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$L-$fuzzy ideals in $\Gamma-$semiring

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Abstract. We introduce the notion of a $L-$fuzzy $\Gamma-$subsemiring, $L-$fuzzy ideal, normal $L-$fuzzy ideal, $L-$fuzzy k ideal, $L-$fuzzy maximal ideal in $\Gamma-$semiring, where L is a complemented distributive lattice and study some of their properties. We prove that if μ is a maximal $L-$ fuzzy ideal of $\Gamma-$semiring M then M_μ is a maximal ideal of $\Gamma-$semiring M.

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1. Introduction

The notion of a semiring was first introduced by H. S. Vandiver [12] in 1934 but semirings had appeared in studies on the theory of ideals of rings. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring. The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a Γ-ring was introduced by N. Nobusawa [9] in 1964. In 1981, M. K. Sen [11] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [4] in 1932, Lister [6] introduced the notion of a ternary ring and Dutta & Kar [1] introduced the notion of a ternary semiring which is a generalization of ternary ring and semiring. In 1995, M. Murali Krishna Rao [7] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring,

A partially ordered set (poset) is a pair $(X, \leq)$, where X is a non-empty set and $\leq$ is a partial order (a reflexive, transitive and antisymmetric binary relation) on X. For any subset A of X and $x \in X$, x is called a lower bound (upper bound) of A if $x \leq a$ ($a \leq x$ respectively) for all $a \in A$. A poset $(X, \leq)$ is called a lattice if every non-empty finite subset of X has greatest lower bound (glb or infimum) and least upper bound (lub or supremum) in X. If $(X, \leq)$ is a lattice and, for any $a, b \in X$, if we define $a \wedge b =$ infimum $\{a, b\}$ and $a \vee b =$ supremum $\{a, b\}$, then $\wedge$ and $\vee$ are binary operations on X which are commutative, associative and idempotent and satisfy the absorption laws $a \wedge (a \vee b) = a = a \vee (a \wedge b)$. Conversely, any algebraic system $(X, \wedge, \vee)$ satisfying the above properties becomes a lattice in which the partial order is defined by $a \leq b \Longleftrightarrow a = a \wedge b \Longleftrightarrow a \vee b = b$. A lattice $(X, \wedge, \vee)$ is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in X$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in X$). A lattice $(X, \wedge, \vee)$ is called a bounded lattice it has the smallest element 0 and largest element 1; that is, there are elements 0 and 1 in X, such that $0 \leq x \leq 1$ for all $x \in X$. A partially ordered set in which every subset has infimum and supremum is called a complete lattice. Two elements a, b of a bounded lattice $(L, \wedge, \vee, 0, 1)$ are complements if $a \wedge b = 0, a \vee b = 1$. In this case each of a, b is the complement of the other. A complemented lattice is a bounded lattice in which every element has a complement.

The fuzzy set theory was developed by L. A. Zadeh [13] in 1965. In 1982, W. J. Liu [5] defined and studied fuzzy subrings as well as fuzzy ideals in rings. In 1988, Zhang [14] studied prime $L-$fuzzy ideals in rings where L is completely distributive lattice. Jun et al. [3] studied fuzzy maximal ideals in $\Gamma-$near rings. The concept of $L-$fuzzy ideal and normal $L-$fuzzy ideal in semirings were studied by Jun, Neggers and Kim in [2]. Y. B. Jun et al. [2] studied normal complete distributive lattice fuzzy ideal in semirings, whereas in this paper we study normal complemented distributive lattice fuzzy ideal in $\Gamma-$semrings. In this paper, we introduce the notion of a $L-$fuzzy $\Gamma-$subsemiring , $L-$fuzzy ideal, normal $L-$fuzzy ideal, $L-$fuzzy k ideal, $L-$fuzzy maximal ideal in $\Gamma-$semiring, where L is a complemented distributive lattice and study some of their properties. We prove that if μ is a maximal $L-$ fuzzy ideal of $\Gamma-$semiring M then M_μ is a maximal ideal of $\Gamma-$semiring M.

2. Preliminaries

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1 ([7])**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i). Addition is a commutative operation.
(ii). Multiplication distributes over addition both from the left and from the right.
(iii). There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

Definition 2.2 ([7]). Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$,

then M is called a $\Gamma-$semiring.

Definition 2.3 ([7]). A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.

Example 2.4 ([7]). Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication

Example 2.5 ([7]). Let M be the additive seimgroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

Definition 2.6 ([7]). Let M be a $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 2.7 ([7]). Let M be a $\Gamma-$semiring. A subset A of M is called a left (right) ideal of M if A is closed under addition and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$. A is called an ideal of M if it is both a left ideal and right ideal.

Definition 2.8 ([13]). Let M be a non-empty set, a mapping $f : M \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.9 ([13]). Let f be a fuzzy subset of a non-empty subset M, for $t \in [0, 1]$ the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called level subset of M with respect to f.

Definition 2.10 ([8]). Let M be a $\Gamma-$semiring . A fuzzy subset μ of M is said to be a fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions
(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.11 ([8]). A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x))$.

Definition 2.12 ([8]). A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if for all $x, y \in M, \alpha \in \Gamma$
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$.

Definition 2.13 ([7]). An ideal I of a $\Gamma-$semiring M is called k ideal if for all $x, y \in M,\ x + y \in I, y \in I \Rightarrow x \in I$.

Definition 2.14 ([13]). A fuzzy subset $\mu : M \to [0, 1]$ is non-empty if μ is not the constant function.

