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On prime cubic bi-ideals of semigroups

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Abstract. As we know that fuzzy sets initiated by Zadeh in 1965, have several important extensions and generalizations, e.g., intuitionistic fuzzy sets, L-fuzzy sets, bipolar fuzzy sets, interval-valued fuzzy sets and n-dimensional fuzzy sets etc. Interval-valued fuzzy sets, also called 2-dimensional fuzzy sets are more suitable than ordinary fuzzy sets (1-dimenional fuzzy sets) in mathematical modeling for uncertainties. Cubic sets (3-dimensinal fuzzy sets) are also an important extension of 1-dimensional fuzzy sets. It is also important to note that the (fuzzy) ideals have an essential role in the study of algebraic structures. In this paper we introduce cubic bi-ideals of semigroups and investigate interesting characterization theorems of these classes in terms of cubic bi-ideals. We introduce and study prime, strongly prime, semiprime, irreducible and strongly irreducible cubic bi-ideals of semigroups and characterize semigroups in terms of semiprime and strongly prime cubic bi-ideals. We study the semigroups in which each cubic bi-ideal is prime.

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1. Introduction

Fuzzy sets are initiated by Zadeh [17]. Since then fuzzy set theory developed by different algebraists and others has created great interest among researchers working in different branches of mathematics. Using this concept, Rosenfeld laid the foundation of fuzzy groupoids/fuzzy ideals [14]. In [10, 11, 12, 13], Kuroki introduced and studied the concepts of fuzzy ideal, fuzzy semiprime ideal and fuzzy bi-ideals in semigroups and characterized different classes of semigroups using these concepts. In [15, 16] Shabir et al. introduced the notions of prime fuzzy ideal and prime fuzzy bi-ideal of semigroups and discussed the class of those semigroups in which every fuzzy ideal and consequently, every fuzzy bi-ideal is prime. In [18], exactly ten years later of the concept of a fuzzy set, Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function instead of a real number. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval [0, 1] are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree by an interval or even by a fuzzy set. In the first case we get an interval-valued fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications [1]. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [8] introduced a new notion, called a cubic set, and investigated several properties of this notion. In [4], Jun et al. introduced cubic subsemigroups and cubic left (resp. right) ideals of semigroups. They studied several properties of cubic subsemigroups and cubic left/right ideals, and discussed the relation between them in semigroups. For further reading on cubic sets, we refer to [2, 3, 4, 5, 6, 7, 8, 9]. Extending the studies of cubic ideals of semigroups, carried out by Jun and Khan, we further studied the concept of cubic prime bi-ideals of semigroups.

In the present work, the concept of cubic bi-ideals of a semigroup is introduced. Using this notion cubic prime, cubic semiprime, cubic strongly prime, cubic irreducible and cubic strongly irreducible bi-deals are introduced and their several properties are investigated. The relation between cubic strongly irreducible, cubic semiprime and cubic strongly prime bi-ideals are discussed. In the last section of this paper, we investigate those semigroups in which each cubic bi-ideal is prime cubic bi-ideal.

2. Preliminaries

A non-empty set S together with an associative binary operation "$\cdot$" is called a *semigroup*. A non-empty subset A of a semigroup S is called a *subsemigroup* if $ab \in A$ for all $a, b \in A$. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. A non-empty intersection of any family of bi-ideals of S is either empty or a bi-ideal of S. Also the product of two bi-ideals of S is a bi-ideal of S. A semigroup S is said to be *regular* if for each element $a \in S$, there exists x in S such that $axa = a$. A semigroup S is said to be *intra-regular* if for each element $a \in S$, there exist x and y in S such that $xa^2y = a$.

