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Categorical relationships of fuzzy topological systems with fuzzy topological spaces and underlying algebras II

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Abstract. This work deals with categorical relationships among variable basis and fixed basis fuzzy topological systems whose underlying sets are fuzzy sets, variable and fixed basis fuzzy topological spaces on fuzzy sets and their underlying algebraic structures. Connection of this study with our future aim i.e., generalizing the notion of geometric logic, is indicated.

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1. Introduction

The title of this paper has been chosen to be the same as our earlier paper [4] with a 'II' meaning thereby that this is the second paper by us of this series submitted for this journal. The approach is basically the same, only here more generalized categories are involved. However, in the last section it will be observed that a somewhat different direction of development has taken place. In this paper we generalize the work submitted in the proceedings of 13th Asian Logic Conference 2013 [5] by taking the value set as any frame L instead of $[0, 1]$. This in turn, generalizes [4]. Further we proceed with the concept of variable basis fuzzy topological spaces on fuzzy sets and propose the notion of variable basis topological systems whose underlying sets are fuzzy sets. Solovyov and Rodabough worked on variable basis fuzzy topological spaces [7] and systems [8] over crisp sets. Hence their case become a particular case of ours. Here we are able to establish adjunction between space and system i.e. spatialization is achieved but adjunction between space and algebra or system and algebra is still unsettled. In the particular case i.e. in [8] Solovyov also

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left this case as an open question. Zadeh introduced fuzzy sets in 1965 [10]. In this paper by a fuzzy set we mean the pair $(X, \tilde{A})$ where X is a non-empty set and $\tilde{A}$ is a mapping from X to a suitable lattice.

To make this paper self-contained we include the following preliminaries.

Preliminaries A

The main aim of this paper is to generalize the notion of topological systems [9] and find the categorical relationships with appropriate spaces and algebraic structures. In this section let us recall the basic concepts related to our work.

Definition 1.1. A *frame* is a partially ordered set such that

(1) every subset has a join,
(2) every finite subset has a meet, and
(3) binary meets distribute over arbitrary joins: i.e. $x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$.

Note that the Lindenbaum algebra of logic of finite observations or geometric logic [9] is a frame. In this paper we are not concern about the logic though it is our future aim. So the details of the logic is not provided here. We only indicate some connections of the logic with the systems and algebras.

Definition 1.2. A *topological system* is a triple $(X, \models, A)$ where X is a nonempty set, A is a frame and $\models$, is a binary relation (i.e. $\models \subseteq X \times A$), satisfying the following conditions.

(1) if S is a finite subset of A, then $x \models \bigwedge S \Leftrightarrow x \models a$ for all $a \in S$,
(2) if S is any subset of A, then $x \models \bigvee S \Leftrightarrow x \models a$ for some $a \in S$.

Topological system was introduced by S. Vickers in 1989. In topological system the binary relation plays the key role and indicates its relationship with the logic of finite observations or geometric logic.

In our previous work [4] we already had some generalization of Vicker's work, known as fuzzy topological systems and defined as follows.

Definition 1.3 ([4])**.** A *fuzzy topological system* is a triple $(X, \models, A)$, where X is a set, A is a frame and $\models$ is a $[0,1]$- fuzzy relation from X to A such that

(1) if S is a finite subset of A, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$,
(2) if S is any subset of A, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$.

Further we generalized the concept of fuzzy topological system.

Definition 1.4 ([5])**.** An *$\mathscr{F}$-topological system* is a quadruple $(X, \tilde{A}, \models, P)$, where $(X, \tilde{A})$ is a non-empty fuzzy set, P is a frame and $\models$ is a $[0,1]$- fuzzy relation from X to P such that

(1) $gr(x \models p) \leq \tilde{A}(x)$,
(2) if S is a finite subset of P, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$,
(3) if S is any subset of P, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$.

where $gr(x \models p)$ is the degree in which x is related with p, $x \in X$, $p \in P$.

Definition 1.5. Let $D = (X, \tilde{A}, \models, P)$ and $E = (Y, \tilde{B}, \models', Q)$ be $\mathscr{F}$ topological systems. A *continuous map* $f : D \longrightarrow E$ is a pair (f_1, f_2) where,

(1) $f_1 : (X, \tilde{A}) \longrightarrow (Y, \tilde{B})$ is a proper function (c.f Definition 1.6) from $(X, \tilde{A})$ to $(Y, \tilde{B})$,
(2) $f_2 : Q \longrightarrow P$ is a frame homomorphism and
(3) $gr(x \models f_2(q)) = gr(f_1(x) \models' q)$, for all $x \in X$ and $q \in Q$.

Definition 1.6 ([2])**.** f_1 is a *proper function* from $(X, \tilde{A})$ to $(Y, \tilde{B})$ meaning that f_1 is a fuzzy relation from $(X, \tilde{A})$ to $(Y, \tilde{B})$ s.t. $\forall x \in \mid \tilde{A} \mid$, $\exists$ *unique* $y \in \mid \tilde{B} \mid$ for which $\tilde{A}(x) = f_1(x, y)$ and $f_1(x, y') = 0$ if $y' \neq y \in \mid \tilde{B} \mid$, where $\mid \tilde{A} \mid = \{x \in X : \tilde{A}(x) > 0\}$ and $\mid \tilde{B} \mid = \{y \in Y : \tilde{B}(y) > 0\}$. For a fixed $x \in \mid \tilde{A} \mid$, we will denote that *unique* $y \in \mid \tilde{B} \mid$ by $f_1(x)$.

Definition 1.7. Let $(X, \tilde{A})$ be a fuzzy set. The map $i_{\tilde{A}} : X \times X \longrightarrow [0, 1]$ (or L, a frame) is said to be an identity proper function iff $i_{\tilde{A}}(x, x) = \tilde{A}(x)$, for any $x \in X$ and $i_{\tilde{A}}(x, x') = 0$ (or 0_L, the least element of the frame L), when $x \neq x'$ in X.

Definition 1.8 ([5])**.** The category $\mathscr{F}$-**TopSys** is defined thus.

