Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 1, (July 2015), pp. 139–160
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

e-Open sets and separation axioms in fuzzifying topology

A. Vadivel, M. Palanisamy

Received 13 November 2014; Revised 3 January 2015; Accepted 9 February 2015

Abstract. The concepts of fuzzy e-open sets and fuzzy e-continuous are introduced and studied in fuzzifying topology and by making use of these concepts, we introduce and study T_0^e-, R_0^e-, T_1^e-, R_1^e-, T_2^e-, T_3^e-, T_4^e-, strong T_3^e- and strong T_4^e- separation axioms in fuzzifying topology and give some of their characterizations as well as the relations of these axioms and other separation axioms in fuzzifying topology introduced by Shen, Fuzzy Sets and Systems, 57 (1993), 111–123.

2010 AMS Classification: 54A40, 54C08

Keywords: Fuzzy logic, Fuzzifying topology, Fuzzyifying e-open sets, Fuzzifying separation axioms.

Corresponding Author: A. Vadivel (avmaths@gmail.com)

1. Introduction

Chang [1], Hutton [3], Lowen [6], Pu and Liu [8], Wong [12]and others have discussed various aspects of fuzzy topology with crisp methods. Ying [13, 14] and Sayed and Zhao [9] introduced fuzzifying topology and elementarily developed fuzzy topology from a new direction with the semantic method of continuous valued logic. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In the framework of fuzzifying topology, Shen [11] introduced and studied T_0-, T_1-, T_2(Hausdorff)-,T_3 (regular)- and T_4(normal)-separation axioms in fuzzifying topology. In [4], the concepts of the R_0- and R_1- separation axioms in fuzzifying topology were added and their relations with the T_1- and T_2- separation axioms, were studied, respectively. In [13], the authers introduced and studied the concepts of fuzzifying negibourhood structure of a point, fuzzifying interior and fuzzifying closure . Erdel Ekici [2] introduced the concepts of e-open sets and e-continuity in general topology. We note the concepts of e-open sets and e-continuity are considered by Seenivasan [10] to fuzzy topology. In the present paper, we define and study the concepts of

e-open sets and e-neighborhood in fuzzifying topology. The main purpose of the present paper, we introduce and study, T_0^e-, R_0^e-, T_1^e-,R_1^e-,T_2^e-(e-Hausdorff)-, T_3^e(e-regularity)-, T_4^e(e-normality)-, strong T_3^e-, strong T_4^e-separation axioms in fuzzifying topology. Also, we give some of their characterizations as well as the relations of these axioms and T_0-, R_0-,T_1-, R_1-, T_2(Hausdorff)-, T_3(regularity)-, T_4(normality)-separation axioms in fuzzifying topology.

2. Preliminaries

First, we display the logical and corresponding set theoretical notions [13, 14] since we need them in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) $[\alpha] = \alpha(\alpha \in [0,\ 1])$; $[\varphi \wedge \psi] :=\min([\varphi], [\psi])$; $[\varphi \to \psi] :=\min(1, 1 - [\varphi] + [\psi])$;
(2) If $\widetilde{A} \in \Im(X)$, where $\Im(X)$ is the family of all fuzzy sets of X, then $[x \in \widetilde{A}] := \widetilde{A}(x)$;
(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$. In addition the following derived formulae are given,
(1) $[\neg\varphi] := [\varphi \to 0] := 1 - [\varphi]$;
(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] := \max([\varphi], [\psi])$;
(3) $[\varphi \leftrightarrow \psi] := [\varphi \to \psi] \wedge [\psi \to \varphi]$;
(4) $[\varphi * \psi] := [\neg(\varphi \to \neg\psi)] := \max(0, [\varphi] + [\psi] - 1)$ $\Big($$*$ is the Lukasiewicz triangular-norm (or $\wedge$)$\Big)$;
(5) $[\varphi \dot{\vee} \psi] := [\neg(\neg\varphi * \neg\psi)] := [\neg\varphi \to \psi)] :=\min(1, [\varphi] + [\psi])$;
(6) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x \in X} [\varphi(x)]$;
(7) If $\widetilde{A}$, $\widetilde{B} \in \Im(X)$, then
$$[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] := \inf_{x \in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x)).$$

Second, we give some definitions and results in fuzzifying topology.

Definition 2.1 ([13])**.** Let X be a universe of discourse, and let $\tau \in \Im(P(X))$, where $P(X)$ is the power set of X satifying the following conditions:

(1) $\models X \in \tau$;
(2) for any $A, B \in P(X)$, $\models (A \in \tau) \wedge (B \in \tau) \to (A \cap B) \in \tau$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X)$, $\models \forall\lambda(\lambda \in \Lambda \to A_\lambda \in \tau) \to \bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.

Then τ is called a fuzzifying topology and $(X,\ \tau)$ is a fuzzifying topological space.

The family of all fuzzifying closed sets will be denoted by F_τ or if there is no confusion by F, and defined as follows: $A \in F := (X - A) \in \tau$, where $X - A$ is the complement of A.

Definition 2.2 ([15])**.** The family of fuzzy regular open sets in fuzzifying topological space (X, τ) is denoted by RO and defined as follows:
$$A \in RO := A \equiv A^{-\circ},$$

ie, $[A \in RO] = [RO(A)] = \min\big(\inf_{x\in A} A^{-\circ}(x), \inf_{x\in X-A}(1 - A^{-\circ}(x))\big)$.

Definition 2.3 ([13])**.** Let $(X,\ \tau)$ be a fuzzifying topological space. The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows:

$$N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$$

Definition 2.4 ([16])**.** Let (X, τ) be a fuzzifying topological space and let $x \in X$. The δ-neighborhood system of x is denoted by $N^\delta_x(A) \in \Im(P(X))$ and defined as follows:

$$N^\delta_x(A) = \sup_{x\in B\subseteq A} RO(B), \forall A \in P(X).$$

Definition 2.5 ([13])**.** Let $(X,\ \tau)$ be a fuzzifying topological space.

(1) The interior (resp. δ-interior) of a set $A \in P(X)$ is denoted by $A^\circ \in \Im(X)$(resp. $A^{\circ_\delta} \in \Im(X)$) and defined as follows:
$$A^\circ(x) = N_x(A) \text{ (resp. } A^{\circ_\delta}(x) = N^\delta_x(A)).$$
(2) The closure (resp. δ-closure) of a set $A \in P(X)$ is denoted by $\bar{A} \in \Im(X)$ (resp. $A^{-_\delta} \in \Im(X)$) and defined as follows:
$$\bar{A}(x) = 1 - N_x(X - A) \text{ (resp. } A^{-_\delta}(x) = 1 - N^\delta_x(X - A)).$$
(3) $\beta \in \Im(P(X))$ is a base of τ iff $\tau = \beta^{(\cup)}$ (Theorem 4.1 [13]), i.e.,
$$\tau(A) = \sup_{\bigcup_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \beta(B_\lambda).$$
(4) $\varphi \in \Im(P(X))$ is a subbase of τ if $\varphi^\cap$ is a base of τ, i.e.,
$$\tau(A) = \sup_{\bigcup_{\lambda\in\Lambda} D_\lambda = A} \inf_{\lambda\in\Lambda} \sup_{\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = D_\lambda} \inf_{\lambda_i\in I_\lambda} \varphi(D_{\lambda_i}).$$

Lemma 2.6 ([5])**.** *If* $[\widetilde{A} \subseteq \widetilde{B}] = 1$, *then*

(1) $\models \overline{\widetilde{A}} \subseteq \overline{\widetilde{B}}$;
(2) $\models (\widetilde{A})^\circ \subseteq (\widetilde{B})^\circ$.

Lemma 2.7 ([5])**.** *Let* (X, τ) *be a fuzzifying topological space. For any* A, B;

(1) $\models X^\circ = X$;
(2) $\models (\widetilde{A})^\circ \subseteq \widetilde{A}$;
(3) $\models (\widetilde{A} \cap \widetilde{B})^\circ \equiv (\widetilde{A})^\circ \cap (\widetilde{B})^\circ$;
(4) $\models (\widetilde{A})^{\circ\circ} \supseteq (\widetilde{A})^\circ$.

Lemma 2.8 ([5])**.** *Let* (X, τ) *be a fuzzifying topological space. For any* $A \in \Im(X)$;

(1) $\models X - (\widetilde{A})^{\circ-} \equiv (X - \widetilde{A})^{-\circ}$;
(2) $\models X - (\widetilde{A})^{-\circ} \equiv (X - \widetilde{A})^{\circ-}$.

Lemma 2.9 ([5])**.** *If* $[\widetilde{A} \subseteq \widetilde{B}] = 1$, *then*

(1) $\models (\widetilde{A})^{\circ-} \subseteq (\widetilde{B})^{\circ-}$;
(2) $\models (\widetilde{A})^{-\circ} \subseteq (\widetilde{B})^{-\circ}$.

Remark 2.10. For simiplicity we use the following notations:

(1) $K(x,y) := \exists A\big((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A)\big)$;
(2) $H(x,y) := \exists B \exists C\big((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C)\big)$;
(3) $M(x,y) := \exists B \exists C(B \in N_x \wedge C \in N_y \wedge B \cap C = \phi)$;
(4) $V(x,D) := \exists A \exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B = \phi)$;
(5) $W(A,B) := \exists G \exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \phi)$.

Definition 2.11 ([11])**.** Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i \in \Im(\Omega)$, $i = 1, ..., 4$, and $R_i \in \Im(X)$, $i = 0, 1$ are defined as follows, respectively

(1) $(X,\tau) \in T_0 := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to K(x,\ y)\big)$;
(2) $(X,\tau) \in T_1 := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to H(x,\ y)\big)$;
(3) $(X,\tau) \in T_2 := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to M(x,\ y)\big)$;
(4) $(X,\tau) \in T_3 := \forall x \forall D\big((x \in X \wedge D \in F \wedge x \notin D) \to V(x,\ D)\big)$;
(5) $(X,\tau) \in T_4 := \forall A \forall B\big((A \in F \wedge B \in F \wedge A \cap B = \phi) \to W(A,\ B)\big)$;
(6) $(X,\tau) \in R_0 := \forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \to \big(K(x,y) \to H(x,\ y)\big)\Big)$;
(7) $(X,\tau) \in R_1 := \forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \to \big(K(x,y) \to M(x,\ y)\big)\Big)$.

