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A new generalization of fuzzy subgroup and level subsets

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Abstract. In this paper, by using the notions of "not belonging" $(\overline{\in})$ and "non quasi-k-coincidence" $(\overline{q_k})$ of a fuzzy point with a fuzzy set, we define the notion of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroups of a group which is a generalization of fuzzy subgroups and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroups. Also, we generalized the concept of the $\overline{\in}$-level set, $(\overline{\in} \vee \overline{q})$-level set and $(\overline{\in}, \overline{\in} \vee \overline{q})$-level set by using "not belonging" $(\overline{\in})$ and "non quasi-k-coincidence" $(\overline{q_k})$ of a fuzzy point with a fuzzy set to obtain the notions of $(\overline{\in} \vee \overline{q_k})$-level set and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-level set. We give characterizations of an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup by the properties of these generalized level sets. The important achievement of the study with an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-level sets is the generalization of that the notions of fuzzy subgroups and level sets.

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1. Introduction

There are several types of complex difficulties in economics, envirment and engineering. To handle these type of complex difficulties, we cannot fruitfully apply classical models because of several uncertainties for these problems. There are some theories as mathematical tools: interval mathematics, theory of probability, and theory of fuzzy sets for dealing with uncertainties. The fuzzy set theory is profitably in these mathematical models. The concept of fuzzy set was introduced by L.A. Zadeh [17] of 1965, providing a natural framework for generalizing several basic notions of algebra. Since then fuzzy sets have been applied to many branches of Mathematics.

Fuzzy logic combines the decision ability of human beings and speed of the computers, and through this combination, an excellent decision making progress is obtained under imprecise, vague and uncertain conditions. The complexity of today's socioeconomic problems require more complex decision making processes. That's why decision makers have to consider many aspects of a problem. The necessity of considering all relevant aspects of a problem forces them to use fuzzy multi-criteria decision making systems. The most important thing in Fuzzy Group Decision Support Systems is to determine the evaluation criteria and their weights in decision process. The knowledge and experience of a human expert is the best source for such kind of information.

It is also useful in particle physics and string theory. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as functional analysis, group, ring, near- ring, vector spaces, automation. There have been wide ranging applications of the theory of fuzzy sets from the design of robots and computer simulation to Engineering and water resources planning.

The fuzzification of algebraic structures was initiated by Rosenfeld in 1971 [12], he defined the concept of fuzzy subgroups. The fuzzy algebraic structures play a vital role in Mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [6, 16]. Murali [10] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. Bhakat and Das generalized the concept of Rosenfeld's fuzzy subgroups and introduced the $(\in, \in \vee q)$-fuzzy subgroups in a group [5] by using the notions of belongingness $(\in)$ and quasi-coincidence (q) of fuzzy point and fuzzy set, which was introduced by Pu and Liu [11]. Liu [8] introduced the concept fuzzy normal subgroup in 1982. A comprehensive study of the fuzzy normal subgroups was defined by Mukherjee and Bhattacharya [9]. This concept was more studied in detail by Bhakat [3, 4], Bhakat and Das [5], and Yuan et al. [16]. In particular, an $(\in, \in \vee q)$-fuzzy subgroup is an imperative and beneficial generalization of Rosenfeld's fuzzy subgroup. In [13], Shabir et al. defined the concept of (α, β)-fuzzy ideals of semigroups. M. Aslam et al introduced generalized fuzzy Γ-ideals in Γ-LA-semigroup and characterized different classes of Γ-LA-semigroup by the properties of generalized fuzzy Γ-ideals [1].

It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. Jun, in [7] generalized the concept of quasi-coincident (q) by quasi-k-coincident (q_k) of a fuzzy point with a fuzzy set, and he introduced an $(\in, q_k)$- fuzzy subalgebra and an $(\in, \in \vee q_k)$- fuzzy subalgebra in BCK/BCI-algebra. Jun et al. studied the generalized form of Bhakat and Das idea and they introduced $(\in, \in \vee q_k)$-fuzzy subgroups, $(\in, \in \vee q_k)$-fuzzy normal subgroups, $(\in, \in \vee q_k)$-fuzzy cosets and generalized the concept of level subsets [7]. In [14], Shabir et al. initiated the concept of $(\in, \in \vee q_k)$-fuzzy ideals in semigroups and characterized the different classes of semigroups. Shabir and Mehmood introduced $(\in, \in \vee q_k)$-fuzzy h-ideals in hemirings and charaterized hemiring by the properieties of $(\in, \in \vee q_k)$-fuzzy h-ideals [15]. In [2], Abdullah et al. gave new generalization of Rosenfeld's fuzzy subgroup. They defined $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy normal subgroups, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy cosets and gave useful characterizations of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy normal subgroups, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy cosets. The present author introduced a new type of fuzzy normal subgroups and fuzzy coset in [2]. The present concept in this article is diffenrent from the concept of Jun et al in [7]. We proved by an example, the Jun concept is different from the present concept.

In this paper, the present authors introduced a new type of a generalization of fuzzy subgroups and its level sets by using the notions of "not belonging" ($\overline{\in}$) and "non quasi-k-coincidence" ($\overline{q_k}$) of a fuzzy point with a fuzzy set, we define the notion of ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroups of a group which is a generalization of fuzzy subgroups and ($\overline{\in}, \overline{\in} \vee \overline{q}$)-fuzzy subgroups. Also, we generalized the concept of $\overline{\in}$-level set, ($\overline{\in} \vee \overline{q}$)-level set and ($\overline{\in}, \overline{\in} \vee \overline{q}$)-level set by using the notions of "not belonging" ($\overline{\in}$) and "non quasi-k-coincidence" ($\overline{q_k}$) of a fuzzy point with a fuzzy set to abtain the notions ($\overline{\in} \vee \overline{q_k}$)-level set and ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-level set. We give characterizations of an ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup by the properties of these generalized level sets. The important achievement of the study with an ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup and ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-level sets is the generalization of that the notions of fuzzy subgroups and level sets. We prove that a fuzzy subset $\mathcal{F}$ of a group G is an ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup of G if and only if it the following conditions hold (1) for all $x, y \in G$, $\max\{\mathcal{F}(xy), \frac{1-k}{2}\} \geq \min\{\mathcal{F}(x), \mathcal{F}(y)\}$ and (2) for all $x \in G$, $\max\{\mathcal{F}(x^{-1}), \frac{1-k}{2}\} \geq \mathcal{F}(x)$. We make a connection between ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup and generalized level set as: a fuzzy subset $\mathcal{F}$ of G is an ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup of G if and only if the $\underline{U_k}\left(\mathcal{F}; t\right) \neq \varnothing$ is a subgroup of G. We also give some related properties of generalized level sets and relation among these generalized level sets. We prove that for a fuzzy subset $\mathcal{F}$ of a set G, we have $U_k\left(\mathcal{F}; t\right) \subseteq \overline{\mathcal{F}^c_{1-t}} \cup \overline{\mathcal{F}^c_{t+k}},$, where $\mathcal{F}^c$ denotes the compliment of $\mathcal{F}$, that is $\mathcal{F}^c\left(x\right) = 1 - \mathcal{F}\left(x\right)$ for all $x \in G$. Also, we show that for a fuzzy subset $\mathcal{F}$ of a set G, we have $\overline{U_k}\left(\mathcal{F}; t\right) \subseteq \mathcal{F}^c_{1-t} \cap \mathcal{F}^c_{t+k},$, where $\mathcal{F}^c$ denotes the compliment of $\mathcal{F}$, that is, $\mathcal{F}^c\left(x\right) = 1 - \mathcal{F}\left(x\right)$, for all $x \in G$. We prove a very good result for group of prime order which is state as if $\mathcal{F}$ is an ($\overline{\in}, \overline{\in} \vee \overline{q_k}$)-fuzzy subgroup of group of prime order G such that $\mathcal{F}\left(a\right) \geq \frac{1-k}{2}$ for some element $a\left(\neq e\right) \in G$, then $\mathcal{F}(x) \geq \frac{1-k}{2}$ for all $x \in G$.

2. Preliminaries

In this section, we will introduced the basic concept of definitions of previous literature.

If $A \subseteq G$, then the characteristic function C_A of A is a function from A into $\{0, 1\}$, defined by

$$C_A\left(x\right) = \begin{cases} 1 & \text{if } x \in A. \\ 0 & \text{if } x \notin A. \end{cases}$$

A fuzzy subset of a universe X is a function f from X into the unit closed interval $[0, 1]$, that is, $f : X \to [0, 1]$. For any two fuzzy subsets f and g of G, $f \leq g$ means that, for all $x \in G$, $f\left(x\right) \leq g\left(x\right)$. The symbols $f \wedge g$, and $f \vee g$ will mean the following fuzzy subsets of G.

$$\begin{aligned} \left(f \wedge g\right)\left(x\right) &= f\left(x\right) \wedge g\left(x\right) \\ \left(f \vee g\right)\left(x\right) &= f\left(x\right) \vee g\left(x\right) \end{aligned}$$

for all $x \in X$. More generally, if $\{f_i : i \in \Omega\}$ is a family of fuzzy subsets of G, then $\wedge_{i \in \Omega} f_i$ and $\vee_{i \in \Omega} f_i$ are defined by

$$\begin{array}{rcl}(\wedge_{i\in\Omega} f_i)(x) & = & \wedge_{i\in\Omega}(f_i(x)) \\ (\vee_{i\in\Omega} f_i)(x) & = & \vee_{i\in\Omega}(f_i(x))\end{array}$$

and are called the intersection and the union of the family $\{f_i : i \in \Omega\}$ of fuzzy subsets of G, respectively.

Let f and g be two fuzzy subsets of G. Then the product $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \vee_{x=yz}\{f(y) \wedge g(z)\}, & \text{if } x = yz, \\ 0 & \text{otherwise.}\end{cases}$$

Definition 2.1. A fuzzy subset f of X of the form

$$f(y) = \begin{cases} t & if \;\; y = x \\ 0 & if \;\; y \neq x\end{cases}$$

is said to be a fuzzy point with support "x" and value "t", where $t \in (0,1]$ and is denoted by x_t or $[x;t]$. A fuzzy point x_t is said to be not belong to (resp, be not quasi-coincidence with) a fuzzy subset f, written as $x_t\overline{\in}f$(resp, $x_t\overline{q}f$) if $f(x) < t$ (resp, $f(x) + t \leq 1$). If $x_t\overline{\in}f$ and (resp, $x_t\overline{q}f$), then we can write $x_t \;\overline{\in} \wedge \overline{q}f$ (resp, $x_t\overline{\in}\vee\overline{q}f$). And x_t be not quasi-k-coincidence with a fuzzy subset f, written as $x_t\overline{q_k}f$ if $f(x)+t+k \leq 1$ or $f(x)+t \leq 1-k$, and $x_t\overline{\in}\vee\overline{q_k}f$ if $x_t\overline{\in}f$ or $x_t\overline{q_k}f$, here $t \in (0,1]$ and $k \in [0,1)$ [7].

Definition 2.2 ([12])**.** A fuzzy subset f of a group G is said to be a fuzzy subgroup of G if for all $x, y \in G$,

(i) $f(xy) \geq \min(f(x), f(y))$ and (ii) $f\left(x^{-1}\right) \geq f(x)$.

or $f\left(x^{-1}y\right) \geq \min\left(f\left(x^{-1}\right), f(y)\right)$ for all $x^{-1}, y \in G$.

Definition 2.3 ([14])**.** Let f be a fuzzy subset of a group G. We define the upper part f^+ as follows, $f^+(x) = f(x) \vee \frac{1-k}{2}$.

Definition 2.4 ([4])**.** A fuzzy subset f of G is called a fuzzy normal subgroup of G if it is fuzzy subgroup of G that satisfies: $f\left(y^{-1}xy\right) \geq f(x)$ $(\forall\ x, y \in G)$ $(t \in (0,1])$.

Definition 2.5 ([4])**.** Let f and μ be two fuzzy subgroups of G. Then f is said to be fuzzy conjugate of μ if for $x \in G$, $f(y) = f\left(x^{-1}yx\right)$ for all $x \in G$.

Definition 2.6 ([5])**.** A fuzzy subset f of a group G is said to be an $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$,

(i) $f(xy) \geq \min\{f(x), f(y), 0.5\}$ and (ii) $f\left(x^{-1}\right) \geq \min\{f(x), 0.5\}$

or $f\left(x^{-1}y\right) \geq \min\left(f\left(x^{-1}\right), f(y), 0.5\right)$ for all $x^{-1}, y \in G$.

Definition 2.7 ([7])**.** A fuzzy subset f of a group G is said to be an $(\in, \in \vee q_k)$-fuzzy subgroup of G if it satisfy:

(i) $(\forall x, y \in G)$ $\left(f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}\right)$ and

(ii) $(\forall x \in G)$ $\left(f(x^{-1}) \geq \min\{f(x), \frac{1-k}{2}\}\right)$.

or $f\left(x^{-1}y\right) \geq \min\left(f\left(x^{-1}\right), f(y), \frac{1-k}{2}\right)$ for all $x^{-1}, y \in G$.

Definition 2.8 ([7])**.** An $(\in, \in \vee q_k)$-fuzzy subgroup f of G is called an $(\in, \in \vee q_k)$-fuzzy normal subgroup of G if f holds

$(\forall x, y \in G)$ $\left(f(y^{-1}xy) \geq \min\{f(x), \frac{1-k}{2}\}\right)$.

or $(\forall\ x, y \in G)$ $(t \in (0,1])$ $\left(x_t \in f \Longrightarrow \left(y^{-1}xy\right)_t \in \vee q_k f\right)$.

Definition 2.9 ([7]). Let f be a fuzzy subset of G. For any $x \in G$, the fuzzy subset $f_x^l : G \to [0,1]$, $y \to f\left(yx^{-1}\right)$ $\left(\text{resp, } f_x^r : G \to [0,1], y \to f\left(x^{-1}y\right)\right)$ is called the fuzzy left (resp. right) coset of G determined by x and f.

Definition 2.10 ([7]). Let f be an $(\in, \in \vee q_k)$-fuzzy subgroup of G. For any $x \in G$, $\overleftarrow{f_x}$ $\left(\text{resp. } \overrightarrow{f_x}\right) : G \to [0,1]$ is defined by $\overleftarrow{f_x}(y) = \min\{f_x^l(y), \frac{1-k}{2}\}$

(resp. $\overrightarrow{f_x}(y) = \min\{f_x^r(y), \frac{1-k}{2}\}$) is called the $(\in, \in \vee q_k)$-fuzzy left (resp. right) coset of G determined by x and f.

Definition 2.11 ([5]). For a fuzzy subgroup f of G the normalizer of f, denoted by $N(f)$ and is defined by $N(f) = \{y \in G; f(y^{-1}xy) \geq \min\{f(x), 0.5\} \ \forall\ x \in G\}$.

Definition 2.12 ([3, 5, 7]). Let f be a fuzzy subset of a set G and $t \in (0,1]$ and $k \in [0,1)$. Then the sets

$f_t = \{x \in G \mid f(x) \geq t\}$ is called the $\in$-level subset of G.

$Q(f;t) = \{x \in G \mid f(x) + t > 1\} = \{x \in G \mid x_t q f\}$ is called the q-level subset of G.

$\underline{Q}(f;t) = \{x \in G \mid f(x) + t \geq 1\} = \{x \in G \mid x_t \underline{q} f\}$ is called the $\underline{q}$-level subset of G.

$Q_k(f;t) = \{x \in G \mid f(x) + t + k > 1\} = \{x \in G \mid x_t q_k f\}$ is called the q_k-level subset of G.

$\underline{Q_k}(f;t) = \{x \in G \mid f(x) + t + k \geq 1\} = \{x \in G \mid x_t \underline{q_k} f\}$ is called the $\underline{q_k}$-level subset of G.

$\underline{U_k}(f;t) = \{x \in G \mid x_t \in \vee \underline{q_k}\ f\} = f_t \cup \underline{Q_k}(f;t)$ is called $\left(\in \vee \underline{q_k}\right)$-level subset of G.

3. Major section

In this section the authors introduced a new generalization of fuzzy subgroups and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroups and give some characterizations of them. In what follows, let G denote a group with identity element e, and k an arbitrary element of $[0,1]$ unless otherwise specified.

The following definition is a definition of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G.

Definition 3.1. A fuzzy subset f of a group G is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G if for all $x, y \in G$ and $t, r \in (0,1]$ and $k \in [0,1)$, the following conditions hold;

(i) $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ and

(ii) $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$.

Similarly, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G can also be defined as following

(i) $(xy)_{M\{t,r\}} \overline{\in} f$ implies $x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$ and

(ii) $x_t^{-1} \overline{\in} f$ implies $x_t \overline{\in} \vee \overline{q_k} f$.

If $k = 0$, then it will be $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G.

Theorem 3.2. *For any $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G, the following relations hold; (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) for all $x, y \in G$ and $t, r \in (0,1]$ and $k \in [0,1)$, where*

(1) $(xy)_{M\{t,r\}} \overline{\in} f$ *implies* $x_t \overline{\in} \vee \overline{q_k} f$ *or* $y_r \overline{\in} \vee \overline{q_k} f$.

(2) $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$.

(3) $x_t^{-1}\overline{\in} f$ *implies* $x_t \overline{\in} \vee \overline{q_k} f$.
(4) $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$.

Proof. Now (1) $\Rightarrow$ (2), Assume that (2) is not true, then there exists some $a, b \in G$ such that $\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\}$ implies there exists some $t \in (\frac{1-k}{2}, 1]$ such that $\max\{f(ab), \frac{1-k}{2}\} < t \leq \min\{f(a), f(b)\}$.

Case-1 : If $\max\{f(ab), \frac{1-k}{2}\} = f(ab)$, then $f(ab) < t \leq \min\{f(a), f(b)\}$ which implies that $f(ab) < t$ and $t \leq \min\{f(a), f(b)\}$ implies $(ab)_t \overline{\in} f$ and $a_t \in f$ or $b_t \in f$.

Case-2 : If $\max\{f(ab), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\frac{1-k}{2} < t \leq \min\{f(a), f(b)\}$ which implies that $\frac{1-k}{2} < t \leq f(a)$ (if $f(a) < f(b)$) or $\frac{1-k}{2} < t \leq f(b)$ (if $f(b) < f(a)$) implies $f(a) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ or $f(b) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $a_t q_k f$ or $b_t q_k f$. Combining case-1 and 2. Then,

$$a_t \in f \text{ and } a_t q_k f \text{ or } b_t \in f \text{ and } b_t q_k f \Longrightarrow a_t \in \wedge q_k f \text{ or } b_t \in \wedge q_k f.$$

Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G implies that $(ab)_{M\{t,r\}} \overline{\in}\ f$ implies $a_t \overline{\in} \vee \overline{q_k} f$ or $b_r \overline{\in} \vee \overline{q_k} f$. So our assumption is wrong and there does not exist any $a, b \in G$ such that

$$\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\}.$$

Hence result (2) is true for all $x, y \in G$.

Conversely (2) $\Rightarrow$ (1). Let for some $x, y \in G$ and $t, r \in (\frac{1-k}{2}, 1]$. Then,

$$(xy)_{M\{t,r\}} \overline{\in} f \Rightarrow f(xy) < \min\{t, r\}.$$

Case-1 : If $\max\{f(xy), \frac{1-k}{2}\} = f(xy)$, then

$$\min\{f(x), f(y)\} \leq f(xy) < \min\{t, r\} \text{ or } \min\{f(x), f(y)\} < \min\{t, r\}$$

which implies that $\min\{f(x), f(y)\} < t$ or $\min\{f(x), f(y)\} < r$ and hence $f(x) < t$ (if $f(x) < f(y)$)or $f(y) < r$ (if $f(y) < f(x)$) implies that $x_t \overline{\in} f$ or $y_r \overline{\in} f$ which implies that $x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$, which is the required result (1).

Case-2 : If $\max\{f(xy), \frac{1-k}{2}\} = \frac{1-k}{2}$, then

$$\min\{f(x), f(y)\} \leq \frac{1-k}{2} \text{ which implies that } f(x) < \frac{1-k}{2} \text{ or } f(y) < \frac{1-k}{2}.$$

Let $x_t \in f \Rightarrow t \leq f(x)$ implies that $f(x) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. So $x_t \overline{q_k} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$.

Or let $y_r \in f \Rightarrow r \leq f(y)$ implies that $f(y) + r + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. So $y_r \overline{q_k} f \Rightarrow y_r \overline{\in} \vee \overline{q_k} f$. Hence $x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$, which is the required result (1).

(3) $\Rightarrow$ (4) Assume that (4) is not true and there exists $a \in G$ implies $a^{-1} \in G$ such that $\max\{f(a^{-1}), \frac{1-k}{2}\} < f(a)$ implies there exist some $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(a^{-1}), \frac{1-k}{2}\} < t \leq f(a).$$

Case-1 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = f(a^{-1})$, then $f(a^{-1}) < t \leq f(a) \Rightarrow a_t^{-1} \overline{\in} f$ and $a_t \in f$.

Case-2 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$, then

$$\frac{1-k}{2} < t \leq f(a) \Rightarrow f(a) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1, \text{ so } a_t q_k f.$$

Combining case-1 and 2 we get

$$a_t \in f \text{ and } a_t q_k f \Rightarrow a_t \in \wedge q_k f.$$

Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G implies that $a_t^{-1} \overline{\in} f$ implies $a_t \overline{\in} \vee \overline{q_k} f$. So our assumption is wrong and there does not exist any $a \in G$ such that

$$\max\{f(a^{-1}), \frac{1-k}{2}\} < f(a).$$

Hence result (4) is true for all $x \in G$.

(4) $\Rightarrow$ (3) Let for some $x^{-1} \in G$ and $t \in (\frac{1-k}{2}, 1]$ we have $(x^{-1})_t \overline{\in} f$ implies $f(x^{-1}) < t$.

Case-1 : If $\max\{f(x^{-1}), \frac{1-k}{2}\} = f(x^{-1})$, then

$$t > f(x^{-1}) \geq f(x) \Rightarrow f(x) < t \Rightarrow x_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f.$$

Case-2 : If $\max\{f(x^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$. So $f(x) \leq \frac{1-k}{2}$. Let

$$x_t \in f \Rightarrow t \leq f(x) \Rightarrow f(x) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

which implies $x_t \overline{q_k} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$. The required result (3). □

The following Corollary is particular case of Theorem 3.2. If $k = 0$, then the following corollary obtain. This mean that Theorem 3.2 is a generalization.

Corollary 3.3. *For any $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G, we have* (1) $\Leftrightarrow$ (2) *and* (3) $\Leftrightarrow$ (4) *for all $x, y \in G$ and $t, r \in (0, 1]$ and $k \in [0, 1)$.*

(1) $(xy)_{M\{t,r\}} \overline{\in} f$ *implies* $x_t \overline{\in} \vee \overline{q} f$ *or* $y_r \overline{\in} \vee \overline{q} f$

(2) $\max\{f(xy), 0.5\} \geq \min\{f(x), f(y)\}$

(3) $x_t^{-1} \overline{\in} f$ *implies* $x_t \overline{\in} \vee \overline{q} f$

(4) $\max\{f(x^{-1}), 0.5\} \geq f(x)$

Proof. The proof follows from Theorem 3.2. □

Theorem 3.4. *A fuzzy subset f of a group G is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G if and only if the following conditions hold for all $x, y \in G$ and $t, r \in (0, 1]$ and $k \in [0, 1)$.*

(1) *For all* $x, y \in G$, $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ *and*

(2) *For all* $x \in G$, $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$.

Proof. Suppose fuzzy subset f of a group G is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G. Now first prove result (1). Assume that there exist some $a, b \in G$ such that $\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\}$ implies there exists some $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(ab), \frac{1-k}{2}\} < t \leq \min\{f(a), f(b)\}.$$

Case-1 : If $\max\{f(ab), \frac{1-k}{2}\} = f(ab)$, then $f(ab) < t \leq \min\{f(a), f(b)\}$ which implies that $f(ab) < t$ and $t \leq \min\{f(a), f(b)\}$, so $(ab)_t \overline{\in} f$ and $t \leq f(a)$ or $t \leq f(b)$ implies $(ab)_t \overline{\in} f$ and $a_t \in f$ or $b_t \in f$.

Case-2 : If $\max\{f(ab), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\frac{1-k}{2} < t \leq \min\{f(a), f(b)\}$ which implies that $\frac{1-k}{2} < t \leq f(a)$(if $f(a) < f(b)$) or $\frac{1-k}{2} < t \leq f(b)$(if $f(b) < f(a)$) and hence $f(a)+t+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ or $f(b)+t+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$, so a_tq_kf or b_tq_kf. Combining case-1 and 2 we get

$$a_t\in f \text{ and } a_tq_kf \text{ or } b_t\in f \text{ and } b_tq_kf\Rightarrow a_t\in\wedge q_kf \text{ or } b_t\in\wedge q_kf$$

Since f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy subgroup of G implies that $(ab)_{M\{t,r\}}\overline{\in}f\Rightarrow a_t\overline{\in}\vee\overline{q_k}f$ or $b_r\overline{\in}\vee\overline{q_k}f$. So our assumption is wrong and there does not exist any $a,b\in G$ such that

$$\max\{f(ab),\frac{1-k}{2}\}<\min\{f(a),f(b)\}.$$

Hence result-(1) is true for all $x,y\in G$.

Now result-(2) Assume that 2 is not true and there exist some $a\in G$ implies $a^{-1}\in G$ such that $\max\{f(a^{-1}),\frac{1-k}{2}\}<f(a)$ implies there exist some $t\in(\frac{1-k}{2},1]$ such that

$$\max\{f(a^{-1}),\frac{1-k}{2}\}<t\leq f(a).$$

Case-1 : If $\max\{f(a^{-1}),\frac{1-k}{2}\}=f(a^{-1})$, then $f(a^{-1})<t\leq f(a)$ implies that $a_t^{-1}\overline{\in}f$ and $a_t\in f$.

Case-2 : If $\max\{f(a^{-1}),\frac{1-k}{2}\}=\frac{1-k}{2}$, then $\frac{1-k}{2}<t\leq f(a)$ which implies that $f(a)+t+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$, so a_tq_kf.

Combining case-1 and 2 we get

$$a_t\in f \text{ and } a_tq_kf\Longrightarrow a_t\in\wedge q_kf.$$

Since f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy subgroup of G implies that $a_t^{-1}\overline{\in}f\Longrightarrow a_t\overline{\in}\vee\overline{q_k}f$.

So our assumption is wrong and there does not exist any $a\in G$ such that

$$\max\{f(a^{-1}),\frac{1-k}{2}\}<f(a).$$

Hence result(2) is true for all $x\in G$.

Conversely; Suppose conditions (1) and (2) are true then show that fuzzy subset f of a group G is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy subgroup of G. Let suppose that there exist some $x,y\in G$ and $t,r\in(\frac{1-k}{2},1]$, such that

$$(xy)_{M\{t,r\}}\overline{\in}f\Rightarrow f(xy)<\min\{t,r\}$$

Case-1 : If $\max\{f(xy),\frac{1-k}{2}\}=f(xy)$, then $\min\{f(x),f(y)\}\leq f(xy)<\min\{t,r\}$ which implies that $\min\{f(x),f(y)\}<\min\{t,r\}$ implies $\min\{f(x),f(y)\}<t$ or $\min\{f(x),f(y)\}<r$ and hence $f(x)<t$ (if $f(x)<f(y)$)or $f(y)<r$ (if $f(y)<f(x)$) implies that $x_t\overline{\in}f$or $y_r\overline{\in}f\Rightarrow x_t\overline{\in}\vee\overline{q_k}f$ or $y_r\overline{\in}\vee\overline{q_k}f$.

Case-2 : If $\max\{f(xy),\frac{1-k}{2}\}=\frac{1-k}{2}$, then $\min\{f(x),f(y)\}\leq\frac{1-k}{2}$ implies that $f(x)\leq\frac{1-k}{2}$ or $f(y)\leq\frac{1-k}{2}$. Let $x_t\in f\Rightarrow t\leq f(x)$. Then, $f(x)+t+k\leq\frac{1-k}{2}+\frac{1-k}{2}+k=1$, so $x_t\overline{q_k}f\Rightarrow x_t\overline{\in}\vee\overline{q_k}f$. Or let $y_r\in f\Rightarrow t\leq f(y)$. Then, $f(y)+t+k\leq\frac{1-k}{2}+\frac{1-k}{2}+k=1$, so $y_r\overline{q_k}f\Rightarrow y_r\overline{\in}\vee\overline{q_k}f$. Combining both cases, then $x_t\overline{\in}\vee\overline{q_k}f$ or $y_r\overline{\in}\vee\overline{q_k}f$. Thus, the required results obtain. Let assume that for some $x^{-1}\in G$ and $t\in(\frac{1-k}{2},1]$. Then,

$$(x^{-1})_t\overline{\in}f\Longrightarrow f(x^{-1})<t.$$

Case-1 : If $\max\{f(x^{-1}),\frac{1-k}{2}\}=f(x^{-1})$, then $t>f(x^{-1})\geq f(x)$ which implies that $f(x)<t\Rightarrow x_t\overline{\in}f\Rightarrow x_t\overline{\in}\vee\overline{q_k}f$.

Case-2 : If $\max\{f(x^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$, so $f(x) \leq \frac{1-k}{2}$. Let

$$x_t \in f \Rightarrow t \leq f(x), \text{ hence } f(x) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1.$$

So $x_t \overline{q_k} f \Longrightarrow x_t \overline{\in} \vee \overline{q_k} f$. This completes the proof. ☐

The following Corollary is particular case of Theorem 3.4. If $k = 0$, then the following corollary obtain. This mean that Theorem 3.4 is a generalization.

Corollary 3.5. *A fuzzy subset f of a group G is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G if and only if the following conditions hold for all $x, y \in G$ and $t, r \in (0, 1]$ and $k \in [0, 1)$.*
(1) *For all $x, y \in G$, $\max\{f(xy), 0.5\} \geq \min\{f(x), f(y)\}$ and*
(2) *For all $x \in G$, $\max\{f(x^{-1}), 0.5\} \geq f(x)$.*

Proof. The proof follows from Theorem 3.4. ☐

Theorem 3.6. *Let f be a fuzzy subset of a group G. Then (1) $\Leftrightarrow$ (2).*
(1) *($t \in (\frac{1-k}{2}, 1]$ implies $f_t \neq \varnothing$ is a subgroup of G).*
(2) *f satisfies the following conditions*
(2.1) *For all $x, y \in G$, $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ and*
(2.2) *For all $x \in G$, $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$.*

Proof. (1) $\Rightarrow$ (2) Assume that condition-(2) is not true and for condition-(2.1) assume that there exists some $a, b \in G$ such that $\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\}$ implies there exists some $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(ab), \frac{1-k}{2}\} < t \leq \min\{f(a), f(b)\}.$$

Case-1 : If $\max\{f(ab), \frac{1-k}{2}\} = f(ab)$, then $f(ab) < t \leq \min\{f(a), f(b)\}$ which implies that $f(ab) < t$ and $t \leq \min\{f(a), f(b)\}$. Hence $(ab) \overline{\in} f_t$ and $t \leq f(a)$(if $f(a) < f(b)$) or $t \leq f(b)$(if $f(b) < f(a)$) implies that a or $b \in f_t$. Since f_t is a subgroup of G. It follows that if $a, b \in f_t \Rightarrow ab \in f_t$ that is $f(ab) \geq t$, a contradiction. Therefore for all $x, y \in G$

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}.$$

Case-2 : If $\max\{f(ab), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $f(ab) < \frac{1-k}{2} < t \leq \min\{f(a), f(b)\}$ which implies that $\frac{1-k}{2} < t \leq f(a)$(if $f(a) < f(b)$) or $\frac{1-k}{2} < t \leq f(b)$(if $f(b) < f(a)$) implies that $f(ab) < t$ and $t \leq f(a)$ or $t \leq f(b) \Longrightarrow (ab) \overline{\in} f_t$ and $a, b \in f_t \Longrightarrow ab \in f_t$. Since f_t is a subgroup of G. It follows that $f(ab) \geq t$, a contradiction. Therefore for all $x, y \in G$

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}.$$

Now (2.2) Assume that there exists some $a \in G$ implies $a^{-1} \in G$ such that

$$\max\{f(a^{-1}), \tfrac{1-k}{2}\} < f(a)$$

implies there exists some $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(a^{-1}), \frac{1-k}{2}\} < t \leq f(a).$$

Case-1 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = f(a^{-1})$, then $f(a^{-1}) < t \leq f(a)$ implies $a^{-1}\overline{\in} f_t$and $a \in f_t$.

Case-2 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $f(a^{-1}) < \frac{1-k}{2} < t \leq f(a)$ which implies that $f(a^{-1}) < t \leq f(a) \Rightarrow a^{-1}\overline{\in} f_t$ and $a \in f_t$, which is a contradiction. Thus for all $x \in G$

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x).$$

Conversely, f satisfies conditions (2.1) and (2.2). Let $t \in (\frac{1-k}{2}, 1]$ such that $f_t \neq \varnothing$ and $x \in f_t$. Then, $f(x) \geq t$ and so $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x) \geq t$ by (2.2). Since $t \in (\frac{1-k}{2}, 1]$, it follows that

$$f\left(x^{-1}\right) \geq t \Longrightarrow x^{-1} \in f_t.$$

Let $x, y \in f_t$ using (2.1), we have

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{t, t\} = t > \frac{1-k}{2}$$

and thus $f(xy) \geq t \Longrightarrow (xy) \in f_t$. Consequently, f_t is a subgroup of G for all $t \in (\frac{1-k}{2}, 1]$. □

The following Corollary is particular case of Theorem 3.6. If $k = 0$, then the following corollary obtain. This mean that Theorem 3.6 is a generalization.

Corollary 3.7. *Let f be a fuzzy subset of a group G. Then* (1) $\Leftrightarrow$ (2)
(1) *($t \in (0.5, 1]$ implies $f_t \neq \varnothing$ is a subgroup of G).*
(2) *f satisfies the following conditions*
(2.1) *For all $x, y \in G$, $\max\{f(xy), 0.5\} \geq \min\{f(x), f(y)\}$ and*
(2.2) *For all $x \in G$, $\max\{f(x^{-1}), 0.5\} \geq f(x)$.*

Proof. The proof follows from Theorem 3.6. □

Theorem 3.8. *For any fuzzy subset f of G the following are equivalent;*
(1) . *f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G.*
(2) *(For all $t \in (\frac{1-k}{2}, 1]$ implies $f_t \neq \varnothing$ is a subgroup of G).*

Proof. (1) $\Rightarrow$ (2) Assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G and let $t \in (\frac{1-k}{2}, 1]$ such that $f_t \neq \varnothing$. Let $x, y \in f_t$,using theorem 3.4(1) we have

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{t, t\} = t > \frac{1-k}{2}.$$

Thus, $f(xy) \geq t \Longrightarrow (xy) \in f_t$. Let $x \in f_t$ then $f(x) \geq t$ and by using theorem 3.4(2). Then

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x) \geq t.$$

Since $t \in (\frac{1-k}{2}, 1]$, it follows that $f\left(x^{-1}\right) \geq t$ so that $x^{-1} \in f_t$. Consequently, f_t is a subgroup of G for all $t \in (\frac{1-k}{2}, 1]$.

Conversely, assume that (2) is valid. Let $(xy)_{\min\{t,r\}}\,\overline{\in} f$ implies $f\,(xy) < \min\{t,r\}$. Then,

If $f\,(xy) = \max\{f\,(xy)\,, \frac{1-k}{2}\}$ then $\min\{t,r\} > f\,(xy) = \max\{f\,(xy)\,, \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ which implies that $\min\{t,r\} > \min\{f(x), f(y)\} \Rightarrow \min\{f(x), f(y)\} < t$ or $\min\{f(x), f(y)\} < r$ which implies that $f(x) < t$ (if $f\,(x) < f\,(y)$) or $f(y) < r$ (if $f\,(y) < f\,(x)$), so $x_t \overline{\in} f$ or $y_r \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$. Also, if $\max\{f(xy), \frac{1-k}{2}\} = \frac{1-k}{2}$, then

$$\min\{f(x), f(y)\} \leq \frac{1-k}{2} \Rightarrow f(x) \leq \frac{1-k}{2} \text{ or } f(y) \leq \frac{1-k}{2}.$$

Let $x_t \in f \Rightarrow t \leq f(x)$ and $f(x) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Then, $x_t \overline{q_k} f \Longrightarrow x_t \overline{\in} \vee \overline{q_k} f$ or let $y_r \in f \Longrightarrow r \leq f(y)$ and $f(y) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Then, $y_r \overline{q_k} f \Longrightarrow y_r \overline{\in} \vee \overline{q_k} f$.

So $x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$. Let assume that for some $x^{-1} \in G$ and $t \in (\frac{1-k}{2}, 1]$. Then,

$$(x^{-1})_t \overline{\in} f \Rightarrow f(x^{-1}) < t.$$

Case-1 : If $\max\{f(x^{-1}), \frac{1-k}{2}\} = f(x^{-1})$, then $t > f(x^{-1}) \geq f(x)$ which implies that $f(x) < t$, so $x_t \overline{\in} f \Longrightarrow x_t \overline{\in} \vee \overline{q_k} f$.

Case-2 : If $\max\{f(x^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$, so $f(x) \leq \frac{1-k}{2}$, let $x_t \in f \Rightarrow t \leq f(x)$ which implies that $f(x) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $x_t \overline{q_k} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G. ∎

The following Corollary is particular case of Theorem 3.8. If $k = 0$, then the following corollary obtain. This mean that Theorem 3.8 is a generalization.

Corollary 3.9. *For any fuzzy subset f of G the following are equivalent;*

(1) *f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G.*

(2) *(For all $t \in (0.5, 1]$ implies $f_t \neq \varnothing$ is a subgroup of G).*

Proof. The proof follows from Theorem 3.8. ∎

Theorem 3.10. *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$- fuzzy subgroup of a group G. Then for all $(t \in (0, \frac{1-k}{2}]$ and $k \in [0, 1))$ implies*

$$(\underline{Q_k}\,(f,t) = \{x \in G \mid x\underline{q_k} f \text{ implies } f\,(x) + t \geq 1-k\} \neq \varnothing \text{ is a subgroup of } G).$$

Proof. Let $x, y \in \underline{Q_k}\,(f,t)$ that is $x_t \underline{q_k} f$ and $y_t \underline{q_k} f$, for some $t \in (0, \frac{1-k}{2}]$, which implies that $f\,(x) + t \geq 1-k$ and $f\,(y) + t \geq 1-k \Rightarrow f\,(x) \geq 1-k-t$ and $f\,(y) \geq 1-k-t$. Now using theorem 3.4(1), we have

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{1-k-t, 1-k-t\} = 1-k-t$$

Since $t \leq \frac{1-k}{2}$, so only possibility is that $f\,(xy) \geq 1-k-t \Rightarrow f\,(xy)+t \geq 1-k$ implies $(xy)_t\, \underline{q_k} f \Rightarrow (xy) \in \underline{Q_k}\,(f,t)$. Now, if $x \in \underline{Q_k}\,(f,t) \Rightarrow x_t \underline{q_k} f$ which implies that $f\,(x) + t \geq 1-k \Rightarrow f\,(x) \geq 1-k-t$. Now using theorem 3.4(2) Then

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x) \geq 1-k-t \Rightarrow \max\{f(x^{-1}), \frac{1-k}{2}\} \geq 1-k-t$$

Only possibility is that $f(x^{-1}) \geq 1-k-t \Rightarrow f(x^{-1})+t \geq 1-k$ implies $x_t^{-1} q_k f \Rightarrow x^{-1} \in \underline{Q_k}(f,t)$. Hence both conditions of Subgroups are satisfied for $\underline{Q_k}(f,t)$. So $\underline{Q_k}(f,t) \neq \varnothing$ is subgroup of G. □

The following Corollary is particular case of Theorem 3.10. If $k=0$, then the following corollary obtain. This mean that Theorem 3.10 is a generalization.

Corollary 3.11. *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$- fuzzy subgroup of a group G. Then for all $(t \in (0, 0.5])$ implies*

$(\underline{Q}(f,t) = \{x \in G \mid x\underline{q}f$ implies $f(x)+t \geq 1\} \neq \varnothing$ is a subgroup of $G)$.

Proof. The proof follows from Theorem 3.10. □

Theorem 3.12. *Let f be a fuzzy subset of G. Then (1)$\Leftrightarrow$ (2);*

(1) *f satisfies the following conditions*

(a) *For all $x, y \in G$, $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ and*

(b) *For all $x \in G$, $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$. (It can also say as f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G)*

(2) *(For all $t \in (0,1]$ implies $\underline{U_k}(f;t) \neq \varnothing$ is a subgroup of G).*

Proof. (1) $\Rightarrow$ (2) : Let $x, y \in \underline{U_k}(f;t)$. We have to show that $xy \in \underline{U_k}(f;t)$ and $x^{-1}, y^{-1} \in \underline{U_k}(f;t)$. Then we can consider the following four cases, (a') $x, y \in f_t$ that is; $f(x) \geq t$ and $f(y) \geq t$.

(b') $x, y \in \underline{Q_k}(f;t)$ that is; $f(x)+t \geq 1-k$ and $f(y)+t \geq 1-k$

(c') $x \in f_t$ and $y \in \underline{Q_k}(f;t)$ that is; $f(x) \geq t$ and $f(y)+t \geq 1-k$

(d') $x \in \underline{Q_k}(f;t)$ and $y \in f_t$ that is; $f(x)+t \geq 1-k$ and $f(y) \geq t$.

We have

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \longrightarrow (A)$$

Now using (A) and (a'),

$$\begin{aligned} \max\{f(xy), \frac{1-k}{2}\} &\geq \min\{f(x), f(y)\} \geq \min\{t,t\} = t \\ \text{implies } \max\{f(xy), \frac{1-k}{2}\} &\geq t \end{aligned}$$

Since, if $t \geq \frac{1-k}{2}$, then only possibility is that $f(xy) \geq t \Rightarrow (xy) \in f_t \subseteq \underline{U_k}(f;t) \Rightarrow (xy) \in \underline{U_k}(f;t)$. Now using (A) and (b'). Then,

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{1-k-t, 1-k-t\} = 1-k-t$$

Which implies that $\max\{f(xy), \frac{1-k}{2}\} \geq 1-k-t$. If $t \leq \frac{1-k}{2}$, then only possibility is that

$$f(xy) \geq 1-k-t \Longrightarrow (xy) \in \underline{Q_k}(f;t) \subseteq \underline{U_k}(f;t) \Longrightarrow (xy) \in \underline{U_k}(f;t).$$

Now using (A) and (c'),

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{t, 1-k-t\} = 1-k-t$$

Which implies that $\max\{f(xy), \frac{1-k}{2}\} \geq 1-k-t$. If $t \leq \frac{1-k}{2}$, then only possibility is that

$$f(xy) \geq 1-k-t \Longrightarrow (xy) \in \underline{Q_k}(f;t) \subseteq \underline{U_k}(f;t) \Longrightarrow (xy) \in \underline{U_k}(f;t).$$

Now using (A) and (d'),

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\} \geq \min\{1-k-t, t\} = 1-k-t$$

Which implies that $\max\{f(xy), \frac{1-k}{2}\} \geq 1-k-t$. If $t \leq \frac{1-k}{2}$, then only possibility is that

$$f(xy) \geq 1-k-t \Rightarrow (xy) \in \underline{Q_k}(f;t) \subseteq \underline{U_k}(f;t)$$

Which implies that $(xy) \in \underline{U_k}(f;t)$. All above cases shows us that $\underline{U_k}(f;t)$ is closed.

Now if $x \in \underline{U_k}(f;t)$, then $x \in f_t \cup \underline{Q_k}(f;t) \Rightarrow x \in f_t$ or $x \in \underline{Q_k}(f;t)$ which implies that $f(x) \geq t$ or $f(x) + t \geq 1-k$.As given $\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x)$, so

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x) \geq t \Rightarrow \max\{f(x^{-1}), \frac{1-k}{2}\} \geq t$$

Since if $t \geq \frac{1-k}{2}$, then only possibility is that

$$f\left(x^{-1}\right) \geq t \Rightarrow x^{-1} \in f_t \subseteq \underline{U_k}(f;t) \Rightarrow x^{-1} \in \underline{U_k}(f;t).$$

Or $f(x) + t \geq 1-k \Rightarrow f(x) \geq 1-k-t$, which implies that

$$\begin{aligned} \max\{f(x^{-1}), \frac{1-k}{2}\} &\geq f(x) \geq 1-k-t \\ \text{implies that } \max\{f(x^{-1}), \frac{1-k}{2}\} &\geq 1-k-t \end{aligned}$$

Since if $t \leq \frac{1-k}{2}$, then only possibility is that

$$f\left(x^{-1}\right) \geq 1-k-t \Rightarrow x^{-1} \in \underline{Q_k}(f;t) \subseteq \underline{U_k}(f;t) \Rightarrow x^{-1} \in \underline{U_k}(f;t).$$

Hence $\underline{U_k}(f;t)$ is a subgroup of G.

(2) $\Rightarrow$ (1) : Assume that condition (1) is not true and for condition (a) Assume that there exist some $a, b \in G$ such that $\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\}$, then

$$\max\{f(ab), \frac{1-k}{2}\} < t \leq \min\{f(a), f(b)\} \text{ for some } t \in (\frac{1-k}{2}, 1].$$

Case-1 : If $\max\{f(ab), \frac{1-k}{2}\} = f(ab)$, then

$$f(ab) < t \leq \min\{f(a), f(b)\} \text{ implies that } f(ab) < t \text{ and } t \leq \min\{f(a), f(b)\}$$

which shows $(ab) \overline{\in} f_t$ and $t \leq f(a)$(if $f(a) < f(b)$) or $t \leq f(b)$(if $f(b) < f(a)$), then $a, b \in f_t \Rightarrow a, b \in \underline{U_k}(f;t)$ and so $(ab) \in \underline{U_k}(f;t)$ that is $(ab) \in f_t$ or $(ab) \in \underline{Q_k}(f;t)$. Since $\underline{U_k}(f;t)$ is a subgroup of G. It follows that $f(ab) \geq t$ or $f(ab) \geq 1-k-t$, a contradiction. Therefore for all $x, y \in G$,

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}.$$

Case-2 : If $\max\{f(ab), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $f(ab) < \frac{1-k}{2} \leq t \leq \min\{f(a), f(b)\}$ which implies that $f(ab) < \frac{1-k}{2} \leq t$ and $\frac{1-k}{2} \leq t \leq f(a)$(if $f(a) < f(b)$) or $\frac{1-k}{2} \leq t \leq f(b)$(if $f(b) < f(a)$) implies $f(ab) < t$ and $t \leq f(a)$ or $t \leq f(b)$ and hence $f(ab) \leq 1-k-t \leq \frac{1-k}{2} \leq t$ and $a \in f_t$ or $b \in f_t$ which implies that $f(ab) \leq 1-k-t \Rightarrow$

$f(ab)+t \leq 1-k \Longrightarrow (ab)_t \overline{q_k} f$. As $(ab)_t \overline{\in} f$ and $(ab)_t \overline{q_k} f \Rightarrow (ab)_t \overline{\in} \wedge \overline{q_k} f \Longrightarrow (ab) \notin \underline{U_k}(f;t)$. As $a, b \in \underline{U_k}(f;t)$ implies $(ab) \in \underline{U_k}(f;t)$ implies $(ab) \in f_t \cup \underline{Q_k}(f;t)$ implies $(ab) \in f_t$ or $(ab) \in \underline{Q_k}(f;t)$. It follows that $f(ab) \geq t$ or $f(ab)+t \geq 1-k$, a contradiction. Therefore for all $x, y \in G$

$$\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}.$$

Now for (b) assume that there exists some $a \in G$ implies $a^{-1} \in G$ such that $\max\{f(a^{-1}), \frac{1-k}{2}\} < f(a)$ implies there exists some $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(a^{-1}), \frac{1-k}{2}\} < t \leq f(a).$$

Case-1 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = f(a^{-1})$, then $\frac{1-k}{2} < f(a^{-1}) < t \leq f(a) \Rightarrow a^{-1} \overline{\in} f_t$ and $a \in f_t$.

Case-2 : If $\max\{f(a^{-1}), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $f(a^{-1}) < \frac{1-k}{2} \leq t \leq f(a) \Rightarrow f(a^{-1}) < t \leq f(a)$ which implies that $a^{-1} \overline{\in} f_t$ and $a \in f_t$ or $f(a^{-1}) < \frac{1-k}{2} \leq t$ and hence $f(a^{-1}) \leq 1-k-t < \frac{1-k}{2} \leq t$ implies that $f(a^{-1}) \leq 1-k-t \Rightarrow f(a^{-1})+t \leq 1-k$, so $a_t^{-1} \overline{q_k} f$. As $a_t^{-1} \overline{\in} f$ and $a_t^{-1} \overline{q_k} f$ implies $a_t^{-1} \overline{\in} \wedge \overline{q_k} f$ or $a^{-1} \overline{\in} f_t \wedge a^{-1} \overline{\in} \underline{Q_k}(f;t)$ implies $a^{-1} \notin \underline{U_k}(f;t)$. Which is a contradiction to our supposition. Since $\underline{U_k}(f;t)$ is a subgroup of G so for any $a \in \underline{U_k}(f;t) \Longrightarrow a^{-1} \in \underline{U_k}(f;t)$. Thus for all $x \in G$.

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x).$$

This completes the proof. □

The following Corollary is particular case of Theorem 3.12. If $k = 0$, then the following corollary obtain. This mean that Theorem 3.12 is a generalization.

Corollary 3.13. *Let f be a fuzzy subset f of G. Then, (1)⇔(2);*

(1) *f satisfies the following conditions*

(a) *For all $x, y \in G$, $\max\{f(xy), 0.5\} \geq \min\{f(x), f(y)\}$ and*

(b) *For all $x \in G$, $\max\{f(x^{-1}), 0.5\} \geq f(x)$. (It can also be say as f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G)*

(2) *(For all $t \in (0,1]$ implies $\underline{U}(f;t) \neq \varnothing$ is a subgroup of G).*

Proof. The proof follows from Theorem 3.12. □

Theorem 3.14. *A non-empty subset S of a group G is a subgroup of G if and only if its characteristic function C_s is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G.*

Proof. Suppose S is a subgroup of G and $x, y \in G$. Also suppose that $x \notin S$ and $y \notin S$ implies $\min\{C_s(x), C_s(y)\} = 0$. So

$$\max\{C_s(xy), \frac{1-k}{2}\} \geq \min\{C_s(x), C_s(y)\}.$$

Now, if $x \in S$ and $y \in S$, then $(xy) \in S$. Since S is a subgroup of G.

$$\min\{C_s(x), C_s(y)\} = 1 = \max\{C_s(xy), \frac{1-k}{2}\}.$$

If $x \in S$ and $y \notin S$ or $x \notin S$ and $y \in S$, then $\min\{C_s(x), C_s(y)\} = 0$. So

$$\max\{C_s(xy), \frac{1-k}{2}\} \geq \min\{C_s(x), C_s(y)\}.$$

Hence in either case

$$\max\{C_s(xy), \frac{1-k}{2}\} \geq \min\{C_s(x), C_s(y)\}.$$

Now, for second Condition suppose $x \notin S$ implies $C_s(x) = 0$ so

$$\max\{C_s(x^{-1}), \frac{1-k}{2}\} \geq C_s(x).$$

If $x \in S$ implies $x^{-1} \in S$ because S is subgroup of G. So $C_s(x) = 1$ implies $C_s\left(x^{-1}\right) = 1$ implies

$$\max\{C_s(x^{-1}), \frac{1-k}{2}\} \geq C_s(x).$$

Hence both conditions are satisfied so C_s is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G.

Conversely, suppose that characteristic function C_s is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G. To prove S is subgroup of G. Let $x, y \in S$, then $\min\{C_s(x), C_s(y)\} = 1$. Since

$$\max\{C_s(xy), \frac{1-k}{2}\} \geq \min\{C_s(x), C_s(y)\} = 1$$

We have $C_s(xy) = 1$ implies $(xy) \in S$.

Suppose $x \in S$ then $f(x) = 1$. Since

$$\begin{aligned} \max\{C_s(x^{-1}), \frac{1-k}{2}\} &\geq C_s(x) = 1 \\ \text{which implies that } C_s(x^{-1}) &= 1 \Longrightarrow x^{-1} \in S. \end{aligned}$$

So S is subgroup of G. Hence proved. □

Theorem 3.15. *Let I be a subgroup of G. Define a fuzzy subset f of G as follows,*

$$f(x) = \begin{cases} < \frac{1-k}{2} & for \quad x \notin I, \\ 1 & Otherwise. \end{cases}$$

Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$- fuzzy subgroup of G.

Proof. Let $x, y \in G$ and $t, r \in (0, 1]$ such that

$$\begin{aligned} (xy)_{\min\{t,r\}} \overline{\in} f &\Longrightarrow f(xy) < \min\{t, r\} \\ \text{which implies that } f(xy) &\neq 1 \text{ implies } f(xy) < \frac{1-k}{2} \\ \text{implies } (xy) &\notin I \Rightarrow x \notin I \text{ or } y \notin I \\ \text{hence } f(x) &< \frac{1-k}{2} \text{ or } f(y) < \frac{1-k}{2}. \end{aligned}$$

Case-1 : If $\min\{t, r\} > \frac{1-k}{2}$, then

$$\begin{aligned} f(x) &< \frac{1-k}{2} < \min\{t, r\} \text{ or } f(y) < \frac{1-k}{2} < \min\{t, r\} \\ \text{which implies that } f(x) &< t \text{ or } f(y) < r \text{ implies } x_t \overline{\in} f \text{ or } y_r \overline{\in} f \\ &\text{implies } x_t \overline{\in} \vee \overline{q_k} f \text{ or } y_r \overline{\in} \vee \overline{q_k} f. \end{aligned}$$

Case-2: If $\min\{t,r\} \leq \frac{1-k}{2}$, then $t \leq \frac{1-k}{2}$ or $r \leq \frac{1-k}{2}$ which implies that $f(x)+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k=1$ or $f(y)+r+k < \frac{1-k}{2}+\frac{1-k}{2}+k=1$

$$\text{implies that } x_t\overline{q_k}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f \text{ or } y_r\overline{q_k}f \Rightarrow y_r\overline{\in} \vee \overline{q_k}f$$

Hence first condition satisfied.

Now for second condition let for $x^{-1} \in G$ and $t \in (0,1]$ such that $x_t^{-1}\overline{\in}f$ then $f(x^{-1}) < t$ implies $f(x^{-1}) \neq 1$

$$\text{implies } f(x^{-1}) < \frac{1-k}{2} \text{ implies } x^{-1} \notin I \text{ so } x \notin I.$$

Case-1: If $t > \frac{1-k}{2}$, then $f(x) < \frac{1-k}{2} < t$ which implies that

$$f(x) < t \text{ implies that } x_t\overline{\in}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f.$$

Case-2: If $t \leq \frac{1-k}{2}$, then $f(x)+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k=1$ which implies that $f(x)+t+k<1$ implies $f(x)+t<1-k$ implies $x_t\overline{q_k}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f$. So second condition also satisfied. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G. □

Theorem 3.16. *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$- fuzzy subgroup of G. Then,* (1) *If there exist $x \in G$ such that $f(x) \geq \frac{1-k}{2}$ then $f(e) \geq \frac{1-k}{2}$.*

Proof. (1) Suppose that $f(x) \geq \frac{1-k}{2}$ for some $x \in G$, then

$$\max\{f(x^{-1}), \frac{1-k}{2}\} \geq f(x) \geq \frac{1-k}{2}$$

Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G implies $f(x^{-1}) \geq \frac{1-k}{2}$ and so

$$\begin{aligned}
\max\{f(e), \frac{1-k}{2}\} &= \max\{f(xx^{-1}), \frac{1-k}{2}\} \geq \min\{f(x), f(x^{-1})\} \\
&\geq \min\{\frac{1-k}{2}, \frac{1-k}{2}\} = \frac{1-k}{2} \\
&\Longrightarrow \max\{f(e), \frac{1-k}{2}\} \geq \frac{1-k}{2} \Longrightarrow f(e) \geq \frac{1-k}{2}.
\end{aligned}$$

This completes the proof. □

The following Corollary is particular case of Theorem 3.16. If $k=0$, then the following corollary obtain. This mean that Theorem 3.16 is a generalization.

Corollary 3.17. *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G.* (1) *If there exist $x \in G$ such that $f(x) \geq 0.5$ then $f(e) \geq 0.5$.*

Proof. The proof follows from Theorem 3.16. □

Theorem 3.18. *Let G be a group of prime order. Then, if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G such that $f(a) \geq \frac{1-k}{2}$ for some element $a\,(\neq e) \in G$, then $f(x) \geq \frac{1-k}{2}$ for all $x \in G$.*

Proof. Let $x \in G$ and assume that there exists an element $a\,(\neq e) \in G$ such that

$$f(a) \geq \frac{1-k}{2}.$$

Then $G =< a >$, and so $x = a^p$ for some positive prime integer p.

Since f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy subgroup of G, then

$$\begin{aligned}\max\{f(a^2),\frac{1-k}{2}\} &= \max\{f(aa),\frac{1-k}{2}\}\geq\min\{f(a),f(a)\}\\ &\geq \min\{\frac{1-k}{2},\frac{1-k}{2}\}=\frac{1-k}{2}\end{aligned}$$

Which implies $\max\{f(a^2),\frac{1-k}{2}\} \geq \frac{1-k}{2}\Longrightarrow f(a^2)\geq\frac{1-k}{2}$.

And also we have

$$\begin{aligned}\max\{f(a^3),\frac{1-k}{2}\} &= \max\{f(a^2a),\frac{1-k}{2}\}\geq\min\{f(a^2),f(a)\}\\ &\geq \min\{\frac{1-k}{2},\frac{1-k}{2}\}=\frac{1-k}{2}\end{aligned}$$

Which implies $\max\{f(a^3),\frac{1-k}{2}\}\geq\frac{1-k}{2}\Longrightarrow f(a^3)\geq\frac{1-k}{2}$. Similarly $f(a^5)\geq\frac{1-k}{2}$ and so on. Generally we get $f(a^p)\geq\frac{1-k}{2}$ for every positive prime integer p.

Thus $f(x)\geq\frac{1-k}{2}$ for all $x\in G$, where G be a group of prime order. □

The following Corollary is particular case of Theorem 3.18. If $k=0$, then the following corollary obtain. This mean that Theorem 3.18 is a generalization.

Corollary 3.19. *Let G be a group of prime order. Then, if f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subgroup of G such that $f(a)\geq 0.5$ for some element $a\,(\neq e)\in G$, then $f(x)\geq 0.5$ for all $x\in G$.*

Proof. The proof follows from Theorem 3.18. □

Theorem 3.20. *Let f be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy subgroup of G and let $x,y\in G$ such that $f(x)<f(y)$, then*

(1) $f(xy)<\frac{1-k}{2}$ or $f(yx)<\frac{1-k}{2}$ implies $f(x)<\frac{1-k}{2}$ for all $x,y\in G$.
(2) $f(xy)\geq\frac{1-k}{2}$ implies $f(xy)\geq f(x)$ or $f(yx)\geq\frac{1-k}{2}$ implies $f(yx)\geq f(x)$.

Proof. (1) If $f(xy)<\frac{1-k}{2}$ implies that $(xy)\overline{\in}f_{\frac{1-k}{2}}$. Since $f_{\frac{1-k}{2}}$ is a subgroup of G, it follows that

$$x\overline{\in}f_{\frac{1-k}{2}}\text{ or }y\overline{\in}f_{\frac{1-k}{2}}\Rightarrow f(x)<\frac{1-k}{2}\text{ or }f(y)<\frac{1-k}{2}$$

for $x,y\in G$. So

$$f(xy)<\frac{1-k}{2}\Longrightarrow f(x)<\frac{1-k}{2}\text{ because }(f(x)<f(y)).$$

Similarly for $f(yx)<\frac{1-k}{2}$ implies $f(x)<\frac{1-k}{2}$.

(2) Suppose that $f(xy)\geq\frac{1-k}{2}$, so by definition

$$\max\{f(xy),\frac{1-k}{2}\}\geq\min\{f(x),f(y)\}\Longrightarrow f(xy)\geq f(x).$$

Similarly for $f(yx)\geq\frac{1-k}{2}$, so by definition

$$\max\{f(yx),\frac{1-k}{2}\}\geq\min\{f(y),f(x)\}\Longrightarrow f(yx)\geq f(x).$$

This completes the proof. □

Definition 3.21. Let f be a fuzzy subset of a group G. We define the upper part f^{+} as follows, $f^{+}(x) = f(x) \vee \frac{1-k}{2}$.

Theorem 3.22. *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of a group G, then f^{+} is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G.*

Proof. Let $x, y \in G \Longrightarrow x^{-1}, y^{-1} \in G$. Then

$$\begin{aligned}
\max\{\left(f \vee \frac{1-k}{2}\right)(xy), \frac{1-k}{2}\} &= \left(\left(f \vee \frac{1-k}{2}\right)(xy)\right) \vee \frac{1-k}{2} \\
&= \left(f(xy) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \\
&\geq (f(x) \wedge f(y)) \vee \frac{1-k}{2} \text{ (since } f \text{ is subgroup)} \\
&\geq \left(f(x) \vee \frac{1-k}{2}\right) \wedge \left(f(y) \vee \frac{1-k}{2}\right) \\
&\geq \min\{f(x) \vee \frac{1-k}{2}, f(y) \vee \frac{1-k}{2}\} \\
&\geq \min\{\left(f \vee \frac{1-k}{2}\right)(x), \left(f \vee \frac{1-k}{2}\right)(y)\}.
\end{aligned}$$

So

$$\max\{\left(f \vee \frac{1-k}{2}\right)(xy), \frac{1-k}{2}\} \geq \min\{\left(f \vee \frac{1-k}{2}\right)(x), \left(f \vee \frac{1-k}{2}\right)(y)\}.$$

Now let

$$\begin{aligned}
\max\{\left(f \vee \frac{1-k}{2}\right)(x^{-1}), \frac{1-k}{2}\} &= \left(\left(f \vee \frac{1-k}{2}\right)(x^{-1})\right) \vee \frac{1-k}{2} \\
&= \left(f(x^{-1}) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \\
&\geq f(x) \vee \frac{1-k}{2} = \left(f \vee \frac{1-k}{2}\right)(x) \\
\text{Hence } \max\{\left(f \vee \frac{1-k}{2}\right)(x^{-1}), \frac{1-k}{2}\} &\geq \left(f \vee \frac{1-k}{2}\right)(x)
\end{aligned}$$

Hence f^{+} is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroup of G. □

Lemma 3.23. *Let A and B be non-empty subsets of a group G. Then the following holds.*

(1) $(C_A \wedge C_B)^{+} = C_{A\cap B}^{+}$
(2) $(C_A \vee C_B)^{+} = C_{A\cup B}^{+}$
(3) $(C_A \circ C_B)^{+} = C_{AB}^{+}$

Proof. (1) Let a be any element of G. Suppose $a \in A \cap B$. Then $a \in A$ and $a \in B$. So

$$\begin{aligned}
(C_A \wedge C_B)^+(a) &= (C_A \wedge C_B)(a) \vee \frac{1-k}{2} \\
&= (C_A(a) \wedge C_B(a)) \vee \frac{1-k}{2} \\
&= \left(\left(C_A(a) \vee \frac{1-k}{2}\right) \wedge \left(C_B(a) \vee \frac{1-k}{2}\right)\right) \\
&= \left(\left(1 \vee \frac{1-k}{2}\right) \wedge \left(1 \vee \frac{1-k}{2}\right)\right) \\
&= (1 \wedge 1) = 1 \\
\text{Thus } (C_A \wedge C_B)^+(a) &= C^+_{A\cap B}(a).
\end{aligned}$$

If $a \notin A \cap B$. Then $a \notin A$ or $a \notin B$.
So

$$\begin{aligned}
(C_A \wedge C_B)^+(a) &= (C_A \wedge C_B)(a) \vee \frac{1-k}{2} \\
&= (C_A(a) \wedge C_B(a)) \vee \frac{1-k}{2} \\
&= 0 \vee \frac{1-k}{2} = \frac{1-k}{2} \\
\text{Thus } (C_A \wedge C_B)^+(a) &= C^+_{A\cap B}(a).
\end{aligned}$$

(2) Let $a \in A \cup B$. Then $a \in A$ or $a \in B$. So

$$\begin{aligned}
(C_A \vee C_B)^+(a) &= (C_A \vee C_B)(a) \vee \frac{1-k}{2} \\
&= C_A(a) \vee C_B(a) \vee \frac{1-k}{2} \\
&= 1 \\
\text{Thus } (C_A \vee C_B)^+(a) &= C^+_{A\cup B}(a)
\end{aligned}$$

If $a \notin A \cup B$. Then $a \notin A$ and $a \notin B$. So

$$\begin{aligned}
(C_A \vee C_B)^+(a) &= (C_A \vee C_B)(a) \vee \frac{1-k}{2} \\
&= C_A(a) \vee C_B(a) \vee \frac{1-k}{2} \\
&= \frac{1-k}{2} \\
&= C^+_{A\cup B}(a).
\end{aligned}$$

(3) Let a be any element of G. Suppose $a \in AB$. Then $a = xy$ for some $x \in A$ and $y \in B$. Thus.

$$\begin{aligned}
(C_A \circ C_B)^+ (a) &= (C_A \circ C_B)(a) \vee \frac{1-k}{2} \\
&= (\vee_{a=uv}\{C_A(u) \wedge C_B(v)\}) \vee \frac{1-k}{2} \\
&\geq \{C_A(x) \wedge C_B(y)\} \vee \frac{1-k}{2} \\
&\geq \{1 \wedge 1\} \vee \frac{1-k}{2} \\
&\geq \{\left(1 \vee \frac{1-k}{2}\right) \wedge \left(1 \vee \frac{1-k}{2}\right)\} \\
&\geq 1 \wedge 1 = 1 \\
\text{And so } (C_A \circ C_B)^+ (a) &= 1.
\end{aligned}$$

Since $a \in AB$, $C^+_{AB}(a) = 1$. So $(C_A \circ C_B)^+ (a) = C^+_{AB}(a)$. Now if $a \notin AB$, then $a \neq xy$ for all $x \in A$ and $y \in B$. If $a = uv$ for some $u, v \in G$. Then,

$$\begin{aligned}
(C_A \circ C_B)^+ (a) &= (C_A \circ C_B)(a) \vee \frac{1-k}{2} \\
&= (\vee_{a=uv}\{C_A(u) \wedge C_B(v)\}) \vee \frac{1-k}{2} \\
&= 0 \vee \frac{1-k}{2} = \frac{1-k}{2} = C^+_{AB}(a). \\
\text{Thus } (C_A \circ C_B)^+ (a) &= C^+_{AB}(a).
\end{aligned}$$

□

4. Generalization of $(\overline{\in}, \overline{\in} \vee \overline{q})$-level subsets

Let $t \in (0,1]$ and $k \in [0,1)$. For a fuzzy point x_t and a fuzzy subset f of a set G,then

(1) $x_t \overline{\in} f$ if $f(x) < t$.
(2) $x_t \overline{q} f$ if $f(x) + t \leq 1$.
(3) $x_t \overline{q_k} f$ if $f(x) + t + k \leq 1$ or $f(x) + t \leq 1 - k$.
(4) $x_t \overline{\underline{q}} f$ if $f(x) + t < 1$.
(5) $x_t \overline{\underline{q_k}} f$ if $f(x) + t + k < 1$ or $f(x) + t < 1 - k$.
(6) $x_t \overline{\overline{\in}} \vee \overline{q_k}\ f$ if $x_t \overline{\in} f$ or $x_t \overline{q_k} f$.
(7) $x_t \overline{\alpha} f$ if $x_t \alpha f$ does not hold such that $\alpha \in \{\in, q, q_k, \underline{q}, \underline{q_k}\}$.

Definition 4.1. Let f be a fuzzy subset of a set G and $t \in (0,1]$ and $k \in [0,1)$. Then the set

$\overline{f_t} = \{x \in G \mid f(x) < t\}$ is called the $\overline{\in}$-Level subset of G.

$\overline{Q}(f;t) = \{x \in G \mid f(x) + t \leq 1\} = \{x \in G \mid x_t \overline{q} f\}$ is called the $\overline{q}$-Level subset of G.

$\overline{\underline{Q}}(f;t) = \{x \in G \mid f(x) + t < 1\} = \{x \in G \mid x_t \overline{\underline{q}} f\}$ is called the $\overline{\underline{q}}$-Level subset of $\overline{G}$.

$\overline{Q_k}(f;t) = \{x \in G \mid f(x) + t + k \leq 1\} = \{x \in G \mid x_t \overline{q_k} f\}$ is called the $\overline{q_k}$-Level subset of G.

$\underline{Q_k}(f;t) = \{x \in G \mid f(x) + t + k \geq 1\} = \{x \in G \mid x_t \underline{q_k} f\}$ is called the $\underline{q_k}$-Level subset of G.

$\overline{U}(f;t) = \{x \in G \mid x_t \overline{\in \vee q} f \text{ or } x_t \overline{\in} \wedge \overline{q} f\} = \overline{f_t \cup Q(f;t)} = \overline{f_t} \cap \overline{Q}(f;t)$ is called $(\overline{\in \vee q})$-Level subset or $(\overline{\in} \wedge \overline{q})$-Level subset of G.

$\overline{U_k}(f;t) = \{x \in G \mid x_t \overline{\in \vee q_k}\ f \text{ or } x_t \overline{\in} \wedge \overline{q_k} f\} = \overline{f_t \cup Q_k(f;t)} = \overline{f_t} \cap \overline{Q_k}(f;t)$ is called $(\overline{\in \vee q_k})$-Level subset or $(\overline{\in} \wedge \overline{q_k})$-Level subset of G.

$\underline{U_k}(f;t) = \{x \in G \mid x_t \in \vee \underline{q_k}\ f\} = f_t \cup \underline{Q_k}(f;t)$ is called $\left(\in \vee \underline{q_k}\right)$-Level subset of G.

Note: $\overline{U_k}(f;t) \subseteq \overline{U}(f;t) \subseteq \overline{f_t}$ for any $t \in (0,1]$ and $k \in [0,1)$, where f are fuzzy subsets of G. However, the reverse inclusion may not be true.

Example 4.2. Let f be a fuzzy subset of a set $G = \{g,h,i,j,k,l\}$ defined by

$$f = \begin{pmatrix} g & h & i & j & k & l \\ 0.7 & 0.3 & 0.8 & 0.2 & 0.5 & 0.6 \end{pmatrix}$$

Then we get for $t = 0.55 \in (0,1]$ and $k = 0.23 \in [0,1)$.

$$\overline{f_{0.55}} = \{x \in G | f(x) < 0.55\} = \{h,j,k\},$$

$$\overline{Q}(f;0.55) = \{x \in G | f(x) + 0.55 \leq 1\} = \{x \in G | f(x) \leq 1 - 0.55 = 0.45\} = \{h,j\}.$$

From here we noted that $\overline{Q}(f;0.55) \subseteq \overline{f_{0.55}}$. Hence for all $t \in (0,1]$ we have $\overline{Q}(f;t) \subseteq \overline{f_t}$. Now for some $k = 0.23 \in [0,1)$

$$\begin{aligned} \overline{Q_{0.23}}(f;0.55) &= \{x \in G | f(x) + 0.55 \leq 1 - 0.23\} \\ &= \{x \in G | f(x) \leq 1 - 0.23 - 0.55 = 0.22\} = \{j\}. \end{aligned}$$

From above results it is clear that $\overline{Q_{0.23}}(f;0.55) \subseteq \overline{Q}(f;0.55) \subseteq \overline{f_{0.55}}$, that is; for all $t \in (0,1]$ and $k \in [0,1)$, we have

$$\overline{Q_k}(f;t) \subseteq \ \overline{Q}(f;t) \subseteq \overline{f_t}.$$

And now we find $\overline{U}(f;t)$ and $\overline{U_k}(f;t)$ for above values, then

$$\overline{U}(f;t) = \overline{U}(f;0.55) = \overline{f_{0.55}} \cap \overline{Q}(f;0.55) = \{h,j,k\} \cap \{h,j\} = \{h,j\}.$$

And now

$$\overline{U_k}(f;t) = \overline{U_{0.23}}(f;0.55) = \overline{f_{0.55}} \cap \overline{Q_{0.23}}(f;0.55) = \{h,j,k\} \cap \{j\} = \{j\}.$$

Hence from the above results we proved that

$$\overline{U_k}(f;t) \subseteq \ \overline{U}(f;t) \subseteq \overline{f_t}$$

for all $t = 0.55 \in (0,1]$ and $k = 0.23 \in [0,1)$. Note that if $m < n$ then $\overline{f_m} \subseteq \overline{f_n}$ but $\overline{Q_k}(f;n) \subseteq \overline{Q_k}(f;m)$ where $m,n \in (0,1]$ and $k \in [0,1)$.

Proposition 4.3. *Let f be a fuzzy subset of a set G. For any $m,n \in [0,1)$ where $m < n$, we have for all $t \in (0,1]$, $\overline{U_n}(f;t) \subseteq \overline{U_m}(f;t)$.*

Proof. If we take $m < n$ that is; $0.23 < 0.4$ and let for $t = 0.55$. By using above example then that $\overline{f_{0.55}} = \{x \in G \mid f(x) < 0.55\} = \{b, d, e\}$ and

$$\begin{aligned}\overline{Q_{0.23}}(f; 0.55) &= \{x \in G | f(x) + 0.55 \leq 1 - 0.23\} \\ &= \{x \in G | f(x) \leq 1 - 0.23 - 0.55 = 0.22\} = \{j\}\end{aligned}$$

And

$$\begin{aligned}\overline{Q_{0.4}}(f; 0.55) &= \{x \in G | f(x) + 0.55 \leq 1 - 0.4\} \\ &= \{x \in G | f(x) \leq 1 - 0.4 - 0.55 = 0.05\} = \{\}\end{aligned}$$

$$\overline{U_n}(f; t) = \overline{U_{0.4}}(f; 0.55) = \overline{f_{0.55}} \cap \overline{Q_{0.4}}(f; 0.55) = \{h, j, k\} \cap \{\} = \{\}.$$

And now

$$\overline{U_m}(f; t) = \overline{U_{0.23}}(f; 0.55) = \overline{f_{0.55}} \cap \overline{Q_{0.23}}(f; 0.55) = \{h, j, k\} \cap \{j\} = \{j\}.$$

Hence For any $m, n \in [0, 1)$ where $m < n$, we have for all $t \in (0, 1]$.we have $\overline{U_n}(f; t) \subseteq \overline{U_m}(f; t)$. □

Note that if $t, r \in (0, 1]$ where $t > r$; then $\overline{U_k}(f; t)$ may or may not be a subset of $\overline{U_k}(f; r)$ for some $k \in [0, 1)$. From above example it can find, because $0.55 > 0.11$ and $k = 0.23$ so

$$\overline{U_{0.23}}(f; 0.55) = \overline{f_{0.55}} \cap \overline{Q_{0.23}}(f; 0.55) = \{h, j, k\} \cap \{j\} = \{j\}.$$

And now

$$\begin{aligned}\overline{U_{0.23}}(f; 0.11) &= \overline{f_{0.11}} \cap \overline{Q_{0.23}}(f; 0.11) \\ &= \{x \in G | f(x) < 0.11\} \cap \{x \in G | f(x) + 0.11 \leq 1 - 0.23\} \\ &= \{\} \cap \{x \in G | f(x) \leq 1 - 0.23 - 0.11 = 0.66\} \\ &= \{\} \cap \{h, j, k, l\} = \{\}.\end{aligned}$$

So $\overline{U_{0.23}}(f; 0.55) \nsubseteq \overline{U_{0.23}}(f; 0.11)$ and if let take $0.55 > 0.44$ then

$$\begin{aligned}\overline{U_{0.23}}(f; 0.44) &= \overline{f_{0,44}} \cap \overline{Q_{0.23}}(f; 0.44) \\ &= \{x \in G | f(x) < 0.44\} \cap \{x \in G | f(x) + 0.44 \leq 1 - 0.23\} \\ &= \{h, j\} \cap \{x \in G | f(x) \leq 1 - 0.23 - 0.44 = 0.33\} \\ &= \{h, j\} \cap \{h, j\} = \{h, j\}.\end{aligned}$$

But here $\overline{U_{0.23}}(f; 0.55) \subseteq \overline{U_{0.23}}(f; 0.44)$.

Proposition 4.4. *Let* f , g *and* h *be fuzzy subsets of a set* G, *and* $t \in (0, 1]$ *and* $k \in [0, 1)$. *Then*

(1) $\overline{U_k}(f \cup g; t) = \overline{U_k}(f; t) \cup \overline{U_k}(g; t)$
(2) $\overline{U_k}(f \cap g; t) = \overline{U_k}(f; t) \cap \overline{U_k}(g; t)$
(3) $\overline{U_k}(f \cup (g \cap h); t) = \overline{U_k}(f \cup g; t) \cap \overline{U_k}(f \cup h; t)$
(4) $\overline{U_k}(f \cap (g \cup h); t) = \overline{U_k}(f \cap g; t) \cup \overline{U_k}(f \cap h; t)$.

Proof. (1) Let

$$\begin{aligned}
x &\in \overline{U_k}\left(f\cup g;t\right) \iff x_t\overline{\in\vee q_k}(f\cup g)\\
&\iff x_t\overline{\in}\wedge\overline{q_k}(f\cup g) \iff x_t\overline{\in}(f\cup g) \text{ and } x_t\overline{q_k}(f\cup g)\\
&\iff (f\cup g)(x)<t \text{ and } (f\cup g)(x)+t\leq 1-k\\
&\iff [f(x)<t \text{ or } g(x)<t] \text{ and } [f(x)+t\leq 1-k \text{ or } g(x)+t\leq 1-k]\\
&\iff [f(x)<t \text{ and } f(x)+t\leq 1-k] \text{ or } [g(x)<t \text{ and } g(x)+t\leq 1-k]\\
&\iff [x_t\overline{\in}f \text{ and } x_t\overline{q_k}f] \text{ or } [x_t\overline{\in}g \text{ and } x_t\overline{q_k}g]\\
&\iff [x_t\overline{\in}\wedge\overline{q_k}f] \text{ or } [x_t\overline{\in}\wedge\overline{q_k}g]\\
&\iff x_t\overline{\in\vee q_k}f \text{ or } x_t\overline{\in\vee q_k}g\\
&\iff x\in\overline{U_k}\left(f;t\right) \text{ or } x\in\overline{U_k}\left(g;t\right)\\
&\iff x\in\overline{U_k}\left(f;t\right)\cup\overline{U_k}\left(g;t\right).
\end{aligned}$$

So

$$\overline{U_k}\left(f\cup g;t\right)=\overline{U_k}\left(f;t\right)\cup\overline{U_k}\left(g;t\right).$$

(2) Let

$$\begin{aligned}
x &\in \overline{U_k}\left(f\cap g;t\right) \iff x_t\overline{\in\vee q_k}(f\cap g)\\
&\iff x_t\overline{\in}\wedge\overline{q_k}(f\cap g) \iff x_t\overline{\in}(f\cap g) \text{ and } x_t\overline{q_k}(f\cap g)\\
&\iff (f\cap g)(x)<t \text{ and } (f\cap g)(x)+t\leq 1-k\\
&\iff [f(x)<t \text{ and } g(x)<t] \text{ and } [f(x)+t\leq 1-k \text{ and } g(x)+t\leq 1-k]\\
&\iff [f(x)<t \text{ and } f(x)+t\leq 1-k] \text{ and } [g(x)<t \text{ and } g(x)+t\leq 1-k]\\
&\iff [x_t\overline{\in}f \text{ and } x_t\overline{q_k}f] \text{ and } [x_t\overline{\in}g \text{ and } x_t\overline{q_k}g]\\
&\iff [x_t\overline{\in}\wedge\overline{q_k}f] \text{ and } [x_t\overline{\in}\wedge\overline{q_k}g]\\
&\iff x_t\overline{\in\vee q_k}f \text{ and } x_t\overline{\in\vee q_k}g\\
&\iff x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(g;t\right)\\
&\iff x\in\overline{U_k}\left(f;t\right)\cap\overline{U_k}\left(g;t\right).
\end{aligned}$$

So

$$\overline{U_k}\left(f\cap g;t\right)=\overline{U_k}\left(f;t\right)\cap\overline{U_k}\left(g;t\right).$$

(3) Let

$$\begin{aligned}
x &\in \overline{U_k}\left(f\cup(g\cap h);t\right) \iff x\in\overline{U_k}\left(f;t\right)\cup\overline{U_k}\left(g\cap h;t\right)\\
&\iff x\in\overline{U_k}\left(f;t\right) \text{ or } x\in\overline{U_k}\left(g\cap h;t\right)\\
&\iff x\in\overline{U_k}\left(f;t\right) \text{ or } \{x\in\overline{U_k}\left(g;t\right)\cap\overline{U_k}\left(h;t\right)\}\\
&\iff x\in\overline{U_k}\left(f;t\right) \text{ or } \{x\in\overline{U_k}\left(g;t\right) \text{ and } x\in\overline{U_k}\left(h;t\right)\}\\
&\iff [x\in\overline{U_k}\left(f;t\right) \text{ or } x\in\overline{U_k}\left(g;t\right)] \text{ and } [x\in\overline{U_k}\left(f;t\right) \text{ or } x\in\overline{U_k}\left(h;t\right)]\\
&\iff [x\in\overline{U_k}\left(f;t\right)\cup\overline{U_k}\left(g;t\right)] \text{ and } [x\in\overline{U_k}\left(f;t\right)\cup\overline{U_k}\left(h;t\right)]\\
&\iff [x\in\overline{U_k}\left(f\cup g;t\right)] \text{ and } [\overline{U_k}\left(f\cup h;t\right)]\\
&\iff x\in\overline{U_k}\left(f\cup g;t\right)\cap\overline{U_k}\left(f\cup h;t\right).
\end{aligned}$$

So

$$\overline{U_k}\left(f\cup(g\cap h);t\right)=\overline{U_k}\left(f\cup g;t\right)\cap\overline{U_k}\left(f\cup h;t\right).$$

(4) Let

$$\begin{aligned} x &\in \overline{U_k}\left(f\cap(g\cup h);t\right) \iff x\in\overline{U_k}\left(f;t\right)\cap\overline{U_k}\left(g\cup h;t\right)\\ &\iff x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(g\cup h;t\right)\\ &\iff x\in\overline{U_k}\left(f;t\right) \text{ and } \{x\in\overline{U_k}\left(g;t\right)\cup\overline{U_k}\left(h;t\right)\}\\ &\iff x\in\overline{U_k}\left(f;t\right) \text{ and } \{x\in\overline{U_k}\left(g;t\right) \text{ or } x\in\overline{U_k}\left(h;t\right)\}\\ &\iff [x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(g;t\right)] \text{ or } [x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(h;t\right)]\\ &\iff [x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(g;t\right)] \text{ or } [x\in\overline{U_k}\left(f;t\right) \text{ and } x\in\overline{U_k}\left(h;t\right)]\\ &\iff [x\in\overline{U_k}\left(f\cap g;t\right)] \text{ or } [\overline{U_k}\left(f\cap h;t\right)]\\ &\iff x\in\overline{U_k}\left(f\cap g;t\right)\cup\overline{U_k}\left(f\cap h;t\right). \end{aligned}$$

So

$$\overline{U_k}\left(f\cap(g\cup h);t\right)=\overline{U_k}\left(f\cap g;t\right)\cup\overline{U_k}\left(f\cap h;t\right).$$

□

The following Corollary is particular case of Proposition 4.4. If $k=0$, then the following corollary obtain. This mean that Proposition 4.4 is a generalization.