- The objects are $\mathscr{F}$-topological systems $(X, \tilde{A}, \models, P)$.
- The morphisms are continuous maps (c.f Definition 1.5).
- The identity on $(X, \tilde{A}, \models, P)$ is the pair (i_A, i_P), where i_A is the identity proper function (c.f Definition 1.7) on the fuzzy set $(X, \tilde{A})$ and i_P is the identity frame homomorphism. This is an $\mathscr{F}$-**TopSys** morphism can be proved.
- If $(f_1, f_2) : (X, \tilde{A}, \models, P) \longrightarrow (Y, \tilde{B}, \models', Q)$ and $(g_1, g_2) : (Y, \tilde{B}, \models', Q) \longrightarrow (Z, \tilde{C}, \models'', R)$ are morphisms in $\mathscr{F}$-**TopSys**, their composition $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$, where $g_1 \circ f_1$ is the composition of proper function between fuzzy sets and $f_2 \circ g_2$ is the composition of frame homomorphism between two frames. It can be verified that $(g_1, g_2) \circ (f_1, f_2)$ is a morphism in $\mathscr{F}$-**TopSys**.

Definition 1.9 ([2])**.** Let $(X, \tilde{A})$ be a fuzzy set and τ a collection of fuzzy subsets of $(X, \tilde{A})$ such that

(1) $(X, \tilde{\phi})$ and $(X, \tilde{A})$ are in τ, where $\tilde{\phi} : X \longrightarrow [0, 1]$ is such that $\tilde{\phi}(x) = 0$, for all $x \in X$;
(2) $(X, \tilde{A}_1)$, $(X, \tilde{A}_2)$ are in τ imply $(X, \tilde{A}_1 \cap \tilde{A}_2)$ is in τ, where $\tilde{A}_1 \cap \tilde{A}_2(x) = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$, for all $x \in X$;
(3) $(X, \tilde{A}_i) \in \tau$ imply $(X, \cup_{i \in I} \tilde{A}_i) \in \tau$, where $\cup_{i \in I} \tilde{A}_i : X \longrightarrow [0, 1]$ is such that $(\cup_{i \in I} \tilde{A}_i)(x) = \vee_{i \in I} \tilde{A}_i(x)$, for all $x \in X$.

Then $(X, \tilde{A}, \tau)$ is an *$\mathscr{F}$-topological space.*

Definition 1.10 ([2])**.** The category $\mathscr{F}$-**Top** is defined thus.

- The objects are $\mathscr{F}$-topological spaces $(X, \tilde{A}, \tau)$ on fuzzy sets $(X, \tilde{A})$.
- The morphisms are proper functions satisfying the following property: If $f : (X, \tilde{A}, \tau) \longrightarrow (Y, \tilde{B}, \tau')$ and $(Y, \tilde{B}_1)$ is a subset of $(Y, \tilde{B})$ such that $(Y, \tilde{B}_1) \in \tau'$ then $(X, f^{-1}(\tilde{B}_1)) \in \tau$, where $f^{-1}(\tilde{B}_1)$ is a fuzzy subset of X given by $f^{-1}(\tilde{B}_1)(x) = \tilde{B}_1(f(x))$.
- The identity on $(X, \tilde{A}, \tau)$ is the identity proper function (c.f Definition 1.7) on the fuzzy set $(X, \tilde{A})$. This is an $\mathscr{F}$-**Top** morphism can be proved.

- If $f : (X, \tilde{A}, \tau) \longrightarrow (Y, \tilde{B}, \tau')$ and $g : (Y, \tilde{B}, \tau') \longrightarrow (Z, \tilde{C}, \tau'')$ are morphisms in $\mathscr{F}$-**Top**, their composition $g \circ f$ is the composition of proper functions between fuzzy sets. It can be verified that $g \circ f$ is a morphism in $\mathscr{F}$-**Top**.

Definition 1.11 ([2]). Let $(X, \tilde{A})$ be an L-fuzzy set and τ a collection of fuzzy subsets of $(X, \tilde{A})$ such that

(1) $(X, \tilde{\phi})$ and $(X, \tilde{A})$ are in τ, where $\tilde{\phi} : X \longrightarrow L$ is such that $\tilde{\phi}(x) = 0_L$, for all $x \in X$, where 0_L is the least element of the frame L;
(2) $(X, \tilde{A}_1)$, $(X, \tilde{A}_2)$ are in τ implies $(X, \tilde{A}_1 \cap \tilde{A}_2)$ is in τ, where $(\tilde{A}_1 \cap \tilde{A}_2)(x) = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$, for all $x \in X$;
(3) $(X, \tilde{A}_i) \in \tau$ implies $(X, \cup_{i \in I} \tilde{A}_i) \in \tau$, where $\cup_{i \in I} \tilde{A}_i : X \longrightarrow L$ is such that $(\cup_{i \in I} \tilde{A}_i)(x) = \vee_{i \in I} \tilde{A}_i(x)$, for all $x \in X$.

Then $(X, \tilde{A}, \tau)$ is an *$\mathscr{L}$-topological space*.

Preliminaries B

The basic notions of category theory are mostly taken from [1]. Let $G : \mathbb{A} \longrightarrow \mathbb{B}$ be a functor, and let B be a $\mathbb{B}$-object.

Definition 1.12. For any category $\mathbb{A} = (O, hom_{\mathbb{A}}, id, \circ)$ the *dual* (or *opposite*) category of $\mathbb{A}$ is the category $\mathbb{A}^{op} = (O, hom_{\mathbb{A}^{op}}, id, \circ^{op})$ where $hom_{\mathbb{A}^{op}}(A, B) = hom_{\mathbb{A}}(B, A)$ and $f \circ^{op} g = g \circ f$. (Thus $\mathbb{A}$ and $\mathbb{A}^{op}$ have the same objects and,except for their direction, the same morphisms).

Definition 1.13 (G-structured arrow and G-costructured arrow)**.**

(1) A *G-structured arrow* with domain B is a pair (f, A) consisting of an $\mathbb{A}$-object A and a $\mathbb{B}$-morphism $f : B \longrightarrow GA$.
(2) A *G-costructured arrow* with codomain B is a pair (A, f) consisting of an $\mathbb{A}$-object A and a $\mathbb{B}$-morphism $f : GA \longrightarrow B$.

Definition 1.14 (G-universal arrow and G-couniversal arrow)**.**

(1) A G-structured arrow (g, A) with domain B is called *G-universal* for B provided that for each G-structured arrow (g', A') with domain B there exists a unique $\mathbb{A}$-morphism $\hat{f} : A \longrightarrow A'$ with $g' = G(\hat{f}) \circ g$ i.e., s.t. the triangle

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "g'"'] & GA \arrow[d, "G\hat{f}"] \\ & GA' \end{tikzcd}$$

commutes.
(2) A G-costructured arrow (A, g) with codomain B is called *G-couniversal* for B provided that for each G-costructured arrow (A', g') with codomain B there exists a unique $\mathbb{A}$-morphism $\hat{f} : A' \longrightarrow A$ with $g' = g \circ G(\hat{f})$ i.e., s.t. the triangle

$$\begin{tikzcd} GA \arrow[r, "g"] & B \\ GA' \arrow[u, "G\hat{f}"] \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes.

Definition 1.15 (Left Adjoint and Right Adjoint)**.**

(1) A functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be *left adjoint* provided that for every $\mathbb{B}$-object B there exists a G-couniversal arrow with codomain B.
i.e. there exist a natural transformation $\eta : A \longrightarrow FGA$ where $F : \mathbb{B} \longrightarrow \mathbb{A}$ is a functor, s.t. for given $f : A \longrightarrow FB$ there exist a unique $\mathbb{B}$-morphism $\hat{f} : GA \longrightarrow B$ s.t. the triangle

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & FGA \arrow[d, "F\hat{f}"] \\ & FB \end{tikzcd}$$

commutes.
This η is called the unit of the adjunction.
Hence we have the diagram of unit as follows

$$\begin{array}{ll|l} \mathbb{A} & & \mathbb{B} \\ \hline \begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & FGA \arrow[d, "F\hat{f}"] \\ & FB \end{tikzcd} & & \begin{tikzcd} GA \arrow[d, "\hat{f}"] \\ B \end{tikzcd} \end{array}$$

The diagram above indicates the fact that $\eta : A \longrightarrow FGA$ is the F-universal arrow provided that for given $f : A \longrightarrow FB$ there exist a unique $\mathbb{B}$-morphism $\hat{f} : GA \longrightarrow B$ s.t. the triangle commutes.

(2) A functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be *right adjoint* provided that for every $\mathbb{B}$-object B there exists a G-universal arrow with domain B.
From the definition above it follows that there exist a natural transformation $\xi : FGA \longrightarrow A$, where $F : \mathbb{B} \longrightarrow \mathbb{A}$ is a functor s.t. for given $f' : FB \longrightarrow A$ there exist a unique $\mathbb{B}$-morphism $\hat{f} : B \longrightarrow GA$ s.t the triangle

$$\begin{tikzcd} FGA \arrow[r, "\xi"] & A \\ FB \arrow[u, "F\hat{f}"] \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes
This ξ is called the co-unit of the adjunction.
Hence we have the diagram of co-unit as follows

$$
\begin{array}{c|c}
\mathbb{A} & \mathbb{B} \\
\hline
\begin{tikzcd}
FGA \arrow[r, "\xi"] & A \\
FB \arrow[u, "F\hat{f}"] \arrow[ur, "f'"']
\end{tikzcd}
&
\begin{tikzcd}
GA \\
B \arrow[u, "\hat{f}"]
\end{tikzcd}
\end{array}
$$

The diagram above indicates the fact that $\xi : FGA \longrightarrow A$ is the F-couniversal arrow provided that for given $f' : FB \longrightarrow A$ there exist a unique $\mathbb{B}$-morphism $\hat{f} : GA \longrightarrow B$ s.t. the triangle commutes.

2. Relationships among $\mathscr{L}$-TopSys, $\mathscr{L}$-Top and Frm

Definition 2.1. Let L be a frame. An *$\mathscr{L}$-topological system* is a quadruple $(X, \tilde{A}, \models, P)$, where $(X, \tilde{A})$ is a non-empty L valued fuzzy set (L-fuzzy set), P is a frame and $\models$ is an L- fuzzy relation from X to P such that

(1) $gr(x \models p) \in L$.
(2) $gr(x \models p) \leq \tilde{A}(x)$,
(3) if S is a finite subset of P, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$,
(4) if S is any subset of P, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$.

Note 1: Because of condition 2, $\models$ is a fuzzy relation on the L-fuzzy set $(X, \tilde{A})$ [3].

Note 2: The notion of topological system introduced in [9] was defined by crisp set and crisp relation whereas the notion of fuzzy topological system defined in [4] consists of crisp set and fuzzy relation. In our new setting the notion of $\mathscr{L}$-topological system is defined by L-fuzzy set and L-fuzzy relation.

Note 3: $\mathscr{L}$-topological system is a natural generalization of $\mathscr{F}$-topological system [5].

The notion of continuous map between these $\mathscr{L}$-topological systems is defined as follows:

Definition 2.2. Let $D = (X, \tilde{A}, \models, P)$ and $E = (Y, \tilde{B}, \models', Q)$ be $\mathscr{L}$-topological systems. A *continuous map* $f : D \longrightarrow E$ is a pair (f_1, f_2) where,

(1) $f_1 : (X, \tilde{A}) \longrightarrow (Y, \tilde{B})$ is a proper function (Definition 1.6) from $(X, \tilde{A})$ to $(Y, \tilde{B})$,
(2) $f_2 : Q \longrightarrow P$ is a frame homomorphism and
(3) $gr(x \models f_2(q)) = gr(f_1(x) \models' q)$, for all $x \in X$ and $q \in Q$.

Let us define identity map and composition of two maps as follows:

Definition 2.3. Let $D = (X, \tilde{A}, \models, P)$ be an $\mathscr{L}$-topological system. The *identity map* $I_D : D \longrightarrow D$ is a pair (I_1, I_2) defined by $I_1 : (X, \tilde{A}) \longrightarrow (X, \tilde{A})$ s.t. $I_1(x_1, x_2) = \tilde{A}(x)$ iff $x_1 = x_2$, otherwise $I_1(x_1, x_2) = 0_L$ and $I_2 : P \longrightarrow P$ *is identity morphism of* P. Let $D = (X, \tilde{A}, \models', P)$, $E = (Y, \tilde{B}, \models'', Q)$, $F = (Z, \tilde{C}, \models''', R)$. Let $(f_1, f_2) : D \longrightarrow E$ and $(g_1, g_2) : E \longrightarrow F$ be continuous maps. The *composition* $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by $g_1 \circ f_1 : (X, \tilde{A}) \longrightarrow (Z, \tilde{C})$, $f_2 \circ g_2 : R \longrightarrow P$ i.e. $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$.

Remark: In fact we can show that the identity map and the composition of two continuous maps are indeed continuous maps. Hence we propose the category $\mathscr{L}$-**TopSys** whose objects are $\mathscr{L}$-topological systems and the morphisms are the above mentioned continuous maps. Thus we get the category $\mathscr{L}$-**TopSys** of fuzzy topological systems whose underlying sets are L-fuzzy sets.

Definition 2.4 ([2])**.** The category $\mathscr{L}$-**Top** is defined thus.