A bi-ideal B of a semigroup S is called *prime (strongly prime)* if $B_1B_2 \subseteq B$ or $(B_1B_2 \cap B_2B_1 \subseteq B)$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals B_1 and B_2 of S. A bi-ideal B of a semigroup S is called *semiprime* if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S. An *interval valued fuzzy set* (IVF set) $\widetilde{\mu}_A$ defined on a non-empty set X is given by

$$\widetilde{\mu}_A = \left\{\left(x, \left[\mu_A^+(x), \mu_A^-(x)\right]\right) : x \in X\right\}$$

which is briefly denoted by $\widetilde{\mu}_A = [\mu_A^+, \mu_A^-]$, where μ_A^+ and μ_A^- are two fuzzy sets in X such that $\mu_A^+(x) \leq \mu_A^-(x)$ for all $x \in X$.

A cubic set $\mathcal{A}$ in a set X is a structure having the form

$$\mathcal{A} = \{\langle x, \widetilde{\mu}_A(x), f_A(x)\rangle : x \in X\}$$

which is briefly denoted by $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle$ where $\widetilde{\mu} = [\mu_A^+, \mu_A^-]$ is an IVF set in X and f is a fuzzy set in X.

Let $C(X)$ denote the family of cubic sets in a set X. The order relation "$\sqsubseteq$" in the set of all cubic sets is defined by

$\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if $\widetilde{\mu}_A(x) \preceq \widetilde{\mu}_B(x)$, $f_A(x) \geq f_B(x)$ for all $x \in X$.

$$\mathcal{A} \circledcirc \mathcal{B} = \left\{\left\langle x, \left(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B\right)(x), (f_A \circ f_B)(x)\right\rangle : x \in S\right\}$$

which is briefly denoted by $\left\langle \left(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B\right), (f_A \circ f_B)\right\rangle$ where

$$\left(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B\right)(x) = \begin{cases} \operatorname*{rsup}_{x=yz} \left[\operatorname{rmin}\{\widetilde{\mu}_A(y), \widetilde{\mu}_B(z)\}\right] & \text{if } x = yz \text{ for some } y, z \in S, \\ [0,0] & \text{otherwise} \end{cases}$$

and

$$(f_A \circ f_B)(x) = \begin{cases} \bigwedge_{x=yz} [\max\{f_A(y), f_B(z)\}] & \text{if } x = yz \text{ for some } y, z \in S, \\ 1 & \text{otherwise} \end{cases}$$

$\mathcal{A} \sqcap \mathcal{B} = \left\langle \widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_{\mathcal{B}}, f_A \vee f_B \right\rangle$ where

$\left\langle \widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_{\mathcal{B}}, f_A \vee f_B \right\rangle(x) = \left\langle \operatorname{rmin}\left\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x)\right\}, \max\{f_A(x), f_B(x)\} : x \in S\right\rangle$

$\mathcal{A} \sqcup \mathcal{B} = \left\langle \widetilde{\mu}_A \widetilde{\cup} \widetilde{\mu}_B, f_A \wedge f_B \right\rangle$ where

$\left\langle \widetilde{\mu}_A \widetilde{\cup} \widetilde{\mu}_B, f_A \wedge f_B \right\rangle(x) = \left\langle \operatorname{rmax}\left\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x)\right\}, \min\{f_A(x), f_B(x)\} : x \in S\right\rangle$

Lemma 2.1 ([4]). *For any cubic sets $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle, \mathcal{B} = \langle \widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C\rangle$ in a semigroup S, we have*

(1) $\mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C})$,
(2) $\mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C})$,
(3)$\mathcal{A} \circledcirc (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \circledcirc \mathcal{B}) \sqcup (\mathcal{A} \circledcirc \mathcal{C})$,
(4)$\mathcal{A} \circledcirc (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \circledcirc \mathcal{B}) \sqcap (\mathcal{A} \circledcirc \mathcal{C})$.

Lemma 2.2 ([4]). *For any cubic sets $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle, \mathcal{B} = \langle \widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C\rangle$ in a semigroup S, if $\mathcal{A} \sqsubseteq \mathcal{B}$, then $\mathcal{A} \circledcirc \mathcal{C} \sqsubseteq \mathcal{B} \circledcirc \mathcal{C}$ and $\mathcal{C} \circledcirc \mathcal{A} \sqsubseteq \mