B) \odot h_C \widetilde{\subseteq} (f_A \odot h_C) \widetilde{\cap} (g_B \odot h_C)).$$

Proof. (i) Let $x \in N$. Then

$$\begin{aligned}
((f_A \odot g_B) \odot h_C)(x) &= \underset{x=yz}{\cup} \{(f_A \odot g_B)(y) \cap h_C(z)\} \\
&= \underset{x=yz}{\cup} \{\{\underset{y=st}{\cup} (f_A(s) \cap g_B(t))\} \cap h_C(z)\} \\
&= \underset{x=yz}{\cup} \underset{y=st}{\cup} \{(f_A(s) \cap g_B(t)) \cap h_C(z)\} \\
&= \underset{x=yz}{\cup} \underset{y=st}{\cup} \{f_A(s) \cap (g_B(t) \cap h_C(z))\} \\
&\subseteq \underset{x=lm}{\cup} \left[f_A(l) \cap \left[\underset{m=cd}{\cup} (g_B(c) \cap h_C(d)) \right] \right] \\
&= \underset{x=lm}{\cup} \{f_A(l) \cap (g_B \odot h_C)(m)\} \\
&= \underset{x=lm}{\cup} \{(f_A \odot (g_B \odot h_C))(x).
\end{aligned}$$

This implies that

$$(f_A \odot g_B) \odot h_C \widetilde{\subseteq} f_A \odot (g_B \odot h_C).$$

Similarly, we can show that

$$f_A \odot (g_B \odot h_C) \widetilde{\subseteq} (f_A \odot g_B) \odot h_C.$$

Hence,

$$(f_A \odot g_B) \odot h_C = f_A \odot (g_B \odot h_C).$$

(ii) As

$$f_A \widetilde{\subseteq} g_B \Rightarrow f_A(y) \subseteq g_B(y)$$

for all $y \in N$.

Let $x \in N$. If $x \neq yz$ for all $y, z \in N$ then

$$(f_A \odot h_C)(x) = 0 = (g_B \odot h_C)(x).$$

Otherwise

$$\begin{aligned}
(f_A \odot h_C)(x) &= \underset{x=yz}{\cup} \{f_A(y) \cap h_C(z)\} \\
&\subseteq \underset{x=yz}{\cup} \{g_B(y) \cap h_C(z)\} \\
&= (g_B \odot h_C)(x).
\end{aligned}$$

Hence,

$$f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \odot h_C) \widetilde{\subseteq} (g_B \odot h_C).$$

Similarly, we can prove $(h_C \odot f_A) \widetilde{\subseteq} (h_C \odot g_B)$.

(iii) Let $x \in N$. If x is not expressible as $x = yz$ for all $y, z \in N$, then

$$(f_A \odot (g_B \widetilde{\cup} h_C))(x) = \widetilde{0} = (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).$$

Otherwise

$$\begin{aligned}(f_A \odot (g_B \widetilde{\cup} h_C))(x) &= \underset{x=yz}{\cup}\{f_A(y) \cap (g_B \widetilde{\cup} h_C)(z)\} \\ &= \underset{x=yz}{\cup}\{f_A(y) \cap (g_B(z) \cup h_C(z))\} \\ &= \underset{x=yz}{\cup}\{(f_A(y) \cap g_B(z)) \cup (f_A(y) \cap h_C(z))\} \\ &= \{\underset{x=yz}{\cup}(f_A(y) \cap g_B(z))\} \cup \{\underset{x=yz}{\cup}(f_A(y) \cap h_C(z))\} \\ &= (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).\end{aligned}$$

This implies that

$$f_A \odot (g_B \widetilde{\cup} h_C) = (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C).$$

Hence,

$$f_A \odot (g_B \widetilde{\cup} h_C) = (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C).$$

Similarly, we can prove $(f_A \widetilde{\cup} g_B) \odot h_C \widetilde{=} (f_A \odot h_C) \widetilde{\cup} (g_B \odot h_C)$.
(iv) Let $x \in N$. If x is not expressible as $x = yz$ for all $y, z \in N$, then

$$(f_A \odot (g_B \widetilde{\cap} h_C))(x) = \widetilde{0} = (f_A \odot g_B)(x) \widetilde{\cap} (f_A \odot h_C)(x).$$

Otherwise

$$\begin{aligned}(f_A \odot (g_B \widetilde{\cap} h_C))(x) &= \underset{x=yz}{\cup}\{f_A(y) \cap (g_B \widetilde{\cap} h_C)(z)\} \\ &= \underset{x=yz}{\cup}\{f_A(y) \cap (g_B(z) \cap h_C(z))\} \\ &= \underset{x=yz}{\cup}\{(f_A(y) \cap g_B(z)) \cap (f_A(y) \cap h_C(z))\} \\ &\subseteq \{\underset{x=yz}{\cup}(f_A(y) \cap g_B(z))\} \cap \{\underset{x=yz}{\cup}(f_A(y) \cap h_C(z))\} \\ &= (f_A \odot g_B)(x) \cap (f_A \odot h_C)(x).\end{aligned}$$

This implies that

$$f_A \odot (g_B \widetilde{\cap} h_C) \;\widetilde{\subseteq} (f_A \odot g_B) \widetilde{\cap} (f_A \odot h_C).$$

Similarly, we can prove that

$$(f_A \widetilde{\cap} g_B) \odot h_C \,\widetilde{\subseteq} (f_A \odot h_C) \widetilde{\cap} (g_B \odot h_C).$$

□

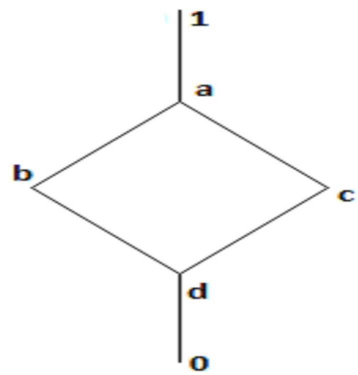
Next example shows that equality does not hold in part (iv) of above Proposition.

Example 3.6. Let $N = \{0, x, y, z\}$ be the nearring with binary operations defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Let $L = \{1, a, b, c, d, 0\}$ be the complete bounded distributive lattice, $U = \{p, q\}$ and $A = B = C = \{x, y, z\}$.



Define f_A, g_B and h_C the L-fuzzy soft sets of N over U as follows:

	p	q
$f_A(x)$	a	d
$f_A(y)$	a	b
$f_A(z)$	c	a

	p	q
$g_B(x)$	b	0
$g_B(y)$	1	b
$g_B(z)$	c	b

	p	q
$h_C(x)$	0	1
$h_C(y)$	1	0
$h_C(z)$	0	d

Now

$$\begin{aligned} (f_A \odot (g_B \widetilde{\cap} h_C))(z) &= \bigcup_{z=xy} \{f_A(x) \cap (g_B \widetilde{\cap} h_C)(y)\} \\ &= \bigcup_{z=xy} \{f_A(x) \cap (g_B(y) \cap h_C(y))\} \\ &= \bigcup_{z=xy} \{f_A(z) \cap (g_B(x) \cap h_C(x)), f_A(z) \cap (g_B(z) \cap h_C(z))\} \\ &= ((c, a) \cap ((b, 0) \cap (0, 1))) \cup ((c, a) \cap ((c, b) \cap (0, d))) \\ &= ((c, a) \cap (0, 0)) \cup ((c, a) \cap (0, d)) = (0, 0) \cup (0, d) = (0, d). \end{aligned}$$

And

$$\begin{aligned} (f_A \odot g_B)(z) \widetilde{\cap} (f_A \odot h_C)(z) &= \{(f_A(z) \cap g_B(x)) \cup (f_A(z) \cap g_B(z))\} \cap \\ &\quad \{(f_A(z) \cap h_C(x)) \cup (f_A(z) \cap h_C(z))\} \\ &= \{((c, a) \cap (b, 0)) \cup ((c, a) \cap (c, b))\} \cap \\ &\quad \{((c, a) \cap (0, 1)) \cup ((c, a) \cap (0, d))\} \\ &= \{(d, 0) \cup (c, b)\} \cap \{(0, a) \cup (0, d)\} \\ &= (c, b) \cap (0, a) = (0, b). \end{aligned}$$

Hence

$$f_A \odot \left(g_B \widetilde{\cap} h_C\right) \widetilde{\neq} \left(f_A \odot g_B\right) \widetilde{\cap} \left(f_A \odot h_C\right).$$

From the above Propositions we conclude that:

Remark 3.7. (1) $\left(S(U), \widetilde{\cup}\right)$ is a commutative semigroup with identity element $\varnothing_A$.
(2) $\left(S(U), \widetilde{\cap}\right)$ is a commutative semigroup with identity element U_A and zero element $\varnothing_A$.
(3) $(S(U), \odot)$ is a semigroup.
(4) $\odot$ distributes over $\widetilde{\cup}$ but not over $\widetilde{\cap}$.
(5) $\left(S(U), \widetilde{\cup}, \odot\right)$ is a semiring.

Definition 3.8. Let $f_A, g_B \in S(U)$. Then their sum $f_A \oplus g_B$ is defined by
$(f_A \oplus g_B)(x) = \underset{x=y+z}{\cup} f_A(y) \cap g_B(z)$ for all $x \in N$.

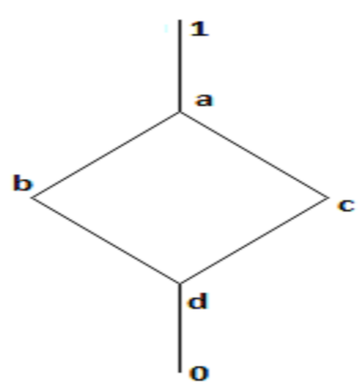
Next we show that $f_A \oplus g_B \neq g_B \oplus f_A$ for L-fuzzy soft sets f_A, g_B of a nearring N over U.