- The objects are $\mathscr{L}$-topological spaces on L-fuzzy sets $(X, \tilde{A}, \tau)$, $(Y, \tilde{B}, \tau')$ etc.
- The morphisms are proper functions satisfying the following continuity property: If $f : (X, \tilde{A}, \tau) \longrightarrow (Y, \tilde{B}, \tau')$ and $(Y, \tilde{B_1})$ is a subset of $(Y, \tilde{B})$ such that $(Y, \tilde{B_1}) \in \tau'$ then $(X, f^{-1}(\tilde{B_1})) \in \tau$.
- The identity on $(X, \tilde{A}, \tau)$ is the identity proper function (c.f Definition 1.7) on the L-fuzzy set $(X, \tilde{A})$. That this is an $\mathscr{L}$-**Top** morphism can be proved.
- If $f : (X, \tilde{A}, \tau) \longrightarrow (Y, \tilde{B}, \tau')$ and $g : (Y, \tilde{B}, \tau') \longrightarrow (Z, \tilde{C}, \tau'')$ are morphisms in $\mathscr{L}$-**Top**, their composition $g \circ f$ is the composition of proper functions between L-fuzzy sets. It can be verified that $g \circ f$ is a morphism in $\mathscr{L}$-**Top**.

It is to note that the category $\mathscr{L}$-**Top** is denoted by Fuz$-$Top (L) in [2] when L is a complete Heyting algebra instead of a frame.

Definition 2.5. Frames together with frame homomorphisms form the category **Frm** [4].

The opposite category of frame is known as the category of locale and denoted by $\mathbf{Frm}^{op}$ or **Loc**.

The interrelation among the categories: $\mathscr{L}$-**TopSys**, $\mathscr{L}$-**Top**, **Loc** via some suitable functors shall be established. Let us propose a notion of extent.

Definition 2.6. Let $(X, \tilde{A}, \models, P)$ be an $\mathscr{L}$- topological system and $p \in P$. For each p, its *extent* in $(X, \tilde{A}, \models, P)$ is given by $ext(p) = (X, ext^*(p))$ where $ext^*(p)$ is a mapping from X to L defined by $ext^*(p)(x) = gr(x \models p)$ for all $x \in X$.
i.e. $ext^*(p) : X \longrightarrow L$ such that $ext^*(p)(x) = gr(x \models p)$ for all $x \in X$.
Also $ext(P) = \{(X, ext^*(p))\}_{p \in P} = (X, ext^*P)$ where $ext^*P = \{ext^*p\}_{p \in P}$.

Now the functor Ext is defined as follows:

Definition 2.7. *Ext* is a functor from $\mathscr{L}$-**TopSys** to $\mathscr{L}$-**Top** defined thus.
Ext acts on the object $(X, \tilde{A}, \models', P)$ as $Ext(X, \tilde{A}, \models', P) = (X, \tilde{A}, ext(P))$ and on the morphism (f_1, f_2) as $Ext(f_1, f_2) = f_1$.

Next we define another functor J as follows:

Definition 2.8. J is a functor from $\mathscr{L}$-**Top** to $\mathscr{L}$-**TopSys** defined thus. J acts on the object $(X, \tilde{A}, \tau)$ as $J(X, \tilde{A}, \tau) = (X, \tilde{A}, \in, \tau)$ where "$\in$" is an L- fuzzy relation such that $gr(x \in \tilde{T}) = \tilde{T}(x)$ for $\tilde{T} \in \tau$ and on the morphism f as $J(f) = (f, f^{-1})$.

To make a connection between $\mathscr{L}$-**TopSys** and **Loc** (opposite category of **Frm**) we introduce two functors viz. Lo, S.

Definition 2.9. Lo is a functor from $\mathscr{L}$-**TopSys** to **Loc** defined thus. Lo acts on the object $(X, \tilde{A}, \models, P)$ as $Lo(X, \tilde{A}, \models, P) = P$ and on the morphism (f_1, f_2) as $Lo(f_1, f_2) = f_2$.

Definition 2.10. S is a functor from **Loc** to $\mathscr{L}$-**TopSys** defined thus. S acts on the object P as $S(P) = (Hom(P, L), \tilde{P}, \models_*, P)$, where
$Hom(P, L) = \{frame\ hom\ v : P \longrightarrow L\}$, $\tilde{P}(v) = \bigvee_{p \in P} v(p)$ and $gr(v \models_* p) = v(p)$, and on the morphism f as $S(f) = (\_ \circ f, f)$.

Finally it can be shown that the following theorems hold.

Theorem 2.11. *Ext is the right adjoint to the functor J.*

Proof. We will prove the theorem by presenting the co-unit of the adjunction. Recall that $J(X, \tilde{A}, \tau) = (X, \tilde{A}, \in \tau)$ and $Ext(X, \tilde{A}, \models, P) = (X, \tilde{A}, ext(P))$.
$So, J(Ext(X, \tilde{A}, \models, P)) = (X, \tilde{A}, \in, ext(P))$.
Let us draw the diagram of co-unit-

$\mathscr{L}$-**TopSys**	$\mathscr{L}$- **Top**
$$\begin{tikzcd} J(Ext(X, \tilde{A}, \models, P)) \arrow[r, "\xi_X"] & (X, \tilde{A}, \models, P) \\ J(Y, \tilde{B}, \tau') \arrow[u, "J(f)(=(f_1, f_1^{-1}))"] \arrow[ur, "\hat{f}(=(f_1, f_2))"'] & \end{tikzcd}$$	$$\begin{tikzcd} Ext(X, \tilde{A}, \models, P) \\ (Y, \tilde{B}, \tau') \arrow[u, "f(=f_1)"'] \end{tikzcd}$$

Let us define co-unit by $\xi_X = (id_X, ext')$
i.e.

$$(X, \tilde{A}, \in, ext(P)) \xrightarrow[(id_X, ext')]{\xi_X} (X, \tilde{A}, \models, P)$$

where ext' is a mapping from P to $ext(P)$ such that, $ext'(p) = (X, ext^*(p))$ for all $p \in P$. It can be shown that $(id_X, ext') : J(Ext(X, \tilde{A}, \models, P)) \longrightarrow (X, \tilde{A}, \models, P)$ is indeed a continuous map of $\mathscr{L}$-Topological System as follows.
According to the definition $ext^*(p)(x) = gr(x \models p)$.
Hence $ext'(p)(x) = gr(x \models p)$.
Consequently $gr(x \in ext'(p)) = gr(id_X(x) \models p)$.

Now define f as follows.
Given $(f_1, f_2) : J(Y, \tilde{B}, \tau') \longrightarrow (X, \tilde{A}, \models, P)$, then $f = f_1$.
Now we will prove that the diagram on the left commutes.
Here $J(f) = (f_1, f_1^{-1})$ and $(f_1, f_2) = \xi_X \circ J(f) = (id_X, ext') \circ (f_1, f_1^{-1}) = (id_X \circ f_1, f_1^{-1} \circ ext')$ Clearly $id_X \circ f_1 = f_1$.
Also we have $f_1^{-1} ext'(p) = f_1^{-1}(p) = f_2(p)$. So, $f_2 = f_1^{-1} \circ ext'$.
Hence $\xi_X(= (id_X, ext')) : J(Ext(X, \tilde{A}, \models, P)) \longrightarrow (X, \tilde{A}, \models, P)$ is the co-unit, consequently Ext is the right adjoint to the functor J. □

Theorem 2.12. *Lo is the left adjoint to the functor S.*

Proof. We will prove the theorem by presenting the unit of the adjunction.

Recall that $S(Q) = (Hom(Q, L), \tilde{Q}, \models_*, Q)$ where $gr(v \models_* q) = v(q)$.

Hence $S(Lo(X, \tilde{A}, \models, P)) = (Hom(P, L), \tilde{P}, \models_*, P)$

$\mathscr{L}$-**TopSys**	**Loc**
$\begin{tikzcd} (X,\tilde{A},\models,P) \arrow[r, "\eta"] \arrow[dr, "f(=(f_1,f_2))"'] & S(Lo(X,\tilde{A},\models,P)) \arrow[d, "S\hat{f}"] \\ & S(Q) \end{tikzcd}$	$\begin{tikzcd} Lo(X,\tilde{A},\models,P) \arrow[d, "\hat{f}(=f_2)"] \\ Q \end{tikzcd}$

Then unit is defined by $\eta = (p*, id_P)$

i.e. $$(X,\tilde{A},\models,P) \xrightarrow[(p^*,\,id_P)]{\eta} S(Lo(X,\tilde{A},\models,P))$$

where

$p^* : (X,\tilde{A}) \longrightarrow (Hom(P,L),\tilde{P})$ s.t. for any $x \in \mid \tilde{A} \mid$, $p^*(x)$ is a mapping from P to L and $p^*(x)(p) = gr(x \models p)$. We can show that $(p^*, id_P) : (X,\tilde{A},\models,P) \longrightarrow S(Lo(X,\tilde{A},\models,P))$ is a continuous map of $\mathscr{L}$-Topological System in the following way.

Here it will be enough to show that $gr(x \models id_P(p)) = gr(p^*(x) \models p)$.
We have $gr(x \models p) = p*(x)(p) = gr(p^*(x) \models p)$.

Let us define $\hat{f}$ as follows-
$(f_1, f_2) : (X,\tilde{A},\models,P) \longrightarrow (Hom(P,L),\tilde{P},\models_*,P)$ then $\hat{f} = f_2$[as f_2 is the frame homomorphism].
Recall that $S(\hat{f}) = (- \circ f_2, f_2)$ Now we have to show that the triangle on the left commute.
We have to show that $(f_1, f_2) = (- \circ f_2, f_2) \circ (p^*, id_P) = ((- \circ f_2)p^*, id_P \circ f_2)$
Clearly $f_2 = id_P \circ f_2$.
It is only left to show that $f_1 = (- \circ f_2)p^* = p_x \circ f_2$.
We have for all $q \in Q$

$$\begin{aligned} p^*(x) \circ f_2(q) &= p^*(x)(f_2(q)) \\ &= gr(x \models f_2(q)) \\ &= gr(f_1(x) \models_* q) \\ &= f_1(x)(q). \end{aligned}$$

So, $(\_ \circ f_2)p^* = f_1$. □

Theorem 2.13. *$Ext \circ S$ is the right adjoint to the functor $Lo \circ J$.*

Proof. Proof follows from Theorem 2.11 and Theorem 2.12. □

3. Relationships among FuzzTopSys, FuzzTop and Frm

Definition 3.1. A *Fuzz topological system* is a quintuple $(X, L, \tilde{A}, \models, P)$, where $(X, L, \tilde{A})$ is a Fuzz-object [2] (i.e. X is a non-empty set, L is a frame, $\tilde{A}$ is a map from X to L), P is a frame and $\models$ is an L- fuzzy relation between X and P such that

(1) $gr(x \models p) \in L$.
(2) $gr(x \models p) \leq \tilde{A}(x)$,

(3) if S is a finite subset of P, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$,
(4) if S is any subset of P, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$.

Note 1: Because of condition 2, $\models$ is a fuzzy relation on the L-fuzzy set $(X, \tilde{A})$ [3].

Note 2: The value set L of $\mathscr{L}$-topological systems is fixed but in the case of Fuzz topological system the value set L may vary. Thus we can consider an $\mathscr{L}$-topological system as a specific instant of Fuzz topological system.

Note 3: In [2] to define Fuzz-object $(X, L, \tilde{A})$, L is taken as a complete Heyting algebra whereas in our work we consider L, a frame.

The notion of continuous map between these Fuzz topological systems is defined as follows:

Definition 3.2. Let $D = (X, L, \tilde{A}, \models, P)$ and $E = (Y, M, \tilde{B}, \models', Q)$ be Fuzz topological systems. A *continuous map* $f : D \longrightarrow E$ is a triple (f, ϕ, g) where,

(1) $(f, \phi) : (X, L, \tilde{A}) \longrightarrow (Y, M, \tilde{B})$ such that
  a) ϕ is a relation from L to M such that $\phi^{-1} : M \longrightarrow L$ is a map preserving finite meet and arbitrary join,
  b) $f : X \times Y \longrightarrow L$ is a map such that $f(x, y) \leq \tilde{A}(x) \wedge \phi^{-1}\tilde{B}(y)$, for all $x \in X$, $y \in Y$ and for any a in $\mid \tilde{A} \mid$, there exist unique b in $\mid \tilde{B} \mid$ with $f(a, b) = \tilde{A}(a)$ and $f(a, b') = 0_L$ with $b' \neq b$ in $\mid \tilde{B} \mid$.
(2) $g : Q \longrightarrow P$ is a frame homomorphism and
(3) $gr(x \models g(q)) = \bigvee_{y \in Y}[\phi^{-1}(gr(y \models' q)) \wedge f(x, y)]$, for all $x \in X$ and $q \in Q$.

Let us define identity map and composition of two maps.

Definition 3.3. Let $D = (X, L, \tilde{A}, \models, P)$ be a Fuzz topological system. The *identity map* $I_D : D \longrightarrow D$ is a triple (I_A, I_L, I_P) defined by $I_A : X \times X \longrightarrow L$ s.t. $I_A(x_1, x_2) = \tilde{A}(x)$ iff $x_1 = x_2$, otherwise $I_A(x_1, x_2) = 0_L$, $I_L : L \longrightarrow L$ is identity morphism of L and $I_P : P \longrightarrow P$ is identity morphism of P. Let $D = (X, L, \tilde{A}, \models', P)$, $E = (Y, M, \tilde{B}, \models'', Q)$, $F = (Z, N, \tilde{C}, \models''', R)$. Let $(f, \phi, g) : D \longrightarrow E$ and $(f_1, \phi_1, g_1) : E \longrightarrow F$ be continuous maps. The *composition* $(f_1, \phi_1, g_1) \circ (f, \phi, g) : D \longrightarrow F$ is defined by $f_1 \circ f : X \times Z \longrightarrow L$ such that $f_1 \circ f(x, z) = \bigvee_{y \in Y}[f(x, y) \wedge \phi^{-1}(f_1(y, z))]$ for all $x \in X$ and $z \in Z$, $\phi_1\phi$ the relation composite of ϕ_1, ϕ.

Remark: In fact we can show that the identity map is a continuous map and the composition of two continuous maps is also so. Hence we propose the category **FuzzTopSys** whose objects are Fuzz topological systems and the morphisms are the above mentioned continuous maps.

Definition 3.4 ([2])**.** Let $(X, L, \tilde{A})$ be a Fuzz-object and τ a collection of maps from X into L such that

(1) if $U \in \tau$, $U(x) \leq \tilde{A}(x)$, for all $x \in X$;
(2) $\tilde{\phi}$ and $\tilde{A}$ are in τ, where $\tilde{\phi} : X \longrightarrow L$ is such that $\tilde{\phi}(x) = 0_L$, for all $x \in X$, where 0_L is the least element of the frame L;
(3) τ is closed under finite infima and arbitrary suprema.

Then $(X, L, \tilde{A}, \tau)$ is a *Fuzz topological space.*

Definition 3.5 ([2])**.** The category **FuzzTop** is defined thus.

- The objects are Fuzz topological spaces.
- The morphisms are pairs $(f, \phi) : (X, L, \tilde{A}, \tau) \longrightarrow (X_1, L_1, \tilde{A_1}, \tau_1)$ satisfying the following properties:
  a) ϕ is a relation from L to L_1 such that $\phi^{-1} : L_1 \longrightarrow L$ is a frame homomorphism.
  b) $f : X \times X_1 \longrightarrow L$ is a map such that $f(x, y) \leq \tilde{A}(x) \wedge \phi^{-1}\tilde{A_1}(y)$, for all $x \in X$, $y \in X_1$ and there exist unique b in $\mid \tilde{A_1} \mid$ with $f(a, b') = \tilde{A}(a)$ for $b' = b$ in $\mid \tilde{A_1}$, otherwise $f(a, b') = 0_L$.
- If $V \in \tau_1$, then $U \in \tau$ where $U(x) = \bigvee_{y \in X_1}[f(x, y) \wedge \phi^{-1}V(y)]$, for all $x \in X$.
- If $(f, \phi) : (X, L, \tilde{A}, \tau) \longrightarrow (X_1, L_1, \tilde{A_1}, \tau_1)$ and $(g, \psi) : (X_1, L_1, \tilde{A_1}, \tau_1) \longrightarrow (X_2, L_2, \tilde{A_2}, \tau_2)$ are morphisms in **FuzzTop**, their composite $(g, \psi) \circ (f.\phi)$ is that of the Fuzz-morphisms $(f, \phi) : (X, L, \tilde{A}) \longrightarrow (X_1, L_1, \tilde{A_1})$ and $(g, \psi) : (X_1, L_1, \tilde{A_1}) \longrightarrow (X_2, L_2, \tilde{A_2})$, viz. $(g \circ f, \psi\phi)$ with $g \circ f : X \times X_2 \longrightarrow L$ as $g \circ f(x, z) = \bigvee_{y \in X_1}[f(x, y) \wedge \phi^{-1}g(y, z)]$, for all $x \in X$, $z \in X_2$ and $\psi\phi$ is the relational composite of ϕ and psi.
- The identity on $(X, L, \tilde{A}, \tau)$ is the identity (i_A, i_L) on the Fuzz-object $(X, L, \tilde{A})$. This is a **FuzzTop** morphism can be proved.

The interrelation between the categories **FuzzTopSys** and **FuzzTop** is now established via some suitable functors. First we propose a notion of extent.

Definition 3.6. Let $(X, L, \tilde{A}, \models, P)$ be a Fuzz topological system and $p \in P$. For each p, its *extent* in $(X, L, \tilde{A}, \models, P)$ is given by $ext(p) : X \longrightarrow L$ such that $ext(p)(x) = gr(x \models p)$ for any $x \in X$, and $ext(P) = \{ext(p)\}_{p \in P}$.

Now the functor Ext is defined as follows:

Definition 3.7. Ext is a functor from **FuzzTopSys** to **FuzzTop** defined thus. Ext acts on the object $(X, L, \tilde{A}, \models', P)$ as $Ext(X, L, \tilde{A}, \models', P) = (X, L, \tilde{A}, ext(P))$ and on the morphism (f, ϕ, g) as $Ext(f, \phi, g) = (f, \phi)$.

Next we define another functor J as follows:

Definition 3.8. J is a functor from **FuzzTop** to **FuzzTopSys** defined thus. J acts on the object $(X, L, \tilde{A}, \tau)$ as $J(X, L, \tilde{A}, \tau) = (X, L, \tilde{A}, \in, \tau)$ where $gr(x \in \tilde{T}) = \tilde{T}(x)$ for $\tilde{T} \in \tau$ and on the morphism (f, ϕ) as $J((f, \phi)) = (f, \phi, f_\phi^{-1})$, where $f_\phi^{-1}(\tilde{T})(x) = \bigvee_{y \in Y}[f(x, y) \wedge \phi^{-1}\tilde{T_1}(y)]$ for all $x \in X$ and $\tilde{T_1} \in \tau_1$.

By routine check it can be shown that Ext and J are indeed functors. Finally it can be shown that the following theorem hold.