Corollary 4.5. *Let f,g and h be fuzzy subsets of a set G, and $t\in(0,1]$ and $k=0$ then*

(1) $\overline{U}\left(f\cup g;t\right)=\overline{U}\left(f;t\right)\cup\overline{U}\left(g;t\right).$
(2) $\overline{U}\left(f\cap g;t\right)=\overline{U}\left(f;t\right)\cap\overline{U}\left(g;t\right).$
(3) $\overline{U}\left(f\cup(g\cap h);t\right)=\overline{U}\left(f\cup g;t\right)\cap\overline{U}\left(f\cup h;t\right).$
(4) $\overline{U}\left(f\cap(g\cup h);t\right)=\overline{U}\left(f\cap g;t\right)\cup\overline{U}\left(f\cap h;t\right).$

Proof. The proof follows from Proposition 4.4. □

Proposition 4.6. *For a fuzzy subset f of a set G, the following holds $U_k\left(f;t\right)\subseteq\overline{Q}\left(f^c;t\right)\cup\overline{f^c_{t+k}}$, where f^c denotes the compliment of f, that is, $f^c\left(x\right)=1-f\left(x\right)$ for all $x\in G$.*

Proof. We get

$$\begin{aligned} x &\in U_k\left(f;t\right)\Rightarrow x_t\in\vee q_k\Rightarrow x_t\in f \text{ or } x_tq_kf\Rightarrow f\left(x\right)\geq t \text{ or } f\left(x\right)+t+k>1\\ &\Rightarrow f^c\left(x\right)=1-f\left(x\right)\leq 1-t \text{ or } f^c\left(x\right)=1-f\left(x\right)<t+k\\ &\Rightarrow f^c\left(x\right)+t\leq 1 \text{ or } x_{t+k}\overline{\in}f^c\Rightarrow x_t\overline{q}f^c \text{ or } x\in\overline{f^c_{t+k}}\\ &\Rightarrow x\in\overline{Q}\left(f^c;t\right) \text{ or } x\in\overline{f^c_{t+k}}\Rightarrow x\in\overline{Q}\left(f^c;t\right)\cup\overline{f^c_{t+k}}. \end{aligned}$$

So

$$U_k\left(f;t\right)\subseteq\overline{Q}\left(f^c;t\right)\cup\overline{f^c_{t+k}}.$$

□

Proposition 4.7. *For a fuzzy subset f of a set G, the following holds $U_k\left(f;t\right)\subseteq\overline{f^c_{1-t}}\cup\overline{f^c_{t+k}}$, Where f^c denotes the compliment of f, that is $f^c\left(x\right)=1-f\left(x\right)$ for all $x\in G$.*

Proof. We get

$$\begin{aligned} x &\in U_k(f;t) \Rightarrow x_t \in \vee q_k f \Rightarrow x_t \in f \text{ or } x_t q_k f \\ &\Rightarrow f(x) \geq t \text{ or } f(x)+t+k>1 \\ &\Rightarrow f^c(x) = 1-f(x) \leq 1-t \text{ or } f^c(x) = 1-f(x) < t+k \\ &\Rightarrow x_{1-t} \overline{\in} f^c \text{ or } x_{t+k} \overline{\in} f^c \\ &\Rightarrow x \in \overline{f^c_{1-t}} \text{ or } x \in \overline{f^c_{t+k}} \\ &\Rightarrow x \in \overline{f^c_{1-t}} \cup \overline{f^c_{t+k}}. \end{aligned}$$

So

$$U_k(f;t) \subseteq \overline{f^c_{1-t}} \cup \overline{f^c_{t+k}}.$$

□

Proposition 4.8. *For a fuzzy subset f of a set G, the following holds $\overline{U_k}(f;t) \subseteq f^c_{1-t} \cap f^c_{t+k}$, Where f^c denotes the compliment of f, that is, $f^c(x) = 1-f(x)$, for all $x \in G$.*

Proof. We get

$$\begin{aligned} x &\in \overline{U_k}(f;t) \Rightarrow x \notin U_k(f;t) \Rightarrow x_t \overline{\in \vee q_k} f \Rightarrow x_t \overline{\in} \wedge \overline{q_k} f \Rightarrow x_t \overline{\in} f \text{ and } x_t \overline{q_k} f \\ &\Rightarrow f(x) < t \text{ and } f(x)+t+k \leq 1 \\ &\Rightarrow f^c(x) = 1-f(x) > 1-t \text{ and } f^c(x) = 1-f(x) \geq t+k \\ &\Rightarrow x \in f^c_{1-t} \text{ and } x \in f^c_{t+k} \Rightarrow x \in f^c_{1-t} \cap f^c_{t+k}. \end{aligned}$$

So

$$\overline{U_k}(f;t) \subseteq f^c_{1-t} \cap f^c_{t+k}\ .$$

□

Proposition 4.9. *For fuzzy subsets f and g of a set G, the following*

$$(\overline{U_k}(f \cap g;t))^c = \left(\overline{U_k}(f;t) \cap \overline{U_k}(g;t)\right)^c \subseteq \overline{f^c_{1-t}} \cup \overline{g^c_{1-t}} \cup \overline{f^c_{t+k}} \cup \overline{g^c_{t+k}}.$$

Proof. Let

$$\begin{aligned} x &\in \left(\overline{U_k}(f;t) \cap \overline{U_k}(g;t)\right)^c \Rightarrow x \notin \overline{U_k}(f;t) \cap \overline{U_k}(g;t) \\ &\Rightarrow x \notin \overline{U_k}(f;t) \text{ or } x \notin \overline{U_k}(g;t) \\ &\Rightarrow x \in U_k(f;t) \text{ or } x \in U_k(g;t) \\ &\Rightarrow x \in \overline{f^c_{1-t}} \cup \overline{f^c_{t+k}} \text{ or } x \in \overline{g^c_{1-t}} \cup \overline{g^c_{t+k}} \\ &\Rightarrow x \in \overline{f^c_{1-t}} \cup \overline{f^c_{t+k}} \cup \overline{g^c_{1-t}} \cup \overline{g^c_{t+k}} \\ &\Rightarrow x \in \overline{f^c_{1-t}} \cup \overline{g^c_{1-t}} \cup \overline{f^c_{t+k}} \cup \overline{g^c_{t+k}}. \end{aligned}$$

So

$$\left(\overline{U_k}(f;t) \cap \overline{U_k}(g;t)\right)^c \subseteq \overline{f^c_{1-t}} \cup \overline{g^c_{1-t}} \cup \overline{f^c_{t+k}} \cup \overline{g^c_{t+k}}.$$

□

Proposition 4.10. *For fuzzy subsets f and g of a set G, the following*

$$(\overline{U_k}(f \cup g;t))^c = \left(\overline{U_k}(f;t) \cup \overline{U_k}(g;t)\right)^c \subseteq \left(\overline{f^c_{1-t}} \cup \overline{f^c_{t+k}}\right) \cap \left(\overline{g^c_{1-t}} \cup \overline{g^c_{t+k}}\right).$$

Proof. Let

$$\begin{aligned} x &\in \left(\overline{U_k}(f;t) \cup \overline{U_k}(g;t)\right)^c \Rightarrow x \notin \overline{U_k}(f;t) \cup \overline{U_k}(g;t) \\ &\Rightarrow x \notin \overline{U_k}(f;t) \text{ and } x \notin \overline{U_k}(g;t) \\ &\Rightarrow x \in U_k(f;t) \text{ and } x \in U_k(g;t) \\ &\Rightarrow x \in \overline{f^c_{1-t}} \cup \overline{f^c_{t+k}} \text{ and } x \in \overline{g^c_{1-t}} \cup \overline{g^c_{t+k}} \\ &\Rightarrow x \in (\overline{f^c_{1-t}} \cup \overline{f^c_{t+k}}) \cap (\overline{g^c_{1-t}} \cup \overline{g^c_{t+k}}). \end{aligned}$$

So

$$\left(\overline{U_k}(f;t) \cup \overline{U_k}(g;t)\right)^c \subseteq \left(\overline{f^c_{1-t}} \cup \overline{f^c_{t+k}}\right) \cap \left(\overline{g^c_{1-t}} \cup \overline{g^c_{t+k}}\right).$$

□

Proposition 4.11. *For fuzzy subsets f and g of a set G, the following hold*

$$\overline{U_k}(f \cap g;t) = \overline{U_k}(f;t) \cap \overline{U_k}(g;t) \subseteq f^c_{1-t} \cap g^c_{1-t} \cap f^c_{t+k} \cap g^c_{t+k}.$$

Proof. Let

$$\begin{aligned} x &\in \overline{U_k}(f;t) \cap \overline{U_k}(g;t) \Rightarrow x \in \overline{U_k}(f;t) \text{ and } x \in \overline{U_k}(g;t) \\ &\Rightarrow x \in f^c_{1-t} \cap f^c_{t+k} \text{ and } x \in g^c_{1-t} \cap g^c_{t+k} \\ &\Rightarrow x \in f^c_{1-t} \cap f^c_{t+k} \cap g^c_{1-t} \cap g^c_{t+k} \\ &\Rightarrow x \in f^c_{1-t} \cap g^c_{1-t} \cap f^c_{t+k} \cap g^c_{t+k}. \end{aligned}$$

So

$$\overline{U_k}(f \cap g;t) = \overline{U_k}(f;t) \cap \overline{U_k}(g;t) \subseteq f^c_{1-t} \cap g^c_{1-t} \cap f^c_{t+k} \cap g^c_{t+k}.$$

□

Proposition 4.12. *For fuzzy subsets f and g of a set G, the following hold*

$$\overline{U_k}(f \cup g;t) = \overline{U_k}(f;t) \cup \overline{U_k}(g;t) \subseteq (f^c_{1-t} \cap f^c_{t+k}) \cup (g^c_{1-t} \cap g^c_{t+k}).$$

Proof. Let

$$\begin{aligned} x &\in \overline{U_k}(f \cup g;t) = \overline{U_k}(f;t) \cup \overline{U_k}(g;t) \Rightarrow x \in \overline{U_k}(f;t) \cup \overline{U_k}(g;t) \\ &\Rightarrow x \in \overline{U_k}(f;t) \text{ or } x \in \overline{U_k}(g;t) \\ &\Rightarrow x \in f^c_{1-t} \cap f^c_{t+k} \text{ or } x \in g^c_{1-t} \cap g^c_{t+k} \\ &\Rightarrow x \in (f^c_{1-t} \cap f^c_{t+k}) \cup (g^c_{1-t} \cap g^c_{t+k}). \end{aligned}$$

So

$$\overline{U_k}(f \cup g;t) = \overline{U_k}(f;t) \cup \overline{U_k}(g;t) \subseteq (f^c_{1-t} \cap f^c_{t+k}) \cup (g^c_{1-t} \cap g^c_{t+k}).$$

□

5. Conclusions and applications

Our aim is to promote research and development of fuzzy technology by studying the generalized fuzzy subgroups. It is well know that groups are basic structure in many applied science. Due to these possibilities of applications, group are currently widely explored in fuzzy setting. Since the notion of fuzzy subgroup of a group play a vital role in the study of group structure, by using the idea of quasi-k-coincidence of a fuzzy point with a fuzzy set, the authors used idea of Jun et al [7]. to group and defined a new generalization of fuzzy groups. The most important generalization of

Rosenfeld fuzzy group of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroups are introduced. The given concept is a generalization of Bhakat $(\in, \in \vee q)$-level subset by using the idea of a non quasi-k-coincidence of a fuzzy point with a fuzzy set to defined $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-level subset. Furthermore, the authors give some characterization theorems of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroups. From these discussion it conclude that the results of this article are generalization of of results of ordinary fuzzy subgroups and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroups. There are some results on connection between $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subgroups and their generalized level subsets. Further the authors studied some basic properties of generalized level subsets. We hope that the research along this direction can be continued and in fact, this work would serve as a foundation for further study of the theory of groups, it will be necessary to carry out more theoretical research to establish a general framework for the practical applications.

In future our study will be focused on 1) $(\in, \in \vee q_k)$-fuzzy solvable groups and $(\in, \in \vee q_k)$-fuzzy nilpotent groups. 2) $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy solvable groups and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy nilpotent groups.

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