Theorem 2.12 ([7])**.** *The mapping $N^\gamma : X \to \Im^N(P(X))$, $x \mapsto N_x^\gamma$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:*

(1) $\models A \in N_x^\gamma \to x \in A$;
(2) $\models A \subseteq B \to (A \in N_x^\gamma \to B \in N_x^\gamma)$;
(3) $\models A \in N_x^\gamma \to \exists H\big(H \in N_x^\gamma \wedge H \subseteq A \wedge \forall y(y \in H \to H \to H \in N_x^\gamma)\big)$.

Theorem 2.13 ([7])**.** $\tau_\gamma(A) = \inf_{x \in A} N_x^\gamma(A)$.

3. Fuzzifying e-open sets

Definition 3.1. Let (X,τ) be a fuzzifying topological space.

(1) The family of fuzzifying e-open sets, denoted by $\tau_e \in \Im(P(X))$, is defined as follows: $A \in \tau_e := \forall x(x \in A \to x \in A^{-\circ\delta} \cup A^{\circ-\delta})$, i.e., $\tau_e(A) = \inf_{x \in A} \max\big(A^{-\circ\delta}(x), A^{\circ-\delta}(x)\big)$.
(2) The family of fuzzifying e-closed sets, denoted by $F_e \in \Im(P(X))$, is defined as follows:

$$A \in F_e := (X - A) \in \tau_e.$$

Example 3.2. Let $X = \{a, b, c\}$ and let τ be a fuzzifying topology on X defined as follows: $\tau(X) = \tau(\phi) = \tau(\{a\}) = \tau(\{a, c\}) = 1$; $\tau(\{b\}) = \tau(\{a, b\}) = 0$; and $\tau(\{c\}) = \tau(\{b, c\}) = \frac{1}{8}$. From the definition of the interior and the closure of a subset of X and the interior and the closure of a fuzzy set of X we have the following: $\tau_e(X) = \tau_e(\phi) = \tau_e(\{a\}) = \tau_e(\{a, c\}) = 1$; and $\tau_e(\{b\}) = \tau_e(\{a, b\}) = \tau_e(\{c\}) = \tau_e(\{b, c\}) = \frac{7}{8}$.

Lemma 3.3. *For any α, β, γ, $\delta \in I$, $(1-\alpha+\beta) \wedge (1-\gamma+\delta) \leq 1-(\alpha \wedge \gamma)+(\beta \wedge \delta)$.*

Theorem 3.4. *Let (X,τ) be a fuzzifying topological space, then*

(1) $\models \tau_e(X) = 1$, $\tau_e(\phi) = 1$;

(2) $\models \tau_e(A \cap B) \geq \tau_e(A) \wedge \tau_e(B)$.
(3) $\models \tau_e(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \tau_e(A_\lambda)$

Proof. The proof of (1) is straightforward.

(2) From Lemma 3.3, we have

$\tau_e(A) \wedge \tau_e(B)$
$= \inf_{x\in A} \min\big(1, 1 - A(x) + (A^{\circ-\delta} \cup A^{-\circ\delta})(x)\big) \wedge \inf_{x\in B} \min\big(1, 1 - B(x) + (B^{\circ-\delta} \cup B^{-\circ\delta})(x)\big)$
$= \inf_{x\in A\cap B} \min\big((1, 1 - A(x) + (A^{\circ-\delta} \cup A^{-\circ\delta})(x)\big) \wedge \big(1, 1 - B(x) + (B^{\circ-\delta} \cup B^{-\circ\delta})(x)\big)\big)$
$= \inf_{x\in A\cap B} \min\big(1, (1-A(x)) + (A^{\circ-\delta} \cup A^{-\circ\delta})(x) \wedge (1-B(x)) + (B^{\circ-\delta} \cup B^{-\circ\delta})(x)\big)$
$\leq \inf_{x\in A\cap B} \min\Big(\big(1, \big(1 - (A\cap B)(x)\big) + (A^{\circ-\delta} \cap B^{\circ-\delta})(x) \cup (A^{-\circ\delta} \cap B^{-\circ\delta})(x)\big)\Big)$
$\leq \inf_{x\in A\cap B} \min\Big(1, \big(1 - (A\cap B)(x)\big) + \big((A\cap B)^{\circ-\delta}(x) \cup (A\cap B)^{-\circ\delta}(x)\big)\Big)$
$= \tau_e(A \cap B)$.

(3) Proof follows from (2).

□

Theorem 3.5. *Let (X, τ) be a fuzzifying topological space, then*
(1) $\models F_e(X) = 1$, $F_e(\phi) = 1$;
(2) $\models F_e(A \cap B) \geq F_e(A) \wedge F_e(B)$.
(3) $\models F_e(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} F_e(A_\lambda)$

Proof. Follows from Theorem 3.4 □

Theorem 3.6. *Let $(X,\ \tau)$ be a fuzzifying topological space. Then, we have*
(1) $\models \tau \subset \tau_e$
(2) $\models F \subseteq F_e$

Proof. (1) $[A \in \tau] = [A \subseteq A^\circ] \leq [A \subseteq (A^{\circ-\delta} \cup A^{\circ-\delta})] = [A \in \tau_e]$.
(2) The proof is obtained from (1). □

4. Fuzzifying e-neighbouihood structure

Definition 4.1. Let $x \in X$. The fuzzifying e-neighborhood system of x, denoted by $N_x^e \in \Im(P(X))$, is defined as follows:

$$A \in N_x^e := \exists B(x \in B \subseteq A \to B \in \tau_e).$$
$$\Big(i.e.,\quad N_x^e(A) = \sup_{x\in B\subseteq A} \tau_e(B)\Big)$$

Theorem 4.2. *The mapping $N^e : X \to \Im^N(P(X))$, $x \mapsto N_x^e$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:*
(1) $\models A \in N_x^e \to x \in A$;
(2) $\models A \subseteq B \to (A \in N_x^e \to B \in N_x^e)$;

(3) $\models A \in N_x^e \wedge B \in N_x^e \to A \cap B \in N_x^e$. *Conversely, if a mapping* N_x^e *satisfies (2) and (3), then* N_x^e *assigns a fuzzifying topology on* X *which is denoted by* $\tau_e \in \Im(P(X))$ *and defined as*

$$A \in \tau_e := \forall x (x \in A \to A \in N_x^e).$$

$$\Big(i.e., \quad \tau_e(A) = \inf_{x \in A} N_x^e(A) \Big)$$

Proof. (1) If $[A \in N_x^e] = \sup_{x \in H \subseteq A} \tau_e(H) > 0$, then there exists H_0 such that $x \in H_0 \subseteq A$. Now, we have $[x \in A] = 1$. Therefore, $[A \in N_x^e] \leq [x \in A]$ always holds.
(2) The proof is immediate.
(3) From Theorem 3.4(2), we have

$$\begin{aligned} [A \cap B \in N_x^e] &= \sup_{x \in H \subseteq A \cap B} \tau_e(H) = \sup_{\substack{x \in H_1 \subseteq A, \\ x \in H_2 \subseteq B}} \tau_e(H_1 \cap H_2) \\ &\geq \sup_{\substack{x \in H_1 \subseteq A, \\ x \in H_2 \subseteq B}} \tau_e(H_1) \wedge \tau_e(H_2) \\ &= \sup_{x \in H_1 \subseteq A} \tau_e(H_1) \wedge \sup_{x \in H_2 \subseteq B} \tau_e(H_2) \\ &= [A \in N_x^e \wedge B \in N_x^e]. \end{aligned}$$

Conversely, we need to prove that $\tau_e(A) = \inf_{x \in A} N_x(A)$ is a fuzzifying topology. From [[13], Theorem 3.2] and since τ_e satisfies properties (2) and (3), τ_e is a fuzzifying topology. □

5. Fuzzifying e-derived sets, fuzzifying e-closure, and fuzzifying e-interior

Definition 5.1. Let (X, τ) be a fuzzifying topological space. The fuzzifying e-derived set of A, denoted by $d_e \in \Im(P(X))$, is defined as

$$d_e(A) = \inf_{B \cap (A - \{x\}) = \phi} (1 - N_x^e(B)).$$

Lemma 5.2. $d_e(A)(x) = 1 - N_x^e((X - A) \cup \{x\})$.

Proof. From Theorem 4.2(2), we have

$$\begin{aligned} d_e(A) &= 1 - \sup_{B \cap (A - \{x\}) = \phi} N_x^e(B) \\ &= 1 - \sup_{B \subseteq ((X - A) \cup \{x\})} N_x^e(B) \\ &= 1 - N_x^e((X - A) \cup \{x\}). \end{aligned}$$ □

Theorem 5.3. *For any* A, $\models A \in F_e \leftrightarrow d_e(A) \subseteq A$.

Proof. From Lemma 5.2, we have

$$\begin{aligned} [d_e(A) \subseteq A] &= \inf_{x \in X - A} (1 - d_e(A)(x)) = \inf_{x \in X - A} N_x^e((X - A) \cup \{x\}) \\ &= \inf_{x \in X - A} N_x^e(X - A) = [X - A \in \tau_e] = [A \in F_e]. \end{aligned}$$ □

Definition 5.4. Let (X, τ) be a fuzzifying topological space. The e-closure of A is denoted and defined as follows:

$$Cl_e(A)(x) = 1 - N_x^e(X - A) \Big(i.e., Cl_e(A)(x) = \inf_{x \notin B \supseteq A}(1 - F_e(B))\Big).$$

Theorem 5.5. (1) $Cl_e(A)(x) = 1 - N_x^e(X - A)$;
(2) $\models Cl_e(\phi) \equiv \phi$;
(3) $\models A \subseteq Cl_e(A)$.

Proof. (1) $Cl_e(A)(x) = \inf_{x \notin B \supseteq A}(1 - F_e(B)) = \inf_{x \in X - B \subseteq X - A}(1 - \tau_e(X - B)) = 1 - \sup_{x \in X - B \subseteq X - A} \tau_e(X - B) = 1 - N_x^e(X - A)$.

(2) $Cl_e(\phi)(x) = 1 - N_x^e(X - \phi) = 0$.

(3) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $N_x^e(A) = 0$. If $x \in A$, then $Cl_e(A)(x) = 1 - N_x^e(X - A) = 1 - 0 = 1$. Then $[A \subseteq Cl_e(A)] = 1$. □

Theorem 5.6. *For any x and A;*
(1) $\models Cl_e(A) \equiv d_e(A) \cup A$;
(2) $\models x \in Cl_e(A) \leftrightarrow \forall B(B \in N_x^e \to A \cap B \neq \varphi)$;
(3) $\models A \equiv Cl_e(A) \leftrightarrow A \in F_e$.

Proof. (1) Applying Lemma 5.2 and Theorem 5.5 (3), we have
$x \in d_e(A) \cup A = \max(1 - N_x^e((X - A) \cup \{x\})), A(x) = Cl_e(A)(x)$.

(2) $[\forall B(B \in N_x^e \to A \cap B \neq \phi)] = \inf_{B \subseteq X - A}(1 - N_x^e(B)) = 1 - N_x^e(X - A)$
$= [x \in Cl_e(A)]$.

(3) From Theorem 5.5(1), we have

$$[A \equiv Cl_e(A)] = \inf_{x \in X - A}(1 - Cl_e(A)(x))$$
$$= \inf_{x \in X - A} N_x^e(X - A) = [(X - A) \in F_e] = [A \in \tau_e].$$ □

Theorem 5.7. *For any A and B, $\models B \equiv Cl_e(A) \to B \in F_e$.*

Proof. If $[A \subseteq B] = 0$, then $[B \equiv Cl_e(A)] = 0$. Now, we suppose $[A \subseteq B] = 1$, then we have $[B \subseteq Cl_e(A)] = 1 - \sup_{x \in B - A} N_x^e(X - A)$ and $[Cl_e(A) \subseteq B] = \inf_{x \in X - B} N_x^e(X - A)$. So,

$$[B \equiv Cl_e(A)] = \max\Big(0, \inf_{x \in X - B} N_x^e(X - A) - \sup_{x \in X - B} N_x^e(X - A)\Big).$$

If $[B \equiv Cl_e(A)] > t$, then $\inf_{x \in X - B} N_x^e(X - A) > t + \sup_{x \in B - A} N_x^e(X - A)$. For any $x \in X - B$, $\sup_{x \in C \subseteq X - A} \tau_e(C) > t + \sup_{x \in B - A} N_x^e(X - A)$, that is, there exists C_x such that $x \in C_x \subseteq X - A$ and $\tau_e(C_x) > t + \sup_{x \in B - A} N_x^e(X - A)$. Now, we want to prove that $C_x \subseteq X - B$. If not, then there exists $x' \in B - A$ such that $x' \in C_x$. Hence, we can obtain that $\sup_{x \in B - A} N_x^e(X - A) \geq N_{x'}^e(X - A) \geq \tau_e(C_x) > t + \sup_{x \in B - A} N_x^e(X - A)$. This is a contradiction. Therefore, $F_e(B) = \tau_e(X - B) = \inf_{x \in X - B} N_x^e(X - B) \geq \inf_{x \in X - B} \tau_e(C_x) > t + \sup_{x \in B - A} N_x^e(X - A) > t$. Since t is arbitrary, it holds that $[B \equiv Cl_e(A)] \leq [B \in F_e]$. □

Definition 5.8. Let (X, τ) be a fuzzifying topological space. For any $A \subseteq X$, the e-interior of A is given as follows:

$$Int_e(A)(x) = N_x^e(A).$$

Theorem 5.9. *For any x, A and B,*

(1) $\models B \in \tau_e \wedge B \subseteq A \to B \subseteq Int_e(A)$;
(2) $\models A \equiv Int_e(A) \leftrightarrow A \in \tau_e$;
(3) $\models x \in Int_e(A) \leftrightarrow x \in A \wedge x \in (X - d_e(X - A))$;
(4) $\models Int_e(A) \equiv X - Cl_e(X - A)$;
(5) $\models B \equiv Int_e(A) \to B \in \tau_e$;
(6) *(a)* $\models Int_e(A) \equiv X$, *(b)* $\models Int_e(A) \subseteq A$.

Proof. (1) If $B \nsubseteq A$, then $[B \in \tau_e \wedge B \subseteq A] = 0$. If $B \subseteq A$, then

$$\begin{aligned}[B \subseteq Int_e(A)] &= \inf_{x \in B} Int_e(A)(x)\\ &= \inf_{x \in B} N_x^e(A) \geq \inf_{x \in B} N_x^e(B)\\ &= [B \in \tau_e] = [B \in \tau_e \wedge B \subseteq A].\end{aligned}$$

(2) $$\begin{aligned}[A \equiv Int_e(A)] &= \min\Big(\inf_{x \in A} Int_e(A)(x), \inf_{x \in X-A}(1 - Int_e(A)(x))\Big)\\ &= \inf_{x \in A} Int_e(A)(x) = \inf_{x \in A} N_x^e(A) = [A \in \tau_e].\end{aligned}$$

(3) If $x \notin A$, then $[x \in Int_e(A)] = 0 = [x \in A \wedge x \in (X - d_e(X - A))]$. If $x \in A$, then $[x \in d_e(X - A)] = 1 - N_x^e(A \cup \{x\}) = 1 - N_x^e(A) = 1 - Int_e(A)(x)$, so that $[x \in A \wedge x \in (X - d_e(X - A)] = [x \in Int_e(A)]$.

(4) It follows from Theorem 5.5(1)

(5) From (4) and Theorem 5.7, we have

$$[B \equiv Int_e(A)] = [X - B \equiv Cl_e(X - A)] \leq [X - B \in F_e] = [B \in \tau_e].$$

(6) (a) It is obtained from (4) above and from Theorem 5.5(2).
(b) It is obtained from (3) above. □

6. Fuzzifying e-continuous functions

Definition 6.1. Let (X, τ) and (Y, U) be two fuzzifying topological spaces. For any $f \in Y^X$, a unary fuzzy predicates $C_e \in \Im(Y^X)$, called e-continuity, is given as

$$C_e(f) := \forall u(u \in U \to f^{-1}(u) \in \tau_e).$$

Definition 6.2. Let (X, τ) and (Y, U) be two fuzzifying topological spaces. For any $f \in Y^X$, we define the unary fuzzy predicates $e_j \in \Im(Y^X)$ where $j = 1, 2, \ldots, 5$ as follows:

(1) $e_1(f) := \forall B(B \in F^Y \to f^{-1}(B) \in F_e^X)$, where F^Y is the family of closed subsets of Y and F_e^X is the family of e-closed subsets of X;
(2) $e_2(f) := \forall x \forall u(u \in N_{f(x)} \to f^{-1}(u) \in N_x^e)$, where N is the neighborhood system of Y and N_x^e is the e-neighborhood system of X;
(3) $e_3(f) := \forall x \forall u(u \in N_{f(x)} \to \exists v(f(v) \subseteq u \to v \in N_x^e))$;
(4) $e_4(f) := \forall A(f(Cl_e^X(A)) \subseteq Cl^Y(f(A)))$;
(5) $e_5(f) := \forall B(Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B)))$.

Theorem 6.3. (1) $\models f \in C_e \leftrightarrow f \in e_1$;
(2) $\models f \in C_e \to f \in e_2$;
(3) $\models f \in e_2 \leftrightarrow f \in e_j$ *for* $j = 3, 4, 5$.

Proof. (1) We prove that $[f \in C_e] = [f \in e_1]$

$$\begin{aligned}[f \in e_1] &= \inf_{A \in P(Y)} \min(1, 1 - F^Y(A) + F_e^X(f^{-1}(A)) \\ &= \inf_{A \in P(Y)} \min(1, 1 - U(Y - A) + \tau_e(X - f^{-1}(A))) \\ &= \inf_{A \in P(Y)} \min(1, 1 - U(Y - A) + \tau_e(f^{-1}(Y - A))) \\ &= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \tau_e(f^{-1}(u))) \\ &= [f \in C_e].\end{aligned}$$

(2) We prove that $e_2(f) \geq C_e(f)$. If $N_{f(x)}(u) \leq N_x^e(f^{-1}(u))$, the result holds. Suppose $N_{f(x)}(u) > N_x^e(f^{-1}(u))$. It is clear that if $f(x) \in A \subseteq u$ then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$$\begin{aligned}N_{f(x)}(u) - N_x^e(f^{-1}(u)) &= \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} \tau_e(B) \\ &\leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} \tau_e(f^{-1}(A)) \\ &\leq \sup_{f(x) \in A \subseteq u} (U(A) - \tau_e(f^{-1}(A))).\end{aligned}$$

So, $1 - N_{f(x)}(u) + N_x^e(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u} (1 - U(A) + \tau_e(f^{-1}(A)))$

and thus

$$\begin{aligned}\min(1, 1 - N_{f(x)}(u) + N_x^e(f^{-1}(u))) &\geq \inf_{f(x) \in A \subseteq u} \min(1, 1 - U(A) + \tau_e(f^{-1}(A))) \\ &\geq \inf_{v \in P(Y))} \min(1, 1 - U(v) + \tau_e(f^{-1}(v))) \\ &= C_e(f).\end{aligned}$$

Hence, $\inf_{x \in X} \min_{u \in P(Y)} \min(1, 1 - N_{f(x)}(u) + N_x^e(f^{-1}(u))) \geq [f \in C_e]$.

(3) (a) We prove that $\models f \in e_2 \leftrightarrow f \in e_3$. Since N_x^e is monotonous (Theorem 4.2 (2)), it is clear that $\sup_{v \in P(X), f(v) \subseteq u} N_x^e(v) = \sup_{v \in P(X), v \subseteq f^{-1}(u)} N_x^e(v) = N_x^e(f^{-1}(u))$.

Then,

$$\begin{aligned}e_3(f) &= \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}(u) + \sup_{v \in P(X), f(v) \subseteq u} N_x^e(v)) \\ &= \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}(u)) + N_x^e(f^{-1}(u))) = e_2(f).\end{aligned}$$

(b) We prove that $\models f \in e_4 \leftrightarrow f \in e_5$.

Frist, for each $B \in P(Y)$, there exists $A \in P(X)$ such that $f^{-1}(B) = A$ and $f(A) \subseteq B$.

So, $[Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] \geq [Cl_e^X(A) \subseteq f^{-1}(Cl^Y(f(A))]$.

Hence,

$$\begin{aligned}e_5(f) &= \inf_{B \in P(Y)} [Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] \\ &\geq \inf_{A \in P(X)} [Cl_e^X(A) \subseteq f^{-1}(Cl^Y(A)))] = e_4(f).\end{aligned}$$

Second, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, $[Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] \leq [Cl_e^X(A) \subseteq f^{-1}(Cl^Y(f(A)))]$. Thus,

$$\begin{aligned}e_4(f) &= \inf_{A \in P(X)} [Cl_e^X(A) \subseteq f^{-1}(Cl^Y(f(A)))] \\ &\geq \inf_{B \in P(Y), B = f(A)} [Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B)))]\end{aligned}$$

$$\geq \inf_{B\in P(Y)}[Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] = e_5(f).$$

(c) We prove that $\models f \in e_5 \leftrightarrow f \in e_2$; from Theorem 5.5(1),

$$e_5(f) = \forall B(Cl_e^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B)))$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-N_x^e(X-f^{-1}(B)))+1-N_{f(x)}(Y-B))$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(Y-B)+N_x^e(X-f^{-1}(B)))$$
$$= \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(u)+N_x^e(f^{-1}(u))) = e_2(f).$$ □

Remark 6.4. In the following theorem, we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous and C_e-continuous. Thus, the symbols (τ, U)-$C(f)$ and (τ, U)-$C_e(f)$, etc. will be understood.

Applying Theorems 3.6 one can deduce the following theorem.

Theorem 6.5. $\models f \in (\tau, U)\text{-}C \to f \in (\tau, U)\text{-}C_e$.

Theorem 6.6. *Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. For any $f \in Y^X$,*

$$\models C(f) \to C_e(f)$$

Proof. The proof is obtained from Theorem 3.6. □

Remark 6.7. In crisp setting, that is, in the case that the underlying fuzzifying topology is the ordinary topology, one can have $C_e(f) \to C(f)$.

But this statement may not be true in general in fuzzifying topology as illustrated by the following example.

Example 6.8. Let (X, τ) be the fuzzifying topological space defined in example 3.2 Consider the identity function f from (X, τ) onto (Y, σ), where σ is a fuzzifying topology on Y defined as follows:

$$\sigma(y) = \begin{cases} 1 & \text{if } \tau = \{X, \phi, \{a, b\}\} \\ 0 & \text{if otherwise.} \end{cases}$$

Then, $\frac{7}{8} = C_e(f) \nleq C(f) = 0$.

7. Fuzzifying e-separation axioms

Remark 7.1. For simplicity we use the following notations:

1. $K_e(x, y) := \exists A\big((A \in N_x^e \wedge y \notin A) \vee (A \in N_y^e \wedge x \notin A)\big)$;
2. $H_e(x, y) := \exists B \exists C\big((B \in N_x^e \wedge y \notin B) \wedge (C \in N_y^e \wedge x \notin C)\big)$;
3. $M_e(x, y) := \exists B \exists C(B \in N_x^e \wedge C \in N_y^e \wedge B \cap C = \phi)$;
4. $V_e(x, D) := \exists A \exists B(A \in N_x^e \wedge B \in \tau_e \wedge D \subseteq B \wedge A \cap B = \phi)$;
5. $W_e(A, B) := \exists G \exists H(G \in \tau_e \wedge H \in \tau_e \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \phi)$.

Definition 7.2. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates e-T_i(T_i^e for short)$\in \Im(\Omega)$, $i = 0, 1, ..., 4$, e-strong-T_i^e($T_i^{e^S}$ for short)$\in \Im(\Omega)$, $i = 3, 4$, and e-R_i(R_i^e for short) $\in \Im(\Omega)$, $i = 0, 1$ are defined as follows, respectively

(1) $(X,\tau) \in T_0^e := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to K_e(x,\ y)\big);$
(2) $(X,\tau) \in T_1^e := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to H_e(x,\ y)\big);$
(3) $(X,\tau) \in T_2^e := \forall x \forall y\big((x \in X \wedge y \in X \wedge x \neq y) \to M_e(x,\ y)\big);$
(4) $(X,\tau) \in T_3^e := \forall x \forall D\big((x \in X \wedge D \in F \wedge x \notin D) \to V_e(x,\ D)\big);$
(5) $(X,\tau) \in T_4^e := \forall A \forall B\big((A \in F \wedge B \in F \wedge A \cap B = \phi) \to W_e(A,\ B)\big);$
(6) $(X,\tau) \in T_3^{e^S} := \forall x \forall D\big((x \in X \wedge D \in F_e \wedge x \notin D) \to V(x,\ D)\big);$
(7) $(X,\tau) \in T_4^{e^S} := \forall A \forall B\big((A \in F_e \wedge B \in F_e \wedge A \cap B = \phi) \to W(A,\ B)\big);$
(8) $(X,\tau) \in R_0^e := \forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \to \big(K_e(x,y) \to H_e(x,\ y)\big)\Big);$
(9) $(X,\tau) \in R_1^e := \forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \to \big(K_e(x,y) \to M_e(x,y)\big)\Big).$

Lemma 7.3. *For any fuzzifying topological space* (X,τ)
(1) $\models K(x,y) \to K_e(x,y)$;
(2) $\models H(x,y) \to H_e(x,y)$;
(3) $\models M(x,y) \to M_e(x,y)$;
(4) $\models V(x,D) \to V_e(x,D)$;
(5) $\models W(A,B) \to W_e(A,B)$.

Proof. From Theorem 3.6(1), $\models \tau \subseteq \tau_e$ and so one can deduce that $N_x(A) \leq N_x^e(A)$ for any $A \in P(X)$, the proof is immediate. □

Theorem 7.4. *For any fuzzifying topological space* $(X,\ \tau)$
(1) $\models (X,\ \tau) \in T_i \to (X,\ \tau) \in T_i^e$*, where* $i = 0, ..., 4$.
(2) $\models (X,\ \tau) \in T_i^{e^S} \to (X,\ \tau) \in T_i$*, where* $i = 3, 4$.
(3) $\models (X,\ \tau) \in T_i^{e^S} \to (X,\ \tau) \in T_i^e$*, where* $i = 3, 4$.

Proof. (1) It is obtain from Lemma 7.3
(2) It follows from Theorem 3.6(2)
(3) It follows from (1) and (2). □

Lemma 7.5. *For any fuzzifying topological space* $(X,\ \tau)$
(1) $\models M_e(x,y) \to H_e(x,y)$;
(2) $\models H_e(x,y) \to K_e(x,y)$;
(3) $\models M_e(x,y) \to K_e(x,y)$;

Proof. (1) If $N_x^e(B) = 0$ or $N_y^e(C) = 0$, then the result holds. Suppose that $N_x^e(B) > 0$ and $N_y^e(C) > 0$. By Theorem 4.2(1) we have $[x \in B] = 1$ and $[y \in C] = 1$. So, $\{B, C \in P(X) : B \cap C = \phi\} \subseteq \{B, C \in P(X) : y \notin B \wedge x \notin C\}$. Thus

$$[M_e(x,y)] = \sup_{B \cap C = \phi} \min(N_x^e(A), N_y^e(C)) \leq \sup_{y \notin B, x \notin C} \min(N_x^e(B), N_y^e(C)) = [H_e(x,y)].$$

(2) We have that

$$[K_e(x,y)] = \max\big(\sup_{y \notin A} N_x^e(A), \sup_{y \notin A} N_y^e(A)\big) \geq \sup_{y \notin A} N_x^e(A) \geq \sup_{y \notin A, x \notin B} \big(N_x^e(A) \wedge N_y^e(B)\big) = [H_e(x,y)].$$

(3) It is obtained from (1) and (2). □

Theorem 7.6. *For any fuzzifying topological spaces* $(X,\ \tau)$
(1) $\models (X,\ \tau) \in T_1^e \to (X,\ \tau) \in T_0^e$;

(2) $\models (X,\ \tau) \in T_2^e \to (X,\ \tau) \in T_1^e$.

Proof. The proof of (1) and (2) are obtained from Lemma 7.5 (2) and (1), respectively. □

Corollary 7.7. *For any fuzzifying topological spaces* $(X,\ \tau)$

$$\models (X,\ \tau) \in T_2^e \to (X,\ \tau) \in T_0^e.$$

Proof. From Theorem 7.6 the proof is immedicate. □

Theorem 7.8. *For any fuzzifying topological space* $(X,\ \tau)$

$$\models (X,\ \tau) \in T_0^e \leftrightarrow \Big(\forall x \forall y\big(x \in X \wedge y \in X \wedge x \neq y \to \big(\neg(x \in Cl_e(\{y\}))\vee \neg(y \in Cl_e(\{x\}))\big)\Big).$$

Proof. Applying Theorem 4.2 (2) we have

$$\begin{aligned}[(X,\ \tau) \in T_0^e] &= \inf_{x\neq y} \max\big(\sup_{y\notin A} N_x^e(A), \sup_{x\notin A} N_y^e(A)\big)\\ &= \inf_{x\neq y} \max\big(N_x^e(X-\{y\}), N_y^e(X-\{x\})\big)\\ &= \inf_{x\neq y} \max\big(1-Cl_e(\{y\})(x), 1-Cl_e(\{x\})(y)\big)\\ &= \inf_{x\neq y} \big(\neg Cl_e(\{y\})(x) \vee \neg Cl_e(\{x\})(y)\big)\end{aligned}$$

$$\Big[\forall x \forall y\Big(x \in X \wedge y \in X \wedge x \neq y \to \big(\neg\big(x \in Cl_e(\{y\})\big)\big) \vee \neg(y \in Cl_e(\{x\}))\Big)\Big].$$ □

Theorem 7.9. *Let* $(X,\ \tau)$ *be a fuzzifying topological space. Then*

$$\models (X,\ \tau) \in T_1^e \leftrightarrow \forall x\big(\{x\} \in F_e\big).$$