Theorem 3.9. *Ext is the right adjoint to the functor J.*

Proof. We will prove the theorem by presenting the co-unit of the adjunction. Recall that $J(X, L, \tilde{A}, \tau) = (X, L, \tilde{A}, \in \tau)$ and $Ext(X, L, \tilde{A}, \models, P) = (X, L, \tilde{A}, ext(P))$.
$So, J(Ext(X, L, \tilde{A}, \models, P)) = (X, L, \tilde{A}, \in, ext(P))$.
Let us draw the diagram of co-unit-

FuzzTopSys	**FuzzTop**

$$\begin{tikzcd} J(Ext(X,L,\tilde{A},\models,P)) \arrow[r, "\xi_X"] & (X,L,\tilde{A},\models,P) \\ J(Y,M,\tilde{B},\tau') \arrow[u, "J(f,\phi)(=(f,\phi,f_\phi^{-1}))"] \arrow[ur, "(f,\phi,g)"'] & \end{tikzcd}$$

$$\begin{tikzcd} Ext(X,L,\tilde{A},\models,P) \\ (Y,M,\tilde{B},\tau') \arrow[u, "(f,\phi)"'] \end{tikzcd}$$

Let us define co-unit by $\xi_X = (i_A, i_L, ext^*)$
i.e.

$$(X, L, \tilde{A}, \in, ext(P)) \xrightarrow[(i_A,\, i_L,\, ext^*)]{\xi_X} (X, L, \tilde{A}, \models, P)$$

where $i_A : X \times X \longrightarrow L$, $i_L : L \longrightarrow L$ and ext^* is a mapping from P to $ext(P)$ such that, $ext^*(p) = ext(p)$ for all $p \in P$. It can be shown that $(i_A, i_L, ext^*) : J(Ext(X, L, \tilde{A}, \models, P)) \longrightarrow (X, L, \tilde{A}, \models, P)$ is indeed a continuous map of Fuzz topological system as follows.
Now we will prove that the diagram on the left commutes.
Here $J(f, \phi) = (f, \phi, f_\phi^{-1})$ and $(f, \phi, g) = \xi_X \circ J(f, \phi) = (i_A, i_L, ext^*) \circ (f, \phi, f_\phi^{-1}) = (i_A \circ f, i_L \circ \phi, f_\phi^{-1} \circ ext^*)$.
We have $i_A \circ f : Y \times X \longrightarrow M$ such that
$i_A \circ f(y, x) = \bigvee_{x' \in X}[f(y, x') \wedge \phi^{-1}(i_A(x', x))] = f(y, x) \wedge \phi^{-1}\tilde{A}(x)$.
Now $f(y, x) \leq \tilde{B}(y) \wedge \phi^{-1}\tilde{A}(x) \leq \phi^{-1}\tilde{A}(x)$.
Hence $f(y, x) \wedge \phi^{-1}\tilde{A}(x) = f(y, x)$.
Therefore $i_A \circ f(y, x) = f(y, x)$ and consequently $i_A \circ f = f$.
Clearly $i_L \circ \phi = \phi$.
Now as (i_A, i_L, ext^*) in continuous so $gr(x \in ext^*(p)) = \bigvee_{x' \in X}[i_L^{-1}(gr(x' \models p)) \wedge i_A(x, x')]$.
So, $ext^*(p)(x) = gr(x \models p) \wedge \tilde{A}(x) = gr(x \models p)$.
Now (f, ϕ, g) is continuous and hence $gr(y \in g(p)) = \bigvee_{x \in X}[\phi^{-1}(gr(x \models p)) \wedge f(y, x)]$.
So, $g(p)(y) = \bigvee_{x \in X}[\phi^{-1}(gr(x \models p)) \wedge f(y, x)]$.
Hence we get $f_\phi^{-1} ext(p)(y) = \bigvee_{x \in X}[f(y, x) \wedge \phi^{-1}(ext(p)(x))] = \bigvee_{x \in X}[\phi^{-1}(gr(x \models p)) \wedge f(y, x)] = g(p)(y)$.
Hence $f_\phi^{-1} \circ ext^*(p)(y) = g(p)(y)$.
So, $g = f_\phi^{-1} \circ ext^*$.
Hence $\xi_X(= (i_A, i_L, ext^*)) : J(Ext(X, L, \tilde{A}, \models, P)) \longrightarrow (X, L, \tilde{A}, \models, P)$ is the co-unit, consequently Ext is the right adjoint to the functor J. $\square$

Note that in this section localification i.e existence of adjunction between FuzzTopSys and Loc×Loc is not established. This can be considered as an interesting open question.

4. Subcategories of α-cuts

In this section we will construct two kinds of subsystems of some $\mathscr{L}$-topological systems. To do so here we will deal with some special kind of subsets of the fuzzy set. We will construct the subsets using the concept of α-cut and fuzzy α-cut of fuzzy set respectively. For the notion of classical α-cut of a fuzzy set we refer to [6]. Here we introduce a notion of fuzzy α-cut of a fuzzy set.

Definition 4.1 (α-cut of a fuzzy set)**.** Let $(X, \tilde{A})$ be an L-fuzzy set. Then for $\alpha \in L$, where L is a frame, the α-cut of $(X, \tilde{A})$ is the ordinary set $\{x \in X \mid \tilde{A}(x) \geq \alpha\}$.

Definition 4.2 (Strict α-cut of a fuzzy set)**.** Let $(X, \tilde{A})$ be an L-fuzzy set. Then for $\alpha \in L$, where L is a frame, the strict α-cut of $(X, \tilde{A})$ is the ordinary set $\{x \in X \mid \tilde{A}(x) > \alpha\}$.

Definition 4.3 (Fuzzy α-cut of a fuzzy set)**.** Let $(X, \tilde{A})$ be an L-fuzzy set. Then for $\alpha \in L$, where L is a frame, the fuzzy α-cut of $(X, \tilde{A})$ is the fuzzy subset $(X, \tilde{A_\alpha})$ such that $\tilde{A_\alpha}$ is defined as follows:

$$\begin{aligned} \tilde{A_\alpha}(x) &= \tilde{A}(x) \ if\ \tilde{A}(x) \geq \alpha \\ &= 0_L \quad otherwise. \end{aligned}$$

4.1. **TopSys$_\alpha$.** Let $(X, \tilde{A}, \models, P)$ be an $\mathscr{L}$-topological system. Let us consider the triple $(\{x \in X \mid \tilde{A}(x) > \alpha\}, \models_\alpha, P)$, where $\models_\alpha \subseteq X \times P$ such that $x \models_\alpha p$ iff $gr(x \models p) > \alpha$. It can be shown that the triple forms a topological system and consequently a subsystem of $\mathscr{L}$-topological system.