Definition 2.15 ([13]). For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition 2.16 ([8]). Let f and g be fuzzy subsets of $\Gamma-$semiring M. Then $f \circ g$, $f+g$, $f \cup g$, and $f \cap g$ are defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=x\alpha y} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise.} \end{cases} ; f+g(z) = \begin{cases} \sup\limits_{z=x+y} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise} \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\} \;\; ; \; f \cap g(z) = min\{f(z), g(z)\}$$

$x, y \in M, \alpha \in \Gamma,$ for all $z \in M$.

Definition 2.17 ([7]). A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be a $\Gamma-$semiring homomorphism if $f(a+b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

Definition 2.18 ([8]). Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.19 ([8]). A fuzzy ideal f of a $\Gamma-$semiring M with zero 0 is said to be a $k-$fuzzy ideal of M if $f(x+y) = f(0)$ and $f(y) = f(0) \Rightarrow f(x) = f(0),$ for all $x, y \in M$.

Definition 2.20 ([8]). A fuzzy ideal f of a $\Gamma-$semiring M is said to be a fuzzy $k-$ideal of M if $f(x) \geq min\{f(x+y), f(y)\},$ for all $x, y \in M$.

3. $L-$fuzzy ideals in $\Gamma-$semiring

In this section, we introduce the notion of a $L-$fuzzy $\Gamma-$subsemiring , $L-$fuzzy ideal, normal $L-$fuzzy ideal, $L-$fuzzy k ideal and $L-$fuzzy maximal ideal in Γ-semiring. Throughout this paper $L = (L, \leq, \wedge, \vee)$ is a complemented distributive lattice and $\Gamma-$semiring M is a $\Gamma-$semiring M with 0.

Definition 3.1. Let M be a $\Gamma-$semiring. A mapping $\mu : M \to L$ is called a $L-$fuzzy subset of $\Gamma-$semiring M, where L is a complemented distributive lattice.

Definition 3.2. A $L-$fuzzy subset μ of a $\Gamma-$semiring M is called a $L-$fuzzy $\Gamma-$subsemiring of M if

(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\},$ for all $x, y \in M, \alpha \in \Gamma$.

Definition 3.3. A $L-$fuzzy $\Gamma-$subsemiring of a $\Gamma-$semiring M is called a $L-$fuzzy left (right) ideal of M if

$$\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big).$$

If μ is a fuzzy left and a fuzzy right ideal of $\Gamma-$semiring M then μ is called a $L-$fuzzy ideal of M.

Theorem 3.4. *Let μ be a $L-$fuzzy ideal of $\Gamma-$semiring M. Then $\mu(x) \leq \mu(0)$ for all $x \in M$.*

Proof. Let $x \in M, \alpha \in \Gamma$. $\mu(0) = \mu(0\alpha x) \geq \mu(x)$. Therefore $\mu(x) \leq \mu(0)$, for all $x \in M$. □

Theorem 3.5. *Let M be a $\Gamma-$semiring. μ is a $L-$fuzzy left ideal of M if and only if for any $t \in L$ such that $\mu_t \neq \phi, \mu_t$ is a left ideal of $\Gamma-$semiring M.*

Proof. Let μ be a $L-$fuzzy left ideal of $\Gamma-$semiring M and $t \in L$ such that $\mu_t \neq \phi$.

$$\begin{aligned} \text{Let } x, y \in \mu_t \Rightarrow & \mu(x), \mu(y) \geq t \\ \Rightarrow & \mu(x+y) \geq min\{\mu(x), \mu(y)\} \geq t \\ \Rightarrow & x + y \in \mu_t. \end{aligned}$$

Let $x \in M, y \in \mu_t, \alpha \in \Gamma$, $\mu(x\alpha y) \geq \mu(y) \geq t$. Then $x\alpha y \in \mu_t$. Therefore μ_t is a left ideal of $\Gamma-$semiring M.

Conversely suppose that μ_t is a left ideal of $\Gamma-$semiring M. Let $x, y \in M$ and $t = min\{\mu(x), \mu(y)\}$. Then $\mu(x), \mu(y) \geq t \Rightarrow x, y \in \mu_t$

$$\begin{aligned} &\Rightarrow x + y \in \mu_t \\ &\Rightarrow \mu(x+y) \geq t \\ &\Rightarrow \mu(x+y) \geq min\{\mu(x), \mu(y)\}. \end{aligned}$$

Let $x, y \in M$, $\mu(y) = s \Rightarrow y \in \mu_s$

$$\begin{aligned} &\Rightarrow x\alpha y \in \mu_s \\ &\Rightarrow \mu(x\alpha y) \geq s = \mu(y). \end{aligned}$$

Therefore μ is a $L-$fuzzy left ideal. □

The proof of the following theorem follows from above theorem.