Proof. For any x_1, x_2, $x_1 \neq x_2$, from Theorem 4.2 we have

$$\begin{aligned}\big[\forall x\big(\{x\} \in F_e\big)\big] &= \inf_{x\in X} F_e(\{x\}) = \inf_{x\in X} \tau_e(X-\{x\}) = \inf_{x\in X}\inf_{y\in X-\{x\}} N_y^e(X-\{x\})\\ &\leq \inf_{y\in X-\{x_2\}} N_y^e(X-\{x_2\}) \leq N_{x_1}^e(X-\{x_2\}) = \sup_{x_2\notin A} N_{x_1}^e(A).\end{aligned}$$

According to the same reason we can prove that

$$\big[\forall x(\{x\} \in F_e)\big] \leq \sup_{x_1\notin B} N_{x_2}^e(A).$$

Therefore

$$\begin{aligned}\big[\forall x(\{x\} \in F_e)\big] &\leq \inf_{x_1\neq x_2} \min\Big(\sup_{x_2\notin A} N_{x_1}^e(A), \sup_{x_1\notin B} N_{x_2}^e(B)\Big)\\ &= \inf_{x_1\neq x_2}\sup_{x_2\notin A, x_1\notin B} \min\Big(N_{x_1}^e(A), N_{x_2}^e(B)\Big) = \Big[(X,\tau) \in T_1^e\Big].\end{aligned}$$

On the other hand

$$\begin{aligned}\big[(X,\tau) \in T_1^e\big] &= \inf_{x_1\neq x_2} \min\Big(\sup_{x_2\notin A} N_{x_1}^e(A), \sup_{x_1\notin B} N_{x_2}^e(B)\Big)\\ &= \inf_{x_1\neq x_2} \min\Big(N_{x_1}^e(X-\{x_2\}), N_{x_1}^e(X-\{x_1\})\Big)\\ &\leq \inf_{x_1\neq x_2} N_{x_1}^e(X-\{x_2\}) = \inf_{x_2\in X}\inf_{x_1\in X-\{x_2\}} N_{x_1}^e(X-\{x_2\})\\ &= \inf_{x_2\in X} \tau_e\big(X-\{x_2\}\big) = \inf_{x\in X} \tau_e\big(X-\{x\}\big) = \big[\forall x(\{x\} \in F_e)\big].\end{aligned}$$

Thus $\big[(X,\tau) \in T_1^e\big] = \big[\forall x(\{x\} \in F_e)\big]$. □

Definition 7.10. The fuzzifying e-local base $e\beta_x$ of x is a function from $P(X)$ into I such that the following conditions are satisfied:

(1) $\models e\beta_x \subseteq N_x^e$;
(2) $\models A \in N_x^e \rightarrow \exists B(B \in e\beta_x \wedge x \in B \subseteq A)$.

Lemma 7.11. $\models A \in N_x^e \leftrightarrow \exists B(B \in e\beta_x \wedge x \in B \subseteq A)$.

Proof. From the condition (1) in Defintion 7.10 and Theorem 4.2 (2) then $N_x^e(A) \geq N_x^e(B) \geq e\beta_x(B)$ for each $B \subseteq X$ such that $x \in B \subseteq A$. So, $N_x^e(A) \geq \sup_{x \in B \subseteq A} e\beta_x(B)$. From condition (2) in Definition 3.1, $N_x^e(A) \leq \sup_{x \in B \subseteq A} e\beta_x(B)$. Hence, $N_x^e(A) = \sup_{x \in B \subseteq A} e\beta_x(B)$. □

Theorem 7.12. *If $e\beta_x$ is a fuzzifying e-local basis of x, then*
$\models (X,\ \tau) \in T_2^e \leftrightarrow \forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \rightarrow \big(\exists B\big(B \in e\beta_x \wedge y \notin Cl_e(B)\big)\big)\Big)$.

Proof. Applying Lemma 7.11 we have

$$\begin{aligned}
&\Big[\forall x \forall y\Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \rightarrow \big(\exists B\big(B \in e\beta_x \wedge y \notin Cl_e(B)\big)\big)\Big)\Big] \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \min\Big(e\beta_x(B), N_y^e(X-B)\Big) \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \min\Big(e\beta_x(B), \sup_{y \in C \subseteq (X-B)} e\beta_x(C)\Big) \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \sup_{y \in C \subseteq (X-B)} \min\Big(e\beta_x(B), e\beta_y(C)\Big) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \sup_{x \in D \subseteq B, y \in E \subseteq C} \min\Big(e\beta_x(B), e\beta_y(E)\Big) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \min\Big(\sup_{x \in D \subseteq B} e\beta_x(D), \sup_{y \in E \subseteq C} e\beta_y(E)\Big) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \min\Big(N_x^e(B), N_y^e(C)\Big) \\
&= \big[(X, \tau) \in T_2^e\big].
\end{aligned}$$

□

Theorem 7.13. *Let $(X,\ \tau)$ be a fuzzifying topological space. Then*

(1) $\models (X,\ \tau) \in R_1^e \rightarrow (X,\ \tau) \in R_0^e$.
(2) *If $T_0(X,\ \tau) = 1$, then*
(a) $\models (X,\ \tau) \in R_0 \rightarrow (X, \tau) \in R_0^e$.
(b) $\models (X,\ \tau) \in R_1 \rightarrow (X, \tau) \in R_1^e$.

Proof. (1) From Lemma 7.5 (1) we have

$$\begin{aligned}
\big[(X, \tau) \in R_0^e\big] &= \inf_{x \neq y} \min\Big(1, 1 - K_e(x,y) + H_e(x,y)\Big) \\
&\geq \inf_{x \neq y} \min\Big(1, 1 - K_e(x,y) + M_e(x,y)\Big) = \big[(X, \tau) \in R_1^e\big].
\end{aligned}$$

(2) Since $T_0(X,\ \tau) = 1$, then for each $x, y \in X$ and $x \neq y$ we have, $K(x,y) = 1$ and so, $K_e(x,y) = 1$.

(a) Applying Lemma 7.3(2) we have

$$\big[(X,\ \tau) \in R_0\big] = \inf_{x \neq y} \min\Big(1, 1 - K(x,y) + H(x,y)\Big)$$

$$\geq \inf_{x\neq y} \min\Big(1, 1-K(x,y)+H_e(x,y)\Big)$$
$$= \inf_{x\neq y} \min\Big(1, 1-K_e(x,y)+H_e(x,y)\Big)$$
$$= \big[(X,\ \tau)\in R_0^e\big].$$

(b) Applying Lemma 7.3 (3) we have

$$\big[(X,\ \tau)\in R_1\big] = \inf_{x\neq y} \min\Big(1, 1-K(x,y)+M(x,y)\Big)$$
$$\geq \inf_{x\neq y} \min\Big(1, 1-K(x,y)+M_e(x,y)\Big)$$
$$= \inf_{x\neq y} \min\Big(1, 1-K_e(x,y)+M_e(x,y)\Big)$$
$$= \big[(X,\ \tau)\in R_1^e\big].$$ □

Theorem 7.14. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models (X,\ \tau)\in T_1^e \to (X,\ \tau)\in R_0^e$;

(2) $\models (X,\ \tau)\in T_1^e \to \big((X,\ \tau)\in R_0^e \wedge (X,\tau)\in T_0^e\big)$;

(3) *If* $T_0^e(X,\tau)=1$, *then* $\models (X,\ \tau)\in T_1^e \leftrightarrow \big((X,\ \tau)\in R_0^e \wedge (X,\tau)\in T_0^e\big)$.

Proof. (1) By some calculations we have

$$T_1^e(X,\tau)=\inf_{x\neq y}[H_e(x,y)] \leq \inf_{x\neq y}\min\big(1, 1-[K_e(x,y)]+[H_e(x,y)]\big)=R_0^e(X,\tau).$$

(2) It is obtained from (1) and from Theorem 7.6 (1).

(3) Since $T_0^e(X,\tau)=1$, then for every $x,y\in X$ such that $x\neq y$ we have $[K_e(x,y)]=1$. Therefore,

$$[(X,\tau)\in R_0^e \wedge (X,\tau)\in T_0^e] = [(X,\tau)\in R_0^e] = \inf_{x\neq y}\min\big(1, 1-[K_e(x,y)]+[H_e(x,y)]\big)$$
$$= \inf_{x\neq y}[H_e(x,y)] = [(X,\tau)\in T_1^e].$$ □

Theorem 7.15. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models ((X,\ \tau)\in R_0^e * (X,\tau)\in T_0^e) \to (X,\tau)\in T_1^e$;

(2) *If* $T_0^e(X,\tau)=1$, *then* $\models ((X,\ \tau)\in R_0^e * (X,\tau)\in T_0^e) \leftrightarrow (X,\tau)\in T_1^e$.

Proof. (1)$[(X,\tau)\in R_o^e * (X,\tau)\in T_0^e] = \max\Big(0, R_0^e(X,\tau)+T_0^e(X,\tau)-1\Big)$

$$= \max\Big(0, \inf_{x\neq y}\min(1, 1-[K_e(x,y)]+[H_e(x,y)])+\inf_{x\neq y}[K_e(x,y)]-1\Big)$$
$$\leq \max\Big(0, \inf_{x\neq y}(\min\big(1, 1-[K_e(x,y)]+[H_e(x,y)]\big)+[K_e(x,y)]-1\big)\Big)$$
$$= \inf_{x\neq y}[H_e(x,y)] = [(X,\tau)\in T_1^e].$$

(2) $[(X,\tau)\in R_o^e * (X,\tau)\in T_0^e] = [R_0^e(X,\tau)] = \inf_{x\neq y}\min\Big(1, 1-[K_e(x,y)]+[H_e(x,y)]\Big)$

$$= \inf_{x\neq y}[H_e(x,y)] = [(X,\tau)\in T_1^e],$$

because $T_0^e(X,\tau)=1$, we have for each $x,y\in X$ such that $x\neq y$ we have $[K_e(x,y)]=1$. □

Theorem 7.16. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_0^e \to \big((X,\tau)\in R_0^e \to (X,\tau)\in T_1^e\big)$;

(2) $\models (X,\tau)\in R_0^e \to \big((X,\tau)\in T_0^e \to (X,\tau)\in T_1^e\big)$.

Proof. (1) From Theorems 7.14(1) and 7.15 (1) we have

$$\begin{aligned}&\big[(X,\tau)\in T_0^e \to \big((X,\tau)\in R_0^e \to (X,\tau)\in T_1^e\big)\big]\\ &\quad = \min\Big(1, 1-\big[(X,\tau)\in T_0^e\big] + \min(1, 1-\big[(X,\tau)\in R_0^e\big] + \big[(X,\tau)\in T_1^e\big])\Big)\\ &\quad = \min\Big(1, 1-\big[(X,\tau)\in T_0^e\big] + 1-\big[(X,\tau)\in R_0^e\big] + \big[(X,\tau)\in T_1^e\big]\Big)\\ &\quad = \min\Big(1, 1-\big(\big[(X,\tau)\in T_0^e\big] + \big[(X,\tau)\in R_0^e\big]-1\big) + \big[(X,\tau)\in T_1^e\big]\Big) = 1.\end{aligned}$$

(2) From Theorem 7.6(1) and 7.15(1) the proof is similar to (1). □

Theorem 7.17. *Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models (X,\ \tau)\in T_2^e \to (X,\ \tau)\in R_1^e$.

(2) $\models (X,\ \tau)\in T_2^e \to \Big((X,\ \tau)\in R_1^e \wedge (X,\tau)\in T_0^e\Big)$.

(3) *If* $T_0^e(X,\tau)=1$, *then* $\models (X,\ \tau)\in T_2^e \leftrightarrow \Big((X,\ \tau)\in R_1^e \wedge (X,\tau)\in T_0^e\Big)$.

Proof. (1) we have

$$T_2^e(X,\tau) = \inf_{x\neq y}[M_e(x,y)] \leq \inf_{x\neq y}[K_e(x,y)\to M_e(x,y)] = R_1^e(X,\tau).$$

(2) It is obtained from (1) and from Corollary 7.7
(3) Since $T_0^e(X,\tau)=1$, then for every $x,y\in X$ such that $x\neq y$ we have $[K_e(x,y)] = 1$. Therefore,

$$\begin{aligned}T_2^e(X,\tau) &= \inf_{x\neq y}[M_e(x,y)] = \inf_{x\neq y}[K_e(x,y)\to M_e(x,y)] = R_1^e(X,\tau)\\ &= R_1^e(X,\tau)\wedge T_0^e(X,\tau).\end{aligned}$$

□

Theorem 7.18. *Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models \big((X,\ \tau)\in R_1^e * (X,\tau)\in T_0^e\big) \to (X,\tau)\in T_2^e$;

(2) *If* $T_0^e(X,\tau)=1$, *then* $\models \big((X,\ \tau)\in R_1^e * (X,\tau)\in T_0^e\big) \leftrightarrow (X,\tau)\in T_2^e$.

Proof. (1) By some calculations we have

$$\begin{aligned}&[(X,\tau)\in R_1^e * (X,\tau)\in T_0^e] = \max\big(0, R_1^e(X,\tau)+T_0^e(X,\tau)-1\big)\\ &\quad = \max\Big(0, \inf_{x\neq y}\min\big(1, 1-[K_e(x,y)]+[M_e(x,y)]\big) + \inf_{x\neq y}[K_e(x,y)]\big) - 1\Big)\\ &\quad \leq \max\Big(0, \inf_{x\neq y}\big(\min\big(1, 1-[K_e(x,y)]+[M_e(x,y)]\big) + [K_e(x,y)]\big) - 1\Big)\\ &\quad = \inf_{x\neq y}[M_e(x,y)] = T_2^e(X,\tau).\end{aligned}$$

(2) Since $T_0^e(X,\tau)=1$, then for every $x,y\in X$ such that $x\neq y$ we have $[K_e(x,y)] = 1$. Therefore,

$$\begin{aligned}[(X,\tau)\in R_1^e * (X,\tau)\in T_0^e] &= [(X,\tau)\in R_1^e] = \inf_{x\neq y}\min\big(1, 1-[K_e(x,y)]+[M_e(x,y)]\big)\\ &= \inf_{x\neq y}[M_e(x,y)] = T_2^e(X,\tau).\end{aligned}$$

□

Theorem 7.19. *Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_0^e \to \big((X,\tau)\in R_1^e \to (X,\tau)\in T_2^e\big)$;
(2) $\models (X,\tau)\in R_1^e \to \big((X,\tau)\in T_0^e \to (X,\tau)\in T_2^e\big)$.

Proof. (1) From Theorems 7.17(1) and 7.18(1) we have
$[(X,\tau)\in T_0^e \to ((X,\tau)\in R_1^e \to (X,\tau)\in T_2^e)] = \min\Big(1, 1-T_0^e(X,\tau)+ \min\big(1, 1-R_1^e(X,\tau)+T_2^e(X,\tau)\big)\Big) = \min\big(1, 1-[(X,\tau)\in T_0^e]+1-[(X,\tau)\in R_1^e]+[(X,\tau)\in T_2^e]\big)$
$= \min\big(1, 1-([(X,\tau)\in T_0^e]+[(X,\tau)\in R_1^e]-1)+[(X,\tau)\in T_2^e]\big) = 1.$
(2) From Corollary 7.7 and Theorem 7.18(1) the proof is similar to (1). □

Theorem 7.20. *Let (X,τ) be a fuzzifying topological space. If $[(X,\tau)\in T_0^e]=1$, then*

(1) $\models \Big((X,\tau)\in T_0^e \to \big((X,\tau)\in R_0^e \to (X,\tau)\in T_1^e\big)\Big) \wedge \Big((X,\tau)\in T_1^e \to \neg\big((X,\tau)\in T_0^e \to \neg\big((X,\tau)\in R_0^e\big)\big)\Big);$

(2) $\models \Big((X,\tau)\in R_0^e \to \big((X,\tau)\in T_0^e \to (X,\tau)\in T_1^e\big)\Big) \wedge \Big((X,\tau)\in T_1^e \to \neg\big((X,\tau)\in T_0^e \to \neg\big((X,\tau)\in R_0^e\big)\big)\Big);$

(3) $\models \Big((X,\tau)\in T_0^e \to \big((X,\tau)\in R_0^e \to (X,\tau)\in T_1^e\big)\Big) \wedge \Big((X,\tau)\in T_1^e \to \neg\big((X,\tau)\in R_0^e \to \neg\big((X,\tau)\in T_0^e\big)\big)\Big);$

(4) $\models \Big((X,\tau)\in R_0^e \to \big((X,\tau)\in T_0^e \to (X,\tau)\in T_1^e\big)\Big) \wedge \Big((X,\tau)\in T_1^e \to \neg\big((X,\tau)\in R_0^e \to \neg\big((X,\tau)\in T_0^e\big)\big)\Big).$

Proof. For simplicity we put $[(X,\ \tau)\in T_0^e]=\eta$, $[(X,\ \tau)\in R_0^e]=\zeta$ and $[(X,\ \tau)\in T_1^e]=\xi$. Now, applying Theorem 7.15 (2), the proof is obtained with some relations in fuzzy logic as follows.
(1) $\big(\eta\to(\zeta\to\xi)\big)\wedge\big(\xi\to\neg(\eta\to\neg\zeta)\big) = \big(\eta\to\neg(\zeta*\neg\xi)\big)\wedge\big(\xi\to\neg(\eta\to\neg\zeta)\big)$
$= \neg\big(\eta*\neg\big(\neg(\zeta*\neg\xi)\big)\big)\wedge\neg\big(\xi*(\eta\to\neg\zeta)\big)$
$= \neg(\eta*\zeta*\neg\xi)\wedge\neg\big(\xi*\neg(\eta*\zeta)\big)$
$= (\eta*\zeta\to\xi)\wedge(\xi\to\eta*\zeta) = \eta\wedge\zeta\leftrightarrow\xi = 1$

Since $*$ is commutative one can have the proof of statements (2)-(4) in a similar way as (1). □

Theorem 7.21. *Let (X,τ) be a fuzzifying topological space. If $[(X,\tau)\in T_0^e]=1$, then*

(1) $\models \Big((X,\tau)\in T_0^e \to \big((X,\tau)\in R_1^e \to (X,\tau)\in T_2^e\big)\Big) \wedge \Big((X,\tau)\in T_2^e \to \neg\big((X,\tau)\in T_0^e \to \neg\big((X,\tau)\in R_1^e\big)\big)\Big);$

(2) $\models \Big((X,\tau)\in R_1^e \to \big((X,\tau)\in T_0^e \to (X,\tau)\in T_2^e\big)\Big) \wedge \Big((X,\tau)\in T_2^e \to \neg\big((X,\tau)\in T_0^e \to \neg\big((X,\tau)\in R_1^e\big)\big)\Big);$

(3) $\models \Big((X,\tau)\in T_0^e \to \big((X,\tau)\in R_1^e \to (X,\tau)\in T_2^e\big)\Big) \wedge \Big((X,\tau)\in T_2^e \to \neg\big((X,\tau)\in R_1^e \to \neg((X,\tau)\in T_0^e)\big)\Big);$

(4) $\models \Big((X,\tau) \in R_1^e \to \big((X,\tau) \in T_0^e \to (X,\tau) \in T_2^e\big)\Big)$
$\wedge \Big((X,\tau) \in T_2^e \to \neg\big((X,\tau) \in R_1^e \to \neg\big((X,\tau) \in T_0^e\big)\big)\Big)$;

Proof. The proof is similar to that of Theorem 7.20 □

Lemma 7.22. *(1) If $D \subseteq B$, then*

$$\sup_{A\cap B=\phi} N_x^e(A) = \sup_{A\cap B=\phi, D\subseteq B} N_x^e(A)$$

(2) $\sup_{A\cap B=\phi} \inf_{y\in D} N_y^e(X-A) = \sup_{A\cap B=\phi, D\subseteq B} \tau_e(B)$.