Example 3.9. Consider $S_3 = \{1, a, b, a^2, ab, a^2b\}$ with the binary operations addition and multiplication as defined below:

$\oplus$	1	a	a^2	b	ab	a^2b
1	1	a	a^2	b	ab	a^2b
a	a	a^2	1	ab	a^2b	b
a^2	a^2	1	a	a^2b	b	ab
b	b	a^2b	ab	1	a^2	a
ab	ab	b	a^2b	a	1	a^2
a^2b	a^2b	ab	b	a^2	a	1

$\odot$	1	a	a^2	b	ab	a^2b
1	1	1	1	1	1	1
a	1	1	1	1	1	1
a^2	1	1	1	1	1	1
b	1	1	1	1	1	1
ab	1	1	1	1	1	1
a^2b	1	1	1	1	1	1

Then $(S_3, \oplus, \odot)$ is a left nearring. Let $L = \{1, a, b, c, d, 0\}$ be the complete bounded distributive lattice, $U = \{p, q\}$, and $A = B = S_3$.



Define L-fuzzy soft sets f_A and g_B of N over U as follows:

	p	q
$f_A(1)$	1	b
$f_A(a)$	c	d
$f_A(a^2)$	a	1
$f_A(b)$	d	c
$f_A(ab)$	b	1
$f_A(a^2b)$	0	a

	p	q
$g_B(1)$	1	a
$g_B(a)$	b	0
$g_B(a^2)$	d	b
$g_B(b)$	0	c
$g_B(ab)$	c	d
$g_B(a^2b)$	a	1

Then

$$\begin{aligned}(f_A \oplus g_B)(a) &= \underset{a=x+y}{\cup}\{f_A(x) \cap g_B(y)\\ &= \cup\{f_A(1) \cap g_B(a), f_A(a) \cap g_B(1), f_A(a^2) \cap g_B(a^2),\\ &\quad f_A(b) \cap g_B(a^2b), f_A(ab) \cap g_B(b), f_A(a^2b) \cap g_B(ab)\}\\ &= \cup\left\{\begin{array}{c}(1,b)\cap(b,0),(c,d)\cap(1,a),(a,1)\cap(d,b),\\ (d,c)\cap(a,1),(b,1)\cap(0,c),(0,a)\cap(c,d)\end{array}\right\}\\ &= \{(b,0)\cup(c,d)\cup(d,b)\cup(d,c)\cup(0,c)\cup(0,d)\} = (a,a).\end{aligned}$$

Similarly,

$$\begin{aligned}(g_B \oplus f_A)(a) &= \underset{a=x+y}{\cup}\{g_B(x) \cap f_A(y)\}\\ &= \cup\{g_B(1) \cap f_A(a), g_B(a) \cap f_A(1),\\ &\quad g_B(a^2) \cap f_A(a^2), g_B(b) \cap f_A(a^2b),\\ &\quad g_B(ab) \cap f_A(b), g_B(a^2b) \cap f_A(ab)\}\\ &= \cup\left\{\begin{array}{c}(1,a)\cap(c,d),(b,0)\cap(1,b),(d,b)\cap(a,1),\\ (0,c)\cap(0,a),(c,d)\cap(d,c),(a,1)\cap(b,1)\end{array}\right\}\\ &= \{(c,d)\cup(b,0)\cup(d,b)\cup(0,c)\cup(d,d)\cup(b,1)\} = (a,1).\end{aligned}$$

Hence, $f_A \oplus g_B \neq g_B \oplus f_A$.

Proposition 3.10. *Let $f_A, g_B, h_C \in S(U)$. Then*
(*i*) $(f_A \oplus g_B) \oplus h_C \widetilde{=} f_A \oplus (g_B \oplus h_C)$.
(*ii*) $f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \oplus h_C) \widetilde{\subseteq} (g_B \oplus h_C)$ *and* $(h_C \oplus f_A) \widetilde{\subseteq} (h_C \oplus g_B)$.
(*iii*) $f_A \oplus (g_B \widetilde{\cup} h_C) \widetilde{=} (f_A \oplus g_B) \widetilde{\cup} (f_A \oplus h_C)$ *and*
$(f_A \widetilde{\cup} g_B) \oplus h_C = (f_A \oplus h_C) \widetilde{\cup} (g_B \oplus h_C)$.
(*iv*) $f_A \oplus (g_B \widetilde{\cap} h_C) \widetilde{\subseteq} (f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C)$ *and*
$(f_A \widetilde{\cap} g_B) \oplus h_C \widetilde{\subseteq} (f_A \oplus h_C) \widetilde{\cap} (g_B \oplus h_C))$.

Proof. (*i*) Let $x \in N$. Then

$$\begin{aligned}((f_A \oplus g_B) \oplus h_C)(x) &= \underset{x=y+z}{\cup}\{(f_A \oplus g_B)(y) \cap h_C(z)\} \\ &= \underset{x=y+z}{\cup}\{\{\underset{y=s+t}{\cup}(g_B(s) \cap f_A(t))\} \cap h_C(z)\} \\ &= \underset{x=y+z}{\cup}\{\{\underset{y=s+t}{\cup}(g_B(s) \cap f_A(t))\} \cap h_C(z)\} \\ &= \underset{x=(s+t)+z}{\cup}\{(f_A(s) \cap g_B(t)) \cap h_C(z)\} \\ &\subseteq \underset{x=s+(t+z)}{\cup}\{f_A(s) \cap (g_B(t) \cap h_C(z))\} \\ &\subseteq \underset{x=s+p}{\cup}\{f_A(s) \cap (g_B \oplus h_C)(p)\} \\ &= \underset{x=s+p}{\cup}\{(f_A \oplus (g_B \oplus h_C))(x)\}.\end{aligned}$$