Theorem 3.6. *Let M be a $\Gamma-$semiring. μ is a $L-$fuzzy ideal of M if and only if for $t \in L$ such that $\mu_t \neq \phi$, μ_t is an ideal of $\Gamma-$semiring M.*

Theorem 3.7. *Let M be a $\Gamma-$semiring and $M_\mu = \{x \in M \mid \mu(x) \geq \mu(0)\}$. If μ is a $L-$fuzzy ideal of M then M_μ is an ideal of $\Gamma-$semiring.*

Proof. Let μ be a $L-$fuzzy ideal of $\Gamma-$semiring M and $x, y \in M_\mu$.

$$\begin{aligned} &\Rightarrow \mu(x) \geq \mu(0), \mu(y) \geq \mu(0) \\ &\Rightarrow \mu(x+y) \geq min\{\mu(x), \mu(y)\} \geq \mu(0) \\ &\Rightarrow x + y \in M_\mu \\ \text{Now } &\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\} \geq \mu(0) \\ &\Rightarrow x\alpha y \in M_\mu. \\ \text{Let } &x \in M_\mu, y \in M, \alpha \in \Gamma \\ &\Rightarrow \mu(x) \geq \mu(0) \\ &\Rightarrow \mu(y\alpha x) \geq \mu(x) \geq \mu(0) \\ &\Rightarrow y\alpha x \in M_\mu. \end{aligned}$$

Similarly $x\alpha y \in M_\mu$. Hence M_μ is an ideal of $\Gamma-$semiring M. □

Theorem 3.8. *Let M and S be $\Gamma-$semirings and $\psi : M \to S$ be an onto homomorphism. If μ is a $L-$fuzzy ideal of S then the pre image of μ under ψ is a $L-$fuzzy ideal of M.*

Proof. Let μ be a $L-$fuzzy ideal of S and γ be the pre image of μ under ψ. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}\gamma(x+y) =& \mu(\psi(x+y))\\ =& \mu(\psi(x)+\psi(y))\\ \geq& min\{\mu(\psi(x)), \mu(\psi(y))\}\\ =& min\{\gamma(x), \gamma(y)\}\\ \text{and } \gamma(x\alpha y) =& \mu(\psi(x\alpha y))\\ =& \mu(\psi(x)\alpha\psi(y))\\ \geq& min\{\mu(\psi(x)), \mu(\psi(y))\}\\ =& min\{\gamma(x), \gamma(y)\}\end{aligned}$$

Hence γ is a $L-$fuzzy subsemiring of $\Gamma-$semiring M.
Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}\gamma(x\alpha y) =& \mu(\psi(x\alpha y))\\ =& \mu(\psi(x)\alpha\psi(y))\\ \geq& \mu\{\psi(x)\} = \gamma(x).\end{aligned}$$

Therefore γ is a $L-$fuzzy left ideal of $\Gamma-$semiring M. Similarly we can prove γ is a $L-$fuzzy right ideal of $\Gamma-$semiring M. Hence γ is a $L-$fuzzy ideal of M. □

Theorem 3.9. *Let M be a $\Gamma-$semiring. If A is an ideal of $\Gamma-$semiring M then there exists a $L-$ fuzzy ideal μ of M such that $\mu_t = A$, for some $t \in L$.*

Proof. Suppose A is an ideal of $\Gamma-$semiring M and $t \in L$. We define $L-$fuzzy subset of M by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ $\mu_t = A$. Let $s \in L$, we have

$$\mu_s = \begin{cases} M, & \text{if } s = 0 \\ A, & \text{if } 0 < s \leq t \\ \phi, & \text{otherwise.} \end{cases}$$

Hence every non-empty subset μ_s of μ is an ideal of $\Gamma-$semiring M. By Theorem 3.5., μ is a $L-$fuzzy ideal of $\Gamma-$semiring M. □

Corollary 3.10. *If A is an ideal of $\Gamma-$semiring M then X_A is a $L-$fuzzy ideal of $\Gamma-$semiring M.*

Theorem 3.11. *Let μ and γ be two $L-$fuzzy ideals of $\Gamma-$semiring M. Then $\mu \cap \gamma$ is a $L-$fuzzy ideal of $\Gamma-$semiring M.*

Proof. Let $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned} \mu \cap \gamma(a+b) &= min\{\mu(a+b), \gamma(a+b)\} \\ &\geq min\big\{min\{\mu(a), \mu(b)\}, min\{\gamma(a), \gamma(b)\}\big\} \\ &= min\big\{min\{\mu(a), \gamma(a)\}, min\{\mu(b), \gamma(b)\}\big\} \\ &= min\{\mu \cap \gamma(a),\ \mu \cap \gamma(b)\}. \\ \mu \cap \gamma(a\alpha b) &= min\{\mu(a\alpha b), \gamma(a\alpha b)\} \\ &\geq min\big\{max\{\mu(a), \mu(b)\}, max\{\gamma(a), \gamma(b)\}\big\} \\ &= max\big\{min\{\mu(a), \gamma(a)\}, min\{\mu(b), \gamma(b)\}\big\} \\ &= max\{\mu \cap \gamma(a),\ \mu \cap \gamma(b)\}. \end{aligned}$$

Hence $\mu \cap \gamma$ is a $L-$fuzzy ideal of $\Gamma-$semiring M. □

Definition 3.12.

(i) A $L-$fuzzy ideal μ of $\Gamma-$semiring M is called $L-$fuzzy k ideal of M if $\mu(x) \geq min\{\mu(x+y), \mu(y)\},$ for all $x, y \in M$.

(ii) A $L-$fuzzy ideal μ of $\Gamma-$semiring M is called $L-k$ fuzzy ideal of M if $\mu(x+y) = 0, \mu(y) = 0 \Rightarrow \mu(x) = 0,$ for all $x, y \in M$.