Thus we get subsystems for each $\alpha < 1_L$ (1_L is the top element of L). Now for $\alpha > \alpha'$, $\{x \in X \mid \tilde{A}(x) > \alpha\}$ is a subset of $\{x \in X \mid \tilde{A}(x) > \alpha'\}$ and hence $(\{x \in X \mid \tilde{A}(x) > \alpha\}, \models_\alpha, P)$ is a subsystem of $(\{x \in X \mid \tilde{A}(x) > \alpha'\}, \models_{\alpha'}, P)$. Hence we get chains of subsystems of $\mathscr{L}$-topological system.

The restriction of a continuous function between two $\mathscr{L}$-topological systems is a continuous function between corresponding subsystems for each $\alpha \in L$.

The above subsystems for a fixed $\alpha < 1_L (\in L)$ together with continuous maps form a category called **TopSys**$_\alpha$, which is a subcategory of $\mathscr{L}$-**TopSys**. Here we will get chains of subcategories of $\mathscr{L}$-**TopSys** by changing the values of α.

It may be noted that for a linear L we will get only one chain.

4.2. **Top$_\alpha$ vs. TopSys$_\alpha$.** Let $(X, \tilde{A}, \tau)$ be an $\mathscr{L}$-topological space and take strict α-cut of $(X, \tilde{A})$ i.e., $\{x \in X \mid \tilde{A}(x) > \alpha\}$. Let τ_α be defined by $\tau_\alpha = \{\{x \in X \mid T_i(x) > \alpha\} \mid \tilde{T_i} \in \tau\}$. Then $(\{x \in X \mid \tilde{A}(x) > \alpha\}, \tau_\alpha)$ also form a topological space and called the topological subspace.

For any fixed $\alpha \in L$, topological subspaces together with continuous maps forms a category, which is a subcategory of the category **Top**, called **Top**$_\alpha$.

By routine check it can be shown that the restriction of the functors (Ext and J) between **TopSys** and **Top** are adjoint functors between **TopSys**$_\alpha$ and **Top**$_\alpha$, for each $\alpha \in L$.

4.3. $\mathscr{L}$**-TopSys$_\alpha$.** Let $(X, \tilde{A}, \models, P)$ be an $\mathscr{L}$-topological system. Let us consider the quadruple $(X, \tilde{A_\alpha}, \models_\alpha, P)$, where $(X, \tilde{A_\alpha})$ is the fuzzy α-cut of $(X, \tilde{A})$, $\models_\alpha$ is an

L-fuzzy relation between X and L such that $gr(x \models_\alpha p) = gr(x \models p)$ for $\tilde{A}(x) \geq \alpha$ and $gr(x \models_\alpha p) = 0_L$ for $\tilde{A}(x) < \alpha$. It can be shown that the quadruple forms an $\mathscr{L}$-topological system and consequently an $\mathscr{L}$-topological subsystem of $\mathscr{L}$-topological system. Hence for each $\alpha \in L$ we will get $\mathscr{L}$-topological subsystems. Furthermore as for $\alpha > \alpha'$, $(X, \tilde{A_\alpha})$ is a fuzzy subset of $(X, \tilde{A_{\alpha'}})$, we will get chains of such $\mathscr{L}$-topological subsystems.

A continuous map between two $\mathscr{L}$-topological subsystems, say $(X, \tilde{A_\alpha}, \models_\alpha, P)$ and $(Y, \tilde{B_\alpha}, \models'_\alpha, Q)$, is the restriction of a continuous map between the $\mathscr{L}$-topological systems $(X, \tilde{A}, \models, P)$ and $(Y, \tilde{B}, \models', Q)$.

It can be shown that for fixed $\alpha \in L$, $\mathscr{L}$-topological subsystems together with the above mentioned continuous maps form a subcategory of $\mathscr{L}$-**TopSys**. Thus we will get chains of such subcategories of $\mathscr{L}$-**TopSys** by changing the values of α in L. For a fixed $\alpha \in L$, let us call the corresponding subcategory by $\mathscr{L}$-$\mathbf{TopSys}_\alpha$.

4.4. **$\mathscr{L}$-$\mathbf{Top}_\alpha$ vs. $\mathscr{L}$-$\mathbf{TopSys}_\alpha$.** Let $(X, \tilde{A}, \tau)$ be an $\mathscr{L}$-topological space and take fuzzy α-cut of $(X, \tilde{A})$ i.e., $(X, \tilde{A_\alpha})$. Let τ' be defined by $\tau' = \{(X, \tilde{T'}) \mid \tilde{T'} = \tilde{A_\alpha} \cap \tilde{T}, \tilde{T} \in \tau\}$. Then $(X, \tilde{A_\alpha}, \tau')$ also form an $\mathscr{L}$-topological space and called the $\mathscr{L}$-topological subspace.

For any fixed $\alpha \in L$, $\mathscr{L}$-topological subspaces together with continuous maps form a category, which is a subcategory of the category $\mathscr{L}$-**Top**, called $\mathscr{L}$-$\mathbf{Top}_\alpha$.

By routine check it can be shown that the restriction of the functors (Ext and J) between $\mathscr{L}$-**TopSys** and $\mathscr{L}$-**Top** are adjoint functors between $\mathscr{L}$-$\mathbf{TopSys}_\alpha$ and $\mathscr{L}$-$\mathbf{Top}_\alpha$, for each $\alpha \in L$.

5. Conclusion

This paper provides a generalization of the notion of $[0,1]$-**Top**, $[0,1]$-**TopSys** [4], **Loc** and their categorical relationships. Furthermore, two ways of constructing subspaces and subsystems of an $\mathscr{L}$-topological space and an $\mathscr{L}$-topological system respectively are provided. Also a concept of fuzzy α-cut is introduced.

In our future work we shall consider

(1) the generalization of $\mathbf{TopSys}_\alpha$ by taking strict α-cut of the set and β-cut for the relation, for α, $\beta < i_L$;
(2) the relationship of α, β in this context in details;
(3) the study of $\mathbf{Top}_\alpha$, $\mathbf{TopSys}_\alpha$, $\mathscr{L}$-$\mathbf{Top}_\alpha$, $\mathscr{L}$-$\mathbf{TopSys}_\alpha$ in details and
(4) the notion of many valued and fuzzy geometric logic as a generalization of geometric logic.

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