This implies

$$(f_A \oplus g_B) \oplus h_C \widetilde{\subseteq} f_A \oplus (g_B \oplus h_C).$$

Similarly, we can show that

$$f_A \oplus (g_B \oplus h_C) \widetilde{\subseteq} (f_A \oplus g_B) \oplus h_C.$$

Hence

$$(f_A \oplus g_B) \oplus h_C \widetilde{=} f_A \oplus (g_B \oplus h_C).$$

(ii) As $f_A \widetilde{\subseteq} g_B \Rightarrow f_A(y) \subseteq g_B(y)$ for all $y \in N$.
Let $x \in N$. Then

$$\begin{aligned}(f_A \oplus g_B)(x) &= \underset{x=y+z}{\cup}\{f_A(y) \cap h_C(z)\} \\ &\subseteq \underset{x=y+z}{\cup}\{g_B(y) \cap h_C(z)\} = (g_B \oplus h_C)(x).\end{aligned}$$

Hence,

$$f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \oplus h_C) \ \widetilde{\subseteq} (g_B \oplus h_C).$$

(iii) Let $x \in N$. Then

$$\begin{aligned}(f_A \oplus (g_B \widetilde{\cup} h_C)(x) &= \underset{x=y+z}{\cup}\{f_A(y) \cap \left(g_B \widetilde{\cup} h_C\right)(z) \\ &= \underset{x=y+z}{\cup}\{f_A(y) \cap (g_B(z) \cup h_C(z)\} \\ &= \underset{x=y+z}{\cup}\{(f_A(y) \cap g_B(z)) \cup (f_A(y) \cap h_C(z))\} \\ &= \{\underset{x=y+z}{\cup}(f_A(y) \cap g_B(z))\} \cup \{\underset{x=y+z}{\cup}(f_A(y) \cap h_C(z))\} \\ &= (f_A \oplus g_B)(x) \cup (f_A \oplus h_C)(x).\end{aligned}$$

This implies that

$$f_A \oplus \left(g_B \ \widetilde{\cup} \ h_C\right) \widetilde{=} (f_A \oplus g_B) \widetilde{\cup} (f_A \oplus h_C).$$

Similarly, we can show that

$$\left(f_A \ \widetilde{\cup} \ g_B\right) \oplus h_C \widetilde{=} (f_A \oplus h_C) \widetilde{\cup} (g_B \oplus h_C).$$

(iv) Let $x \in N$. Then

$$
\begin{aligned}
(f_A \oplus (g_B \widetilde{\cap} h_C)(x) &= \underset{x=yz}{\cup}\{f_A(y) \cap \left(g_B \widetilde{\cap} h_C\right)(z)\right) \\
&= \underset{x=yz}{\cup}\{f_A(y) \cap (g_B(z) \cap h_C(z)\} \\
&= \underset{x=yz}{\cup}\{(f_A(y) \cap g_B(z)) \cap (f_A(y) \cap h_C(z)\} \\
&\subseteq \{\underset{x=yz}{\cup}(f_A(y) \cap g_B(z))\} \cap \{\underset{x=yz}{\cup}(f_A(y) \cap h_C(z))\} \\
&= (f_A \oplus g_B)(x) \cap (f_A \oplus h_C)(x).
\end{aligned}
$$

This implies that .

$$f_A \oplus \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} (f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C).$$

Similarly, we can show that

$$\left(f_A \widetilde{\cap} g_B\right) \oplus h_C \widetilde{\subseteq} (f_A \oplus h_C) \widetilde{\cap} (g_B \oplus h_C).$$

□

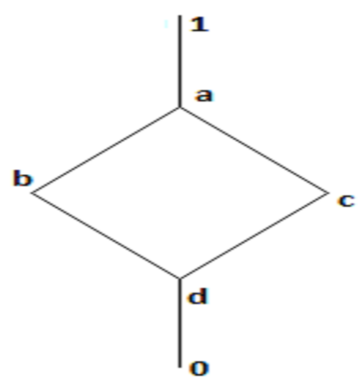
Next we show that equality does not hold in (iv).

Example 3.11. Let $N = \{0, x, y, z\}$ be the nearring with the binary operations as defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider a complete bounded distributive lattice $L = \{1, a, b, c, d, 0\}$, $U = \{p, q\}$ and $A = B = C = \{0, x, y, z\}$.



Define f_A, g_B and h_C the L-fuzzy soft sets of N over U as follows:

	p	q
$f_A(0)$	0	0
$f_A(x)$	a	d
$f_A(y)$	a	b
$f_A(z)$	c	a

	p	q
$g_B(0)$	0	0
$g_B(x)$	b	0
$g_B(y)$	1	b
$g_B(z)$	c	b

	p	q
$h_C(0)$	0	0
$h_C(x)$	0	1
$h_C(y)$	1	0
$h_C(z)$	0	d

Now

$$
\begin{aligned}
& (f_A \oplus (g_B \widetilde{\cap} h_C)(z) = \underset{z=x+y}{\cup}\{f_A(x) \cap (g_B \widetilde{\cap} h_C)(y) \\
= \ & \underset{z=x+y}{\cup}\{f_A(x) \cap ((g_B(y) \cap h_C(y))\} \\
= \ & \underset{z=x+y}{\cup}\{f_A(0) \cap ((g_B(z) \cap h_C(z)), f_A(x) \cap ((g_B(y) \cap h_C(y)), \\
& f_A(y) \cap ((g_B(x) \cap h_C(x)), f_A(z) \cap ((g_B(0) \cap h_C(0))\} \\
= \ & \cup \left\{ \begin{array}{c} (0,0) \cap ((c,b) \cap (0,d)), (a,d) \cap ((1,b) \cap (1,0)), \\ (a,b) \cap ((b,0) \cap (0,1)), (c,a) \cap ((0,0) \cap (0,0)) \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
= \ & \cup\{((0,0) \cap (0,d)), ((a,d) \cap (1,0)), ((a,b) \cap (0,0)), ((c,a) \cap (0,0))\} \\
= \ & (0,0) \cup (a,0) \cup (0,0) \cup (0,0) = (a,0).
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \{(f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C)\}(z) = (f_A \oplus g_B)(z) \cap (f_A \oplus h_C)(z) \\
= \ & \{\underset{z=x+y}{\cup}((f_A(y) \cap (g_B(z))\} \cap \{\underset{z=x+y}{\cup}(f_A(y) \cap (h_C(z))\} \\
= \ & \{\underset{z=x+y}{\cup}((f_A(0) \cap g_B(z)), (f_A(x) \cap g_B(y)), \\
& (f_A(y) \cap g_B(x)), (f_A(z) \cap g_B(0)))\} \\
& \cap\{\underset{z=x+y}{\cup}(\ (f_A(0) \cap h_C(z)), (f_A(x) \cap h_C(y)), \\
& (f_A(y) \cap h_C(x)), (f_A(z) \cap h_C(0)))\}
\end{aligned}
$$

$$
\begin{aligned}
= \ & \cup \left\{ \begin{array}{c} ((0,0) \cap (c,b)), ((a,d) \cap (1,b)), \\ ((a,b) \cap (b,0)), ((c,a) \cap (0,0)) \end{array} \right\} \\
& \cap \left\{ \cup \left\{ \begin{array}{c} ((0,0) \cap (0,d)), ((a,d) \cap (1,0)), \\ ((a,b) \cap (0,1)), ((c,a) \cap (0,0)) \end{array} \right\} \right\} \\
= \ & \{(0,0) \cup (a,d) \cup (b,0) \cup (0,0)\} \cap \{(0,0) \cup (a,0) \cup (0,b) \cup (0,0)\} \\
= \ & (a,d) \cap (a,b) = (a,d).
\end{aligned}
$$

Hence, $f_A \oplus (g_B \ \widetilde{\cap} \ h_C) \ \widetilde{\neq} \ (f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C)$.

Remark 3.12. (1) $(S(U), \oplus)$ is semigroup.
(2) $\oplus$ distributes over $\widetilde{\cup}$ but not over $\widetilde{\cap}$.
(3) $(S(U), \widetilde{\cup}, \oplus)$ is a semiring.

4. L-fuzzy Soft Sub-nearring and L-fuzzy soft Ideals

In this section we define L-fuzzy soft sub-nearring and L-fuzzy soft ideal of a nearring N over U and prove some related results.

Definition 4.1. An L-fuzzy soft subset f_A of a nearring N over U is called an L-fuzzy soft subnearring of N if
(1) $f_A(x-y) \supseteq f_A(x) \cap f_A(y)$
(2) $f_A(xy) \supseteq f_A(x) \cap f_A(y)$, for all $x, y \in N$, where $A \subseteq N$.

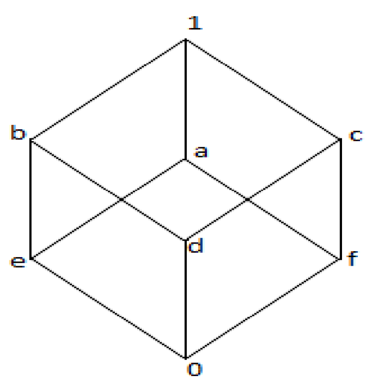
Definition 4.2. An L-fuzzy soft subset f_A of a nearring N over U is called an L-fuzzy soft ideal of N if f_A is an L-fuzzy soft subnearring of N and
(3) $f_A(x) = f_A(y+x-y)$
(4) $f_A(xy) \supseteq f_A(y)$
(5) $f_A((x+i)y - xy) \supseteq f_A(i)$, for any $x, y, i \in N$, where $A \subseteq N$.
f_A is an L- fuzzy soft left ideal of N if it satisfies $(1), (3)$ and (4); f_A is an L-fuzzy soft right ideal of N if it satisfies $(1), (2), (3)$ and (5).

Example 4.3. Let $N = \{0, x, y, z\}$ be the nearring with binary operations as defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ and $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A of N over U as follows:

	j	k
$f_A(0)$	1	a
$f_A(x)$	b	f
$f_A(y)$	c	0
$f_A(z)$	a	e

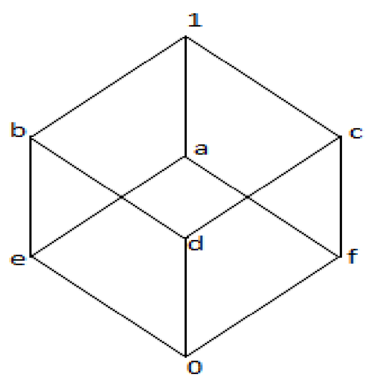
Simple calculations show that f_A is an L-fuzzy soft subnearring of N over U but $f_A(xz) \ngeq f_A(z)$ because $(b,f) \ngeq (a,e)$ and $f_A((x+z)x - xx) \nsupseteq f_A(z)$ because $(b,f) \ngeq (a,e)$. Thus it is neither an L-fuzzy soft left ideal nor an L-fuzzy soft right ideal of N and hence not an L-fuzzy soft ideal of N.

Example 4.4. Let $N = \{0, x, y, z\}$ be the nearring with binary operations as defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ and $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A of N over U as follows:

	j	k
$f_A(0)$	1	f
$f_A(x)$	a	0
$f_A(y)$	a	0
$f_A(z)$	1	f

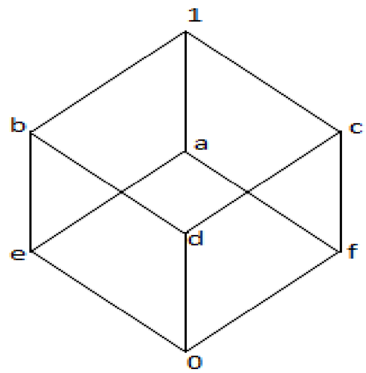
Simple calculations show that f_A is an L-fuzzy soft left ideal but $f_A((x+z)x - xx) \nsupseteq f_A(z)$ because $(a,0) \ngeq (1,f)$. Thus it is not an L-fuzzy soft right ideal and hence not an L-fuzzy soft ideal of N.

Example 4.5. Let $N = \{0, x, y, z\}$ be the nearring with binary operations as defined below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	x	0
z	z	y	0	x

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	0	x	x

Consider the Complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ and $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A of N over U as follows:

	j	k
$f_A(0)$	1	1
$f_A(x)$	a	b
$f_A(y)$	a	e
$f_A(z)$	a	e

Simple calculations show that f_A is an L-fuzzy soft ideal of N.

Lemma 4.6. *The intersection of two L-fuzzy soft ideals of a nearring N over U is again an L-fuzzy soft ideal of N over U.*

Proof. Let f_A and g_B be two L-fuzzy soft ideals of a nearring N over U and $x, y \in N$. Then

(1)

$$\begin{aligned}(f_A \tilde{\cap} g_B)(x-y) &= f_A(x-y) \cap g_B(x-y) \\ &\supseteq \{f_A(x) \cap f_A(y)\} \cap \{g_B(x) \cap g_B(y)\} \\ &= \{f_A(x) \cap g_B(x)\} \cap \{f_A(y) \cap g_B(y)\} \\ &= \{(f_A \tilde{\cap} g_B)(x)\} \cap \{(f_A \tilde{\cap} g_B)(y)\}.\end{aligned}$$

(2)

$$\begin{aligned}(f_A \tilde{\cap} g_B)(xy) &= f_A(xy) \cap g_B(xy) \\ &\supseteq \{f_A(x) \cap f_A(y)\} \cap \{g_B(x) \cap g_B(y)\} \\ &= \{f_A(x) \cap g_B(x)\} \cap \{f_A(y) \cap g_B(y)\} \\ &= \{(f_A \tilde{\cap} g_B)(x)\} \cap \{(f_A \tilde{\cap} g_B)(y)\}.\end{aligned}$$

(3)

$$\begin{aligned}(f_A \tilde{\cap} g_B)(y+x-y) &= f_A(y+x-y) \cap g_B(y+x-y) \\ &\supseteq f_A(x) \cap g_B(x) = (f_A \tilde{\cap} g_B)(x).\end{aligned}$$

(4)

$$(f_A \tilde{\cap} g_B)(xy) = f_A(xy) \cap g_B(xy) \supseteq f_A(x) \cap g_B(y) = (f_A \tilde{\cap} g_B)(y).$$

(5)

$$\begin{aligned}(f_A \widetilde{\cap} g_B)((x+i)y - xy) &= f_A((x+i)y - xy) \cap g_B((x+i)y - xy) \\ &\supseteq f_A(i) \cap g_B(i) = (f_A \widetilde{\cap} g_B)(i).\end{aligned}$$

for all $x, y, i \in N$.

Consequently, $(f_A \widetilde{\cap} g_B)$ is an L-fuzzy soft ideal of N. □

Generally the union of two L-fuzzy soft ideals of a nearring N is not necessarily an L-fuzzy soft ideal of N.

Lemma 4.7. *If an L-fuzzy soft set f_A of a nearring N over U satisfies the property $f_A(x-y) \supseteq f_A(x) \cap f_A(y)$ for all $x, y \in N$, then*
(i) $f_A(0_N) \supseteq f_A(x)$
(ii) $f_A(-x) = f_A(x)$ for all $x, y \in N$.

Proof. (i) For any $x \in N$,

$$f_A(0_N) = f_A(x-x) \supseteq f_A(x) \cap f_A(x) = f_A(x).$$

Hence, $f_A(0_N) \supseteq f_A(x)$.

(ii) For all $x \in N$,

$$f_A(-x) = f_A(0_N - x) \supseteq f_A(0_N) \cap f_A(x) = f_A(x).$$

Since x is arbitrary, we conclude that

$$f_A(-x) = f_A(x).$$

□

Proposition 4.8. *Let f_A and g_B be two L-fuzzy soft ideals of a zero symmetric nearring N. Then $f_A \oplus g_B$ is the smallest L-fuzzy soft ideal of N containing both f_A and g_B.*

Proof. For any $x, y \in N$,

$$\begin{aligned}(f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y) &= \left[\bigcup_{x=a+b}[f_A(a) \cap g_B(b)]\right] \cap \left[\bigcup_{y=c+d}[f_A(c) \cap g_B(d)]\right] \\ &= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap g_B(b)] \cap [f_A(c) \cap g_B(d)]] \\ &= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap f_A(c)] \cap [g_B(b) \cap g_B(d)]]\end{aligned}$$

$$\left(\begin{array}{c}\text{Since } x-y = a+b-(c+d) = a+b-d-c = a-c+(c+b-c)+(c-d-c) \\ \text{and } g_B(c+b-c) = g_B(b),\ g_B(c-d-c) = g_B(-d) = g_B(d) \text{ we have}\end{array}\right)$$

$$\begin{aligned}&= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap f_A(c)] \cap [g_B(c+b-c) \cap g_B(c-d-c)]] \\ &\subseteq \bigcup_{x-y=e+f}[f_A(e) \cap g_B(f) \widetilde{=} (f_A \oplus g_B)(x-y).\end{aligned}$$

Thus, $(f_A \oplus g_B)(x-y) \supseteq (f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y)$.

Now,

$$\begin{aligned}(f_A \oplus g_B)(y) &= \underset{y=a+b}{\cup}[f_A(a) \cap g_B(b)] \\ &\subseteq \underset{y=a+b}{\cup}[f_A(xa) \cap g_B(xb)] \\ &\subseteq \underset{xy=c+d}{\cup}[f_A(c) \cap g_B(d)] = (f_A \oplus g_B)(xy).\end{aligned}$$

Thus, $(f_A \oplus g_B)(xy) \supseteq (f_A \oplus g_B)(y)$.
Hence, $(f_A \oplus g_B)(xy) \supseteq (f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y)$.
Now,

$$\begin{aligned}(f_A \oplus g_B)(x) &= \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)] \\ &= \underset{x=a+b}{\cup}[f_A(y+a-y) \cap g_B(y+b-y)] \\ &\cong \underset{y+x-y=c+d}{\cup}[f_A(c) \cap g_B(d)]\end{aligned}$$

$$\left(\begin{array}{c}\text{Because for each } x=a+b \text{, we have } y+x-y=y+a-y+y+b-y \\ =(y+a-y)+(y+b-y) \text{ and for each } y+x-y=c+d, \\ \text{we have } x=-y+c+d+y=(-y+c+y)+(-y+d+y).\end{array}\right)$$

$$= (f_A \oplus g_B)(y+x-y).$$

Hence, $(f_A \oplus g_B)(x) = (f_A \oplus g_B)(y+x-y)$.

Let $i = a+b$. Then $i = a+b = b-b+a+b$ and $g_B(-b+a+b) = g_B(a)$. Hence whenever $f_A(a) \cap g_B(b)$ is present, then $f_A(b) \cap g_B(a)$ is also present. Now

$$(x+i)y - xy = (x+(a+b))y - xy = (x+(a+b))y - (x+a)y + (x+a)y - xy.$$

Thus $f_A(b) \subseteq f_A((x+a)y+b) - (x+a)y)$.
Now,

$$\begin{aligned}(f_A \oplus g_B)(i) &= \underset{i=a+b}{\cup}[f_A(a) \cap g_B(b)] = \underset{i=a+b}{\cup}[f_A(b) \cap g_B(a)] \\ &\subseteq \underset{i=a+b}{\cup}[f_A((x+(a+b))y-(x+a)y) \cap g_B((x+a)y-xy)] \\ &\subseteq \underset{(x+i)y-xy=c+d}{\cup}[f_A(c) \cap g_B(d)] \\ &= (f_A \oplus g_B)((x+i)y-xy).\end{aligned}$$

Hence, $f_A \oplus g_B$ is an L-fuzzy soft ideal of N.
Now, $(f_A \oplus g_B)(x) = \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)]$

As $x = x+0$ and $x = 0+x$, so $(f_A \oplus g_B)(x) \supseteq f_A(x)$ and also $(f_A \oplus g_B)(x) \supseteq g_B(x)$. If h_C is an L-fuzzy soft ideal of N such that $h_C(x) \supseteq g_B(x)$ and $h_C(x) \supseteq f_A(x)$ for all $x \in N$, then

$$\begin{aligned}(f_A \oplus g_B)(x) &= \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)] \\ &\subseteq \underset{x=a+b}{\cup}[h_C(a) \cap h_C(b)] \\ &= \underset{x=a+b}{\cup}[h_C(a) \cap h_C(-b)] \\ &\subseteq \underset{x=a+b}{\cup} h_C(a+b) = h_C(x).\end{aligned}$$

Thus, $f_A \oplus g_B \widetilde{\subseteq} h_C$. □

Recall that a left nearrin N is called a zero-symmetric nearring if $0a = 0$ for all $a \in N$.