Theorem 3.13. *Let f and g be $L-$fuzzy k ideals of M. Then $f \cap g$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M.*

Proof. Let f and g be $L-$fuzzy k ideals of M. By Theorem 3.11., $f \cap g$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. Let $x, y \in M$. We have

$$\begin{aligned} f \cap g(a) =& min\{f(x), g(x)\} \\ \geq& min\big\{min\{f(x+y), f(y)\}, min\{g(x+y), g(y)\}\big\} \\ =& min\big\{min\{f(x+y), g(x+y)\}, min\{f(y), g(y)\}\big\} \\ =& min\big\{f \cap g(x+y)\}, f \cap g(y)\big\}. \end{aligned}$$

Hence $f \cap g$ is a $L-$fuzzy k ideal of M. □

Theorem 3.14. *A $L-$fuzzy subset μ of M is a $L-$fuzzy k ideal of $\Gamma-$semiring M if and only if μ_t is a k ideal of $\Gamma-$semiring M for any $t \in L, \mu_t \neq \phi$.*

Proof. Let μ be a $L-$fuzzy k ideal of $\Gamma-$semiring M. By Theorem 3.6, $\mu_t \neq \phi$ then μ_t is an ideal of $\Gamma-$semiring M for any $t \in L$. Suppose $a,\ a + x \in \mu_t \Rightarrow \mu(a) \geq t, \mu(a+x) \geq t$. Since μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M.

$$\begin{aligned} \text{We have } \mu(x) &\geq min\{\mu(a+x), \mu(a)\} \\ &\Rightarrow \mu(x) \geq t \\ &\Rightarrow x \in \mu_t. \end{aligned}$$

Hence μ_t is a k ideal of $\Gamma-$semiring M.

Conversely assume that μ_t is a k ideal of $\Gamma-$semiring M with $\mu_t \neq \phi$. Let $\mu(a) = t_1, \mu(x+a) = t_2$. Let $t = min\{t_1, t_2\}$. Then $a \in \mu_t, x + a \in \mu_t$ for some $x \in M \Rightarrow x \in \mu_t \Rightarrow \mu(x) \geq t = min\{t_1, t_2\} = min\{\mu(x+a), \mu(a)\}$. Therefore μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. □

Theorem 3.15. *Let M be a $\Gamma-$semiring. If μ is a $L-$fuzzy k ideal of M then μ is a $L-k$ fuzzy ideal of M.*

Proof. Let μ be a $L-$fuzzy k ideal of M. Let $x, y \in M$ and $\mu(0) = t \in L$.
$\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$.
$\mu(0) = t \Rightarrow x+y \in \mu_t, y \in \mu_t$. By Theorem 3.14, μ_t is a k ideal of M.

$$\begin{aligned}&\Rightarrow x \in \mu_t\\&\Rightarrow \mu(x) \geq t = \mu(0).\end{aligned}$$

We have $\mu(x) \leq \mu(0)$, for all $x \in M$. Hence $\mu(x) = \mu(0)$. Therefore μ is a $L-k$ fuzzy ideal of $\Gamma-$semiring M. □

The proof of the following theorem is similar to [7, Proposition 3.3].

Theorem 3.16. *Intersection of a non-empty collection of L-fuzzy ideals of Γ-semiring is a L-fuzzy ideal.*

Theorem 3.17. *The set of all $L-$fuzzy ideals of $\Gamma-$semiring M forms a complete lattice.*

Proof. Suppose the set of all $L-$fuzzy ideals denoted by $LFI(M)$. Let $\mu_1, \mu_2 \in LFI(M)$. Define a relation $\leq$ such that $\mu_1 \leq \mu_2$ if and only if $\mu_1 \subseteq \mu_2$. Then $LFI(M)$ is a poset with respect to a relation $\leq$. Obviously $\mu_1 + \mu_2$ is the least upper bound of μ_1 and μ_2 and $\mu_1 \cap \mu_2$ is the greatest lower bound of μ_1 and μ_2. Therefore $LFI(M)$ is a lattice. Suppose ψ is a $L-$fuzzy subset of M such that $\psi(x) = 1$, for all $x \in M$. then $\psi \in LFI(M)$ and $\mu(x) \leq \psi(x)$ for all $x \in M, \mu \in LFI(M)$. Therefore ψ is the greatest element. Let $\{\mu_i \mid i \in I\}$ be a non-empty family of $L-$fuzzy ideals of M. Then $\bigcap\limits_{i \in I} \mu_i \in LFI(M)$. Hence $LFI(M)$ is a complete lattice. □

Definition 3.18. Let μ be a $L-$fuzzy subset of X and $a, b \in L$. the mapping $\mu_a^T : X \to L$, $\mu_b^M : X \to L$ and $\mu_{b,a}^{MT} : X \to L$ are called fuzzy translation, fuzzy multiplication and fuzzy magnified translation of μ respectively, if

$$\mu_a^T(x) = \mu(x) \vee a; \quad \mu_b^M(x) = b \wedge \mu(x); \quad \mu_{b,a}^{MT}(x) = \big(b \wedge \mu(x)\big) \vee a, \text{ for all } x \in X.$$

Theorem 3.19. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. Then $a \in L, \mu$ is a $L-$fuzzy ideal of $\Gamma-$semiring M if and only if μ_a^T, the fuzzy translation is a $L-$fuzzy ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a $L-$fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}\mu_a^T(x+y) &= \mu(x+y) \vee a\\&\geq min\{\mu(x), \mu(y)\}\\&= min\{\mu(x) \vee a, \mu(y) \vee a\}\\&= min\{\mu_a^T(x), \mu_a^T(y)\}.\\ \mu_a^T(x\alpha y) &= \mu(x\alpha y) \vee a\\&\geq min\{\mu(x), \mu(y)\} \vee a\\&= min\{\mu(x) \vee a, \mu(y) \vee a\}\\&= min\{\mu_a^T(x), \mu_a^T(y)\}.\end{aligned}$$

Hence μ_a^T, the fuzzy translation is a $L-$fuzzy ideal of $\Gamma-$semiring M.