Proof. (1) Since $D \subseteq B$, then we have

$$\sup_{A\cap B=\phi} N_x^e(A) = \sup_{A\cap B=\phi} N_x^e(A) \wedge [D \subseteq B] = \sup_{A\cap B=\phi, D\subseteq B} N_x^e(A).$$

(2) Let $y \in D$ and $A \cap B = \phi$. Then

$$\begin{aligned}\sup_{A\cap B=\phi, D\subseteq B} \tau_e(B) &= \sup_{A\cap B=\phi, D\subseteq B} \tau_e(B) \wedge [y \in D] = \sup_{y\in D\subseteq B\subseteq X-A} \tau_e(B).\\ &= \sup_{y\in B\subseteq X-A} \tau_e(B) = N_y^e(X-A)\\ &= \inf_{y\in D} N_y^e(X-A) = \sup_{A\cap B=\phi} \inf_{y\in D} N_y^e(X-A).\end{aligned}$$

□

Definition 7.23. Let (X,τ) be a fuzzifying topological space. Then
$eT_3^{(1)}(X,\tau) := \forall x \forall D\Big(\big(x \in X \wedge D \in F \wedge x \notin D\big) \to \exists A\big(A \in N_x^e \wedge$
$\big(Cl_e(A) \cap D \equiv \phi\big)\big)\Big)$.

Theorem 7.24. *Let (X,τ) be a fuzzifying topological space.*
$\models (X,\tau) \in T_3^e \leftrightarrow (X,\tau) \in eT_3^{(1)}$.

Proof. Now,

$$\begin{aligned}(X,\tau) \in T_3^{(1)} &= \inf_{x\notin D} \min\Big(1, 1-\tau(X-D) + \sup_{A\in P(X)} \min\big(N_x^e(A), \inf_{y\in D}(1-Cl_e(A)(y))\big)\Big)\\ &= \inf_{x\notin D} \min\Big(1, 1-\tau(X-D) + \sup_{A\in P(X)} \min\big(N_x^e(A), \inf_{y\in D} N_y^e(X-A)\big)\Big).\end{aligned}$$

and

$$[(X,\tau) \in T_3^e] = \inf_{x\notin D} \min\Big(1, 1-\tau(X-D) + \sup_{A\cap B=\phi, D\in B} \min\big(N_x^e(A), \tau_e(B)\big)\Big).$$

So, the result holds if we prove that

$$\sup_{A\in P(X)} \min(N_x^e(A), \inf_{y\in D} N_y^e(X-A)) = \sup_{A\cap B=\phi, D\subseteq B} \min(N_x^e(A), \tau_e(B)) \tag{7.1}$$

In fact, in the left side of 7.1 when $A \cap D \neq \phi$ then there exists $y \in X$ such that $y \in D$ and $y \in A$. Namely, $y \in D$ and $y \notin X - A$. So, $\inf_{y\in D} N_y^e(X-A) = 0$ and thus 7.1 becomes

$$\sup_{A\in P(X), A\cap B=\phi} \min\big(N_x^e(A), \inf_{y\in D} N_y^e(X-A)\big) = \sup_{A\cap B=\phi, D\subseteq B} \min\big(N_x^e(A), \tau_e(B)\big),$$

which is obtained from Lemma 7.22 □

Definition 7.25. Let (X,τ) be a fuzzifying topological space. Then
$eT_3^{(2)}(X,\tau) := \forall x \forall B\big((x \in B \wedge B \in \tau) \to \big(\exists A(A \in N_x^e \wedge Cl_e(A) \subseteq B)\big)\big)$.

Theorem 7.26. *Let (X,τ) be a fuzzifying topological space. Then*
$$\models (X,\tau)\in T_3^e \leftrightarrow (X,\tau)\in eT_3^{(2)}.$$

Proof. From Theorem 7.24, we have

$\big[(X,\tau)\in T_3^e\big] = \inf_{x\notin D} \min\Big(1, 1-\tau(X-D)+\sup_{A\in P(X)} \min\big(N_x^e(A), \inf_{y\in D} N_y^e(X-A)\big)\Big)$

Now, if we put $B = X-D$, then

$[(X,\tau)\in T_3^{(2)}] = \inf_{x\in B} \min\Big(1, 1-\tau(B)+\sup_{A\in P(X)} \min\big(N_x^e(A), \inf_{y\in(X-B)} N_y^e(X-A)\big)\Big).$

$= \inf_{x\notin D} \min\Big(1, 1-\tau(X-D)+\sup_{A\in P(X)} \min\big(N_x^e(A), \inf_{y\in D} N_y^e(X-A)\big)\Big).$

$= [(X,\tau)\in T_3^e].$ □

Definition 7.27. Let (X,τ) be a fuzzifying topological space and φ be a subbase of τ then $(X,\tau)\in eT_3^{(3)} := \forall x\forall D\Big(x\in D\wedge D\in\varphi\rightarrow \exists B\big(B\in N_x^e \wedge Cl_e(B)\subseteq D\big)\Big).$

Theorem 7.28. $\models (X,\tau)\in T_3^e \leftrightarrow (X,\tau)\in eT_3^{(3)}.$

Proof. Since $[\varphi\subseteq\tau]=1$, and with regards to Theorem 7.24 and 7.26 we have $eT_3^{(3)}(X,\tau)\geq eT_3^{(2)}(X,\tau) = T_3^e(X,\tau)$. So, it remains to prove that $eT_3^{(3)}(X,\tau)\leq eT_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x\in A$,

$\min\Big(1, 1-\tau(A)+\sup_{B\in P(X)} \min\big(N_x^e(B), \inf_{y\in X-A} N_y^e(X-B)\big)\Big) \geq \big[(X,\tau)\in eT_3^{(3)}\big].$

Set $\big[(X,\tau)\in eT_3^{(3)}\big]=\delta$. Then, for any $x\in X$ and any $D_{\lambda_i}\in P(X)$, $\lambda_i\in I_\lambda$(I_λ denotes a finite index set), $\lambda\in\Lambda$,

$$\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$$

We have

$1-\varphi(D_{\lambda_i})+\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in X-D_{\lambda_i}} N_y^e(X-B)\Big)\geq\delta>\delta-\epsilon$, where ϵ is any positive number. Thus

$\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in X-D_{\lambda_i}} N_y^e(X-B)\Big) > \varphi(D_{\lambda_i})-1+\delta-\epsilon.$

Set $\beta_{\lambda_i}=\{B : B\subseteq D_{\lambda_i}\}$. Then

$\inf_{\lambda_i\in I_\lambda}\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in(X-D_{\lambda_i})} N_y^e(X-B)\Big).$

$= \sup_{f\in\Pi\{\beta_{\lambda_i}:\lambda_i\in I_\lambda\}} \inf_{\lambda_i\in I_\lambda} \min\Big(N_x^e(f(\lambda_i)), \inf_{y\in(X-D_{\lambda_i})} N_y^e\big(X-f(\lambda_i)\big)\Big)$

$= \sup_{f\in\Pi\{\beta_{\lambda_i}:\lambda_i\in I_\lambda\}} \min\Big(\inf_{\lambda_i\in I_\lambda} N_x^e(f(\lambda_i)), \inf_{\lambda_i\in I_\lambda}\inf_{y\in(X-D_{\lambda_i})} N_y^e\big(X-f(\lambda_i)\big)\Big)$

$= \sup_{f\in\Pi\{\beta_{\lambda_i}:\lambda_i\in I_\lambda\}} \min\Big(\inf_{\lambda_i\in I_\lambda} N_x^e(f(\lambda_i)), \inf_{y\in\bigcup_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-f(\lambda_i))\Big)$

$= \sup_{B\in P(X)} \min\Big(\inf_{\lambda_i\in I_\lambda} N_x^e(B), \inf_{y\in\bigcup_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-B)\Big)$

$$= \sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in \bigcup\limits_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-B)\Big),$$

where $B = f(\lambda_i)$.

Similary, we can prove

$$\inf_{\lambda\in\Lambda}\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in \bigcup\limits_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-B)\Big)$$
$$= \sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in \bigcup\limits_{\lambda\in\Lambda}\bigcup\limits_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-B)\Big)$$
$$\leq \sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in \bigcap\limits_{\lambda\in\Lambda}\bigcup\limits_{\lambda_i\in I_\lambda}(X-D_{\lambda_i})} N_y^e(X-B)\Big)$$
$$\leq \sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in X-A} N_y^e(X-B)\Big),$$

we have

$$\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in X-A} N_y^e(X-B)\Big)$$
$$\geq \inf_{\lambda\in\Lambda}\inf_{\lambda_i\in I_\lambda}\sup_{B\in P(X)} \min\Big(N_x^e(B), \inf_{y\in (X-D_{\lambda_i})} N_y^e(X-B)\Big)$$
$$\geq \inf_{\lambda\in\Lambda}\inf_{\lambda_i\in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

For any I_λ and Λ that satisfy

$$\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$$

the above inequality is true. So,

$$\sup_{B\in P(X)} \min\big(N_x^e(B), \inf_{y\in(X-A)} N_y^e(X-B)\big) \geq$$
$$\sup_{\bigcup\limits_{\lambda\in\Lambda} D_\lambda = A}\inf_{\lambda\in\Lambda}\sup_{\bigcap\limits_{\lambda_i\in I_\lambda} D_{\lambda_i}=D_\lambda}\inf_{\lambda_i\in I_\lambda}\varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$
$$= \tau(A) - 1 + \delta - \epsilon,$$

i.e., $\min\Big(1, 1-\tau(A) + \sup_{B\in P(X)} \min\big(N_x^e(B), \inf_{y\in(X-A)} N_y^e(X-B)\big)\Big) \geq \delta - \epsilon.$

Because ϵ is any positive number, when $\epsilon \to 0$ we have

$$[(X,\tau) \in eT_3^{(2)}] \geq \delta = [(X,\tau)\in eT_3^{(3)}].$$

So, $\models (x,\tau) \in T_3^e \leftrightarrow (x,\tau) \in eT_3^{(3)}$. □

Definition 7.29. Let (X,τ) be the fuzzifying topological space and let

1. $(X,\tau) \in eST_3^{(1)} = \forall x\forall D\Big(\big((x\in X)\wedge(D\in F_e)\wedge(x\notin D)\big) \to \exists A\big(A\in N_x \wedge (Cl(A)\cap D \equiv \phi)\big)\Big);$
2. $(X,\tau) \in eST_3^{(2)} = \forall x\forall B\Big(\big((x\in B)\wedge(B\in\tau)\big) \to \exists A\big(A\in N_x^e \wedge (Cl(A)\cap B)\big)\Big);$

3. $(X,\tau) \in eT_4^{(1)} = \forall A \forall B\Big(\big((A \in \tau) \wedge (B \in F) \wedge (A \cap B \equiv \phi)\big) \rightarrow \exists G\big((G \in \tau) \wedge (A \subseteq G) \wedge (Cl_e(G) \cap B \equiv \phi)\big)\Big)$;

4. $(X,\tau) \in eT_4^{(2)} = \forall A \forall B\Big(\big((A \in F) \wedge (B \in \tau) \wedge (A \subseteq B)\big) \rightarrow \exists G\big((G \in \tau) \wedge (A \subseteq G) \wedge (Cl_e(G) \cap B)\big)\Big)$;

5. $(X,\tau) \in eST_4^{(1)} = \forall A \forall B\Big(\big((A \in \tau) \wedge (B \in F_e) \wedge (A \cap B \equiv \phi)\big) \rightarrow \exists G\big((G \in \tau) \wedge (A \subseteq G) \wedge (Cl_e(G) \cap B \equiv \phi)\big)\Big)$;

6. $(X,\tau) \in eST_4^{(2)} = \forall A \forall B\Big(\big((A \in F) \wedge (B \in \tau_e) \wedge (A \subseteq B)\big) \rightarrow \exists G\big((G \in \tau) \wedge (A \subseteq G) \wedge (Cl_e(G) \cap B)\big)\Big)$.