Proposition 4.9. *Let N be a zero-symmetric nearring and f_A and g_B be L-fuzzy soft ideals of N over U. Then $f_A \odot g_B \widetilde{\subseteq} f_A \cap g_B$.*

Proof. Let f_A and g_B be L-fuzzy soft ideals of N and $x \in N$. Then

$$(f_A \odot g_B)(x) = \begin{cases} \underset{x=yz}{\cup} f_A(y) \cap g_B(z) & \text{if } \exists\, y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases}.$$

As f_A is an L-fuzzy soft ideal, so $f_A(z) \subseteq f_A(yz) = f_A(x)$. As N is a zerosymmetric nearring, so $yz = (0+y)z - 0z$. Hence,

$$g_B(x) = g_B(yz) = g_B((0+y)z - 0z) \supseteq g_B(y).$$

Thus,

$$\begin{aligned}(f_A \odot g_B)(x) &= \begin{cases} \underset{x=yz}{\cup} f_A(yz) \cap g_B(yz) & \text{if } \exists\, y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases} \\ &\subseteq (f_A \cap g_B)(x).\end{aligned}$$

□

Let N be a nearring. Let $F(N)$ denote the set of all L-fuzzy soft subsets of N over U. Let $F^{\star}(N)$ be the set of all L-fuzzy soft ideals of N. Let $f_A \in F(N)$. Then the L-fuzzy soft ideal generated by f_A, denoted by $< f_A >$, is the intersection of all L-fuzzy soft ideals of N which contain f_A. Now, onwards N will denote a zerosymmetric left nearring.

Definition 4.10. A nearring N is called fully L-fuzzy soft idempotent if for each L-fuzzy soft ideal f_A of N, $f_A \widetilde{=} < f_A^2 >$.

Proposition 4.11. *The following assertions for a zero-symmetric nearring N are equivalent*:

(*i*) *N is fully L-fuzzy soft idempotent.*

(*ii*) *For each pair of L-fuzzy soft ideals f_A, g_B of N, $f_A \cap g_B \widetilde{=} < f_A \odot g_B >$.*

(*iii*) *The set of L-fuzzy soft ideals of N form a lattice $(F^{\star}(N), \cup, \cap)$ with $f_A \cup g_B \widetilde{=} f_A \oplus g_B$ and $f_A \cap g_B \widetilde{=} < f_A \odot g_B >$ for each pair of L-fuzzy soft ideals f_A, g_B of N.*

Proof. $(i) \Rightarrow (ii)$ By Proposition 4.10, for each pair of L-fuzzy soft ideals f_A, g_B of N

$$f_A \odot g_B \widetilde{\subseteq} f_A \cap g_B,$$

thus

$$< f_A \odot g_B > \widetilde{\subseteq} f_A \cap g_B.$$

For reverse inclusion, as $f_A \cap g_B$ is an L-fuzzy soft ideal and

$$f_A \cap g_B \widetilde{\subseteq} f_A$$

and

$$f_A \cap g_B \widetilde{\subseteq} g_B, (f_A \cap g_B)^2 \widetilde{\subseteq} f_A \odot g_B,$$

we have

$$f_A \cap g_B \widetilde{=} < (f_A \cap g_B)^2 > \widetilde{\subseteq} < f_A \odot g_B > .$$

Thus,

$$f_A \cap g_B \widetilde{=} < f_A \odot g_B > .$$

$(ii) \Rightarrow (iii)$ The set of all L-fuzzy soft ideals of a nearring N ordered by inclusion forms a lattice under the sum and intersection of L-fuzzy soft ideals. Thus for each pair of L-fuzzy soft ideals f_A, g_B of N,

$$f_A \cup g_B \widetilde{=} f_A \oplus g_B$$

and

$$f_A \cap g_B \widetilde{=} < f_A \odot g_B > .$$

$(iii) \Rightarrow (i)$ By assumption

$$f_A \cap g_B \widetilde{=} < f_A \odot g_B > .$$

Now taking $f_A \widetilde{=} g_B$, we have

$$f_A \widetilde{=} < f_A \odot f_A > \widetilde{=} < (f_A)^2 > .$$

Hence, N is fully L-fuzzy soft idempotent. □

Theorem 4.12. *The set of all L-fuzzy soft ideals of a zero-symmetric fully L-fuzzy soft idempotent nearring N (ordered by inclusion) forms a distributive lattice under the sum and intersection of ideals.*

Proof. Follows from Proposition 4.11. □

5. Conclusion

In this paper, We have proceeded L-fuzzy soft sets of nearrings over the common universe U. It is tried to show that the some of the properties and results of Fuzzy soft sets of nearrings are also valid in L-fuzzy soft sets of nearrings over U. Moreover, these properties and results in L-fuzzy soft sets of nearrings over U hold due to existence of the corresponding properties in Lattices. We are hoping to introduce some results and problems of L-fuzzy soft sets of different algebraic structures over U in near future.

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