Conversely suppose that $a \in L, \mu_a^T$, the fuzzy translation is a $L-$fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$,

$$\begin{aligned}
\mu_a^T(x+y) &\geq min\{\mu_a^T(x), \mu_a^T(y)\} \\
&= min\{\mu(x) \vee a, \mu(y) \vee a\} \\
\mu_a^T(x+y) &= \mu(x+y) \vee a \\
\Rightarrow \mu_a^T(x+y) \vee a &\geq min\{\mu(x) \vee a, \mu(y)\} \vee a \\
\Rightarrow \mu_a^T(x+y) &\geq min\{\mu(x), \mu(y)\}. \\
\text{Now } \mu_a^T(x\alpha y) &\geq max\{\mu_a^T(x), \mu_a^T(y)\} \\
\Rightarrow \mu(x\alpha y) \vee a &\geq max\{\mu(x) \vee a, \mu(y) \vee a\} \\
\Rightarrow \mu(x\alpha y) \vee a &\geq max\{\mu(x), \mu(y)\} \vee a \\
\Rightarrow \mu(x\alpha y) &\geq max\{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence μ is a $L-$fuzzy ideal of $\Gamma-$semiring M. □

Theorem 3.20. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M and $a \in L$. Then μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M if and only if μ_a^T, the fuzzy translation is a $L-$fuzzy k ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. By Theorem 3.19, μ_a^T is a $L-$fuzzy ideal of $\Gamma-$semiring M.

$$\begin{aligned}
\mu_a^T(x) =& \mu(x) \vee a \\
\geq& min\{\mu(x+y), \mu(y)\} \vee a \\
=& min\{\mu(x+y) \vee a, \mu(y) \vee a\} \\
=& min\{\mu_a^T(x+y), \mu_a^T(y)\}, \text{ for all } x, y \in M.
\end{aligned}$$

Hence μ_a^T is a $L-$fuzzy k ideal of $\Gamma-$semiring M.

Conversely suppose that $a \in L, \mu_a^T$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M.

$$\begin{aligned}
\mu(x) \vee a =& \mu_a^T(x) \geq min\{\mu_a^T(x+y), \mu_a^T(y)\} \\
=& min\{\mu(x+y) \vee a, \mu(y) \vee a\} \\
=& min\{\mu(x+y),\ \mu(y)\} \vee a \\
\mu(x) \geq& min\{\mu(x+y),\ \mu(y)\}, \text{ for all } x, y \in M.
\end{aligned}$$

Therefore μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. □

Theorem 3.21. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. Then μ is a $L-$fuzzy ideal of $\Gamma-$semiring M if and only if $b \in L, \mu_b^M$, fuzzy multiplication is a $L-$fuzzy ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a $L-$fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma, b \in L$.

$$\begin{aligned}
\mu_b^M(x+y) &= \mu(x+y) \wedge b \\
&\geq min\{\mu(x), \mu(b)\} \wedge b \\
&= min\{\mu(x) \wedge b, \mu(b) \wedge b\} \\
\mu_b^M(x+y) &= min\{\mu_b^M(x), \mu_b^M(y)\}. \\
\text{Now } \mu_b^M(x\alpha y) &= \mu_a(x\alpha y) \wedge b \\
&\geq max\{\mu(x), \mu(y)\} \wedge b \\
&= max\{\mu(x) \wedge b, \mu(y) \wedge b\} \\
\mu_b^M(x\alpha y) &= max\{\mu_b^M(x), \mu_b^M(y)\}.
\end{aligned}$$

Hence μ_b^M, fuzzy multiplication is a $L-$fuzzy ideal of $\Gamma-$semiring M.

Conversely suppose that μ_b^M, fuzzy multiplication is a $L-$fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}
\mu_b^M(x+y) &\geq min\{\mu_b^M(x), \mu_b^M(y)\} \\
\mu(x+y) \wedge b &\geq min\{\mu(x) \wedge b, \mu(y) \wedge b\} \\
\mu(x+y) \wedge b &\geq min\{\mu(x), \mu(y)\} \wedge b \\
\mu(x+y) &\geq min\{\mu(x), \mu(y)\}. \\
\text{Now } \mu_b^M(x\alpha y) &\geq max\{\mu_b^M(x), \mu_b^M(y)\} \\
\mu(x\alpha y) \wedge b &\geq max\{\mu(x) \wedge b, \mu(y) \wedge b\} \\
\mu(x\alpha y) \wedge b &\geq max\{\mu(x), \mu(y)\} \wedge b \\
\mu(x\alpha y) &\geq max\{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence μ is a $L-$fuzzy ideal of $\Gamma-$semiring M. □

Theorem 3.22. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. Then μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M if and only if $b \in L, \mu_b^M$, fuzzy multiplication is a $L-$fuzzy k ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. By Theorem 3.21, μ_b^M is a $L-$fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma, b \in L$.

$$\begin{aligned}
\mu_b^M(x) =& \mu(x) \wedge b \\
\geq& min\{\mu(x+y), \mu(y)\} \wedge b \\
=& min\{\mu(x+y) \wedge b, \mu(y) \wedge b\} \\
=& min\{\mu_b^M(x+y), \mu_b^M(y)\}, \text{ for all } x, y \in M.
\end{aligned}$$

Hence μ_b^M is a $L-$fuzzy k ideal of $\Gamma-$semiring M. Conversely suppose that $a \in L, \mu_b^M$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M.

$$\begin{aligned}\mu(x) \wedge b =& \mu_b^M(x) \geq min\{\mu_b^M(x+y), \mu_b^M(y)\}\\ =& min\{\mu(x+y) \wedge b, \mu(y) \wedge b\}\\ =& min\{\mu(x+y),\ \mu(y)\} \wedge b\\ \mu(x) \geq& min\{\mu(x+y),\ \mu(y)\},\ \text{for all } x, y \in M.\end{aligned}$$

Therefore μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M. □

Theorem 3.23. *Let μ be a $L-$fuzzy k subset of $\Gamma-$semiring M. Then μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M if and only if $\mu_{b,a}^{MT}$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M.*

Proof. Let μ be a $L-$fuzzy k ideal of $\Gamma-$semiring M.
$\Leftrightarrow$ μ_b^M is a $L-$fuzzy k ideal of $\Gamma-$semiring M, by Theorem 3.22.
$\Leftrightarrow$ $\mu_{b,a}^{MT}$ is a $L-$fuzzy k ideal of $\Gamma-$semiring M, by Theorem 3.20. □

Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. Then the set $\{x \in M \mid \mu(x) = \mu(0)\}$, is denoted by M_μ.

Theorem 3.24. *If μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M then M_μ is a k ideal of $\Gamma-$semiring M.*

Proof. Let μ be a $L-$fuzzy k ideal of $\Gamma-$semiring M and $x, y \in M_\mu$. Then $\mu(x) = \mu(0) = \mu(y)$.

$$\begin{aligned}\mu(x+y) &\geq min\{\mu(x), \mu(y)\}\\ &= min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

We have $\mu(x+y) \leq \mu(0)$. Therefore $\mu(0) = \mu(x+y)$. Hence $x+y \in M_\mu$.
Let $x \in M_\mu, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}\mu(x\alpha y) &\geq max\{\mu(x), \mu(y)\}\\ &= max\{\mu(0), \mu(y)\} = \mu(0).\end{aligned}$$

We have $\mu(x\alpha y) \leq \mu(0)$. Therefore $\mu(0) = \mu(x\alpha y)$. Hence $x\alpha y \in M_\mu$. M_μ is an ideal of $\Gamma-$semiring M. Let $x+y, x \in M_\mu$. Then $\mu(x+y) = \mu(0) = \mu(x)$. Since μ is a $L-$fuzzy k ideal of $\Gamma-$semiring M, we have $\mu(y) \geq min\{\mu(x+y), \mu(x)\} = min\{\mu(0), \mu(0)\} = \mu(0)$. Therefore $\mu(y) = \mu(0) \Rightarrow y \in M_\mu$. Hence M_μ is a k ideal of $\Gamma-$semiring M. □

Theorem 3.25. *If $t \in L$ such that $\mu_t \neq \phi, \mu_t$ is a k ideal of $\Gamma-$semiring M then μ is a $L-k$ fuzzy ideal of M.*

Proof. Let μ_t be k ideal of $\Gamma-$semiring $M, t \in L$ and $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$. $\Rightarrow x+y \in M_\mu$. since M_μ is a k ideal of $\Gamma-$semiring $M \Rightarrow x \in M_\mu$. Therefore $\mu(x) \geq \mu(0)$. By Theorem 3.4, we have $\mu(x) \leq \mu(0)$. Hence $\mu(x) = \mu(0)$. Therefore μ is a $L-k$ fuzzy ideal of M. □

Definition 3.26. A $L-$fuzzy subset μ of a $\Gamma-$semiring M is said to be normal if $\mu(0) = 1$.

Definition 3.27. Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. We define μ^+ on S by

$$\mu^+(x) = \mu(x) \vee \big(\mu(0)\big)', \text{where } \big(\mu(0)\big)' \text{ is the complement of } \mu(0)$$

The proofs of the following theorems are straight forward.

Theorem 3.28. *Let A be an ideal of $\Gamma-$semiring M. If we define $L-$fuzzy subset on M by*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad , \text{for all } x \in M.$$

Then χ_A is a normal $L-$fuzzy ideal of M and $M_{\chi_A} = A$.

Theorem 3.29. *If μ and λ are normal $L-$fuzzy ideals of $\Gamma-$semiring M then $\mu \cap \lambda$ is a normal $L-$fuzzy ideal of $\Gamma-$semiring.*

Theorem 3.30. *If μ is a normal $L-$fuzzy ideal of $\Gamma-$semiring M then μ_a^T, fuzzy translation is a normal $L-$fuzzy ideal.*

Proof. Let μ be a normal $L-$fuzzy ideal of $\Gamma-$semiring M. By Theorem 3.19, μ_a^T, the fuzzy translation is a $L-$fuzzy ideal of $\Gamma-$semiring M.

$$\begin{aligned} \mu_a^T(x) =& \mu(x) \vee a, \text{ for all } x \in M. \\ \Rightarrow \mu_a^T(0) =& \mu(0) \vee a \\ =& 1 \vee a = 1. \end{aligned}$$

Hence the theorem. □

Theorem 3.31. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. Then*

(i) *μ^+ is a normal $L-$fuzzy subset of M containing μ.*
(ii) *$(\mu^+)^+ = \mu$.*
(iii) *μ is a normal if and only if $\mu^+ = \mu$.*
(iv) *If there exists a $L-$fuzzy subset γ of M satisfying $\gamma^+ \subseteq \mu$ then μ is a normal.*

Proof. (i)

$$\begin{aligned} \text{We have } \ \mu^+(x) &= \mu(x) \vee (\mu(0))', \quad \text{for all } x \in M. \\ \mu^+(0) &= \mu(0) \vee (\mu(0))' = 1, \\ \text{and } \mu(x) &\leq \mu(x) \vee (\mu(0))' = \mu^+(x), \text{ for all } x \in M. \end{aligned}$$

Hence μ^+ is a normal $L-$fuzzy subset of M containing μ.

(ii)

$$\begin{aligned} (\mu^+)^+(x) &= \mu^+(x) \vee (\mu^+(0))' \\ &= \mu^+(x) \vee 1' \\ &= \mu^+(x) \vee 0 = \mu^+(x), \text{ for all } x \in M. \end{aligned}$$

Therefore $(\mu^+)^+ = \mu$.

(iii) Suppose $\mu = \mu^+$.

$$\begin{aligned} \text{We have } \ \mu^+(x) &= \mu(x) \vee (\mu(0))', \quad \text{for all } x \in M \\ \Rightarrow \mu(x) &= \mu(x) \vee (\mu(0))' \\ \Rightarrow \mu(0) &= \mu(0) \vee (\mu(0))' = 1. \end{aligned}$$

Hence μ is a normal.

Conversely suppose that μ is a normal. Then

$$\begin{aligned} \mu^+(x) &= \mu(x) \vee (\mu(0))' \\ &= \mu(x) \vee (1') = \mu(x) \\ \text{Therefore } \ \mu^+ &= \mu. \end{aligned}$$

(iv)

$$\begin{aligned} \text{We have } \ \gamma^+(x) &= \gamma(x) \vee (\gamma(0))' \\ \Rightarrow \gamma^+(0) &= \gamma(0) \vee (\gamma(0))' \\ &= 1 \\ \text{We have } \ \gamma^+ &\subseteq \mu \\ \Rightarrow \gamma^+(0) &\leq \mu(0) \\ \Rightarrow 1 &\leq \mu(0) \\ \mu(0) &= 1. \end{aligned}$$

Hence μ is a normal.

□

Theorem 3.32. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring M. If μ is a $L-$fuzzy ideal of M then μ^+ is a normal $L-$fuzzy ideal of M containing μ.*

Proof. Let $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} \mu^+(x+y) &= \mu(x+y) \vee (\mu(0))' \\ &\geq min\{\mu(x), \mu(y)\} \vee (\mu(0))' \\ &= min\{\mu(x) \vee (\mu(0))', \mu(y) \vee (\mu(0))'\} \\ &= min\{\mu^+(x), \mu^+(y)\}. \\ \mu^+(x\alpha y) &= \mu(x\alpha y) \vee (\mu(0))' \\ &\geq \mu(y) \vee (\mu(0))' \\ &= \mu^+(y). \end{aligned}$$

Similarly $\mu^+(x\alpha y) \geq \mu^+(x)$. By Theorem 3.31, μ^+ is a normal $L-$fuzzy ideal containing μ. □

Corollary 3.33. *Let μ be a $L-$fuzzy subset of $\Gamma-$semiring and $x \in M$. If $\mu^+(x) = 0$ then $\mu(x) = 0$.*

Proof. By Theorem 2.28, we have

$$\begin{aligned} \mu(x) &\leq \mu^{+}(x) \\ \Rightarrow \mu(x) &\leq \mu^{+}(x) = 0 \\ \Rightarrow \mu(x) &\leq 0. \end{aligned}$$

Therefore $\mu(x) = 0$ □

Theorem 3.34. *Let M be a $\Gamma-$semiring, $\psi : M \to M$ be an onto homomorphism and μ be a $L-$fuzzy subset of M. Define $\mu^{\psi} : M \to L$ by $\mu^{\psi}(x) = \mu(\psi(x))$, for all $x \in M$. If μ is a $L-$fuzzy ideal of M then μ^{ψ} is a $L-$fuzzy ideal of M.*

Proof. Let M be a $\Gamma-$semiring and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \mu^{\psi}(x+y) &= \mu(\psi(x+y)) \\ &= \mu(\psi(x) + \psi(y)) \\ &\geq min\{\mu(\psi(x)), \mu(\psi(y))\} \\ &= min\{\mu^{\psi}(x), \mu^{\psi}(y)\} \\ \text{and } \mu^{\psi}(x\alpha y) &= \mu(\psi(x\alpha y)) \\ &= \mu(\psi(x)\alpha\psi(y)) \\ &\geq min\{\mu(\psi(x)), \mu(\psi(y))\} \\ &= min\{\mu^{\psi}(x), \mu^{\psi}(y)\}. \end{aligned}$$

Therefore μ^{ψ} is a $L-$fuzzy $\Gamma-$subsemiring of M.

$$\begin{aligned} \mu^{\psi}(x\alpha y) &= \mu(\psi(x\alpha y)) \\ &= \mu(\psi(x)\alpha\psi(y)) \\ &\geq \{\mu(\psi(y))\} \\ &= \mu^{\psi}(y). \\ \text{Similarly } \mu^{\psi}(x\alpha y) &\geq \mu^{\psi}(x). \end{aligned}$$

Hence μ^{ψ} is an ideal of $\Gamma-$semiring M. □

Theorem 3.35. *Let μ and γ be $L-$fuzzy ideals of $\Gamma-$semiring M. If $\mu \subseteq \gamma$ and $\mu(0) = \gamma(0)$ then $M_{\mu} \subseteq M_{\gamma}$.*