By a similar proof of Theorem 7.24 and 7.26 we have the following theorem.

Theorem 7.30. *Let (X,τ) be any fuzzifying topological space. Then*

(1) $\models (X,\tau) \in T_3^{e^S} \leftrightarrow (X,\tau) \in eST_3^{(1)}$;
(2) $\models (X,\tau) \in T_3^{e^S} \leftrightarrow (X,\tau) \in eST_3^{(2)}$;
(3) $\models (X,\tau) \in T_4^{e} \leftrightarrow (X,\tau) \in eT_4^{(1)}$;
(4) $\models (X,\tau) \in T_4^{e} \leftrightarrow (X,\tau) \in eT_4^{(2)}$;
(5) $\models (X,\tau) \in T_4^{e^S} \leftrightarrow (X,\tau) \in eST_4^{(1)}$;
(6) $\models (X,\tau) \in T_4^{e^S} \leftrightarrow (X,\tau) \in eST_4^{(2)}$.

8. Relation among separation axioms

Theorem 8.1. $\models \big((X,\tau) \in T_3^e * (X,\tau) \in T_1\big) \rightarrow (X,\tau) \in T_2^e$;

Proof. From Theorem 2.2 [11] we have, $T_1(X,\tau) = \inf_{z \in X} \tau(X - \{z\})$. So, $[(X,\tau) \in T_3^e] + [(X,\tau) \in T_1]$

$$= \inf_{x \notin D} \min\big(1, 1 - \tau(X - D) + \sup_{A \cap B = \phi, D \subseteq B} \min(N_x^e(A), \tau_e(B))\big) + \inf_{z \in X} \tau(X - \{z\})$$

$$\leq \inf_{x \in X, x \neq y} \inf_{y \in X} \min\big(1, 1 - \tau(X - \{y\}) + \sup_{A \cap B = \phi} \min(N_x^e(A), N_y^e(B))\big) + \inf_{z \in X} \tau(X - \{z\})$$

$$= \inf_{x \in X, x \neq y} \Big(\inf_{y \in X} \min\big(1, 1 - \tau(X - \{y\}) + \sup_{A \cap B = \phi} \min(N_x^e(A), N_y^e(B))\big) + \inf_{z \in X} \tau(X - \{z\})\Big)$$

$$\leq \inf_{x \in X, x \neq y} \inf_{y \in X} \Big(\min\big(1, 1 - \tau(X - \{y\}) + \sup_{A \cap B = \phi} \min(N_x^e(A), N_y^e(B))\big) + \tau(X - \{y\})\Big)$$

$$\leq \inf_{x \neq y} \Big(1 + \sup_{A \cap B = \phi} \min(N_x^e(A), N_y^e(B))\Big)$$

$$\leq 1 + \inf_{x \neq y} \sup_{A \cap B = \phi} \min\big(N_x^e(A), N_y^e(B)\big) = 1 + [(X,\tau) \in T_2^e],$$

namely, $[(X,\tau)\in T_2^e]\geq[(X,\tau)\in T_3^e]+[(X,\tau)\in T_1]-1$. Thus, $[(X,\tau)\in T_2^e]\geq \max\big(0,[(X,\tau)\in T_3^e]+[(X,\tau)\in T_1]-1\big)$. □

Theorem 8.2. $\models\big((X,\tau)\in T_4^e*(X,\tau)\in T_1\big)\to(X,\tau)\in T_3^e$;

Proof. It is equivalent to prove that $[(X,\tau)\in T_3^e]\geq[(X,\tau)\in T_4^e]+[(X,\tau)\in T_1]-1$. In fact,

$$\begin{aligned}
[(X,\tau)\in T_4^e]+[(X,\tau)\in T_1] &= \inf_{E\cap D\equiv\phi}\min\Big(1,1-\min\big(\tau(X-E),\tau(X-D)\big)\\
&\qquad+\sup_{A\cap B=\phi,E\subseteq A,D\subseteq B}\min\big(\tau_e(A),\tau_e(B)\big)\Big)+\inf_{z\in X}\tau(X-\{z\})\\
&\leq \inf_{x\notin D}\min\Big(1,1-\min\big(\tau(X-\{x\}),\tau(X-D)\big)\\
&\qquad+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\Big)+\inf_{z\in X}\tau(X-\{z\})\\
&\leq \inf_{x\notin D}\min\Big(1,\max(1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big),\\
&\qquad 1-\tau(X-\{x\})+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_y^e(B)\big))\Big)+\inf_{z\in X}\tau(X-\{z\})\\
&= \inf_{x\notin D}\max\Big(\min\big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\big),\\
&\qquad \min\big(1,1-\tau(X-\{x\})+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\big)\Big)+\inf_{z\in X}\tau(X-\{z\})\\
&\leq \inf_{x\notin D}\max\Big(\min\big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)+\tau(X-\{x\}),\\
&\qquad \min\big(1,1-\tau(X-\{x\}\big)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\big)\Big)+\\
&\tau(X-\{x\})\\
&\leq \inf_{x\notin D}\max\Big(\min\big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\big)+\tau(X-\{x\}),\\
&\qquad 1+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\Big)\\
&\leq \inf_{x\notin D}\Big(\min\big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\big)+1\Big)\\
&\leq \inf_{x\notin D}\Big(1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^e(A),\tau_e(B)\big)\Big)+1=[(X,\tau)\in T_3^e]+1.
\end{aligned}$$

By a similar procedures of Theorems 8.1-8.2 we have the following Theorems respectively. □

Theorem 8.3. $\models\big((X,\tau)\in T_3^{e^S}*(X,\tau)\in T_1^e\big)\to(X,\tau)\in T_2$;

Theorem 8.4. $\models((X,\tau)\in T_4^{e^S}*(X,\tau)\in T_1^e)\to(X,\tau)\in T_3^{e^S}$;

From the above discussion one can have the following diagram

$$\begin{array}{ccccc}
 & & T_4^{e^S} * T_1^e & \leftarrow T_4^{e^S} * T_1^e & \\
 & & & \downarrow & \\
 & & \downarrow & T_3^{e^S} & \leftarrow T_4^{e^S} \\
 & & & \downarrow & \downarrow \\
T_0 & \leftarrow T_1 & \leftarrow T_2 & \leftarrow T_3 & \leftarrow T_4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
T_0^e & \leftarrow T_1^e & \leftarrow T_2^e & \leftarrow T_3^e & \leftarrow T_4^e .
\end{array}$$

References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 190–201.

[2] Erdal Ekici, On e-open sets, DP^*-sets and DPE^*-sets and decompositions of continity, Arabian J. Sci. 33 (2) (2008) 269–282.

[3] B. Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl. 50 (1975) 74–79.

[4] F. H. Khedr, F. M. Zeyada and O. R. Sayed, On spearation axioms in fuzzifying topology, Fuzzy sets and Systems 119 (2001) 439–458.

[5] F. H. Khedr, F. M. Zeyada and O. R. Sayed, Fuzzy semi-continuity and fuzzy csemi-continuity in fuzzifying topology, The Journal of Fuzzy Mathematics 7 (1) (1999) 105–124.

[6] R. Lowen, Fuzzy topological spaces and compactness, J. Math. Anal. Appl. 65 (1976) 621–633.

[7] T. Noiri and O. R. Sayed, Fuzzy γ-open sets and fuzzy γ-continuity in fuzzifying topology, Sci. Math. Jpn. 55(2) (2002) 439–458.

[8] P. M. Pu and Y. M. Liu, Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.

[9] O. R. Sayed and Hu Zhao, Some covering properties in semantic method of continuous valued logic, Journal of Intelligent and Fuzzy Systems 27 (2014) 1419–1431.

[10] V. Seenivasan and K. Kamala, Fuzzy e-continuity and fuzzy e-open sets, Ann.Fuzzy Math. Inform. 8 (1) (2014) 141–148.

[11] J. Shen, Separation axiom in fuzzifying topology, Fuzzy Sets and Systems 57 (1993) 111–123.

[12] C. K. Wong, Fuzzy point and local properties of fuzzy topology, J. Math. Anal. Appl. 46 (1974) 316–328.

[13] M. S. Ying, A new approach for fuzzy topology (I), Fuzzy Sets and Systems 39 (1991) 303–321.

[14] M. S. Ying, A new approach for fuzzy topology (II), Fuzzy Sets and Systems 47 (1992) 221–232.

[15] A. M. Zahran, Almost continuity and δ-continuity in fuzzifying topology, Fuzzy Sets and Systems 116 (2000) 339–352.

[16] F. M. Zeyada and A. K. Mousa, Regular open sets in fuzzifying topology redefined, J. Egyptian Math. Soc.(2014)

A. Vadivel (avmaths@gmail.com)
Assistant Professor, Mathematics Section (FEAT), Annamalai University, Annamalainagar, Tamil Nadu-608 002

M. Palanisamy (palaniva26@gmail.com)
Research Scholar, Department of Mathematics, Annamalai University, Annamalainagar 608 002