Proof. Suppose that $\mu \subseteq \gamma$ and $\mu(0) = \gamma(0)$.

$$\begin{aligned} \text{If } x \in M_{\mu} &\Rightarrow \mu(x) = \mu(0) = \gamma(0) \\ &\Rightarrow \gamma(0) = \mu(x) \leq \gamma(x), \text{ for all } x \in M. \\ \text{We have } \gamma(x) &\leq \gamma(0), \text{ for all } x \in M \\ &\Rightarrow \gamma(x) = \gamma(0). \end{aligned}$$

Therefore $x \in M_{\gamma}$. Hence $M_{\mu} \subseteq M_{\gamma}$. □

Corollary 3.36. *Let μ and γ be normal $L-$fuzzy ideals of $\Gamma-$semiring M. If $\mu \subseteq \gamma$ then $M_{\mu} \subseteq M_{\gamma}$.*

Theorem 3.37. *If μ and γ are normal $L-$fuzzy ideals of $\Gamma-$semiring M then $M_{\mu\cap\gamma} = M_\mu \cap M_\gamma$.*

Proof. Let μ and γ be normal $L-$fuzzy ideals of $\Gamma-$semiring M.

$$\begin{aligned}
\text{Suppose } & x \in M_{\mu\cap\gamma} \\
\text{if and only if } & \mu\cap\gamma(x) = \mu\cap\gamma(0) \\
\text{if and only if } & min\{\mu(x),\gamma(x)\} = min\{\mu(0),\gamma(0)\} = 1 \\
\text{if and only if } & \mu(x) = 1 \text{ and } \gamma(x) = 1 \\
\text{if and only if } & \mu(x) = \mu(0) \text{ and } \gamma(x) = \gamma(0) \\
\text{if and only if } & x \in M_\mu \cap M_\gamma \\
\text{Hence } & M_{\mu\cap\gamma} = M_\mu \cap M_\gamma.
\end{aligned}$$

□

$\big(N(M), \subseteq\big)$ denotes the partially ordered set of normal $L-$fuzzy ideals of $\Gamma-$semiring M under set inclusion.

Definition 3.38. A non constant $L-$fuzzy ideal μ of $\Gamma-$semiring M is said to be a maximal $L-$fuzzy ideal if μ^+ is a maximal element of $\big(N(M), \subseteq\big)$

Theorem 3.39. *If μ be a non constant maximal normal $L-$fuzzy ideal of $\Gamma-$semiring M then μ takes the values only 0 and 1.*

Proof. Let $y \in M, \mu$ be a maximal normal $L-$fuzzy ideal of $\Gamma-$semiring M, $0 < \mu(y) < 1$ and $\mu(y) = a$. Define $L-$fuzzy subset γ of M by $\gamma(x) = \mu(x) \vee a$, for all $x \in M$. Then $\gamma(x) = \mu_a^T(x)$ and $\gamma(x) \geq \mu(x)$, for all $x \in M$. By Theorem 3.30, γ is a normal $L-$fuzzy ideal of $\Gamma-$semiring M. If $x \neq 0, \mu(x) < \gamma(x)$. Therefore μ is not a maximal, which is a contradiction. Hence the theorem. □

Theorem 3.40. *If μ is a maximal $L-$fuzzy ideal of $\Gamma-$semiring M then M_μ is a maximal ideal of $\Gamma-$semiring M.*

Proof. Let μ be a maximal $L-$fuzzy ideal of $\Gamma-$semiring M. Then μ^+ is a maximal element of $\big(N(M), \subseteq\big)$. By Theorem 3.39, μ^+ takes only the values 0 and 1.

$$\begin{aligned}
\text{If } \mu^+(x) = 1 &\Rightarrow \mu(x) \vee \big(\mu(0)\big)' = 1 \\
&\Rightarrow \mu(0) = 1, \text{since} \mu(0) \geq \mu(x), \text{ for all } x \in M. \\
\text{We have } & \mu(x) \leq \mu^+(x), \text{ for all } x \in M. \\
\text{If } \mu^+(x) = 0 &\Rightarrow \mu(x) \vee \big(\mu(0)\big)' = 0 \\
&\Rightarrow \mu(x) = 0 \text{ and } \big(\mu(0)\big)' = 0 \\
&\Rightarrow \mu(0) = 1.
\end{aligned}$$

Therefore μ is a normal $L-$fuzzy ideal of $\Gamma-$semiring M. Now M_μ is a proper ideal of $\Gamma-$semiring M, since μ is a non constant. Let A be an ideal of $\Gamma-$semiring M such that $M_\mu \subseteq A \Rightarrow \chi_{M_\mu} \subseteq \chi_A \Rightarrow \ \mu = \chi_{M_\mu} \subseteq \chi_A$. Since μ and χ_A are normal $L-$fuzzy ideals of M and $\mu = \mu^+ \Rightarrow \ \mu$ is a maximal element of $N(M) \Rightarrow \ \mu = \chi_A$ or $\chi_A = \mathbf{1}$, where $\mathbf{1} : M \Rightarrow \ L, \mathbf{1}(x) = 1$, for all $x \in M$ is a $L-$fuzzy ideal. If $\chi_A = \mathbf{1}$

then $A = M$. If $\mu = \chi_A$ then $M_\mu = M_{\chi_A} = A$. Hence M_μ is a maximal ideal of $\Gamma-$semiring M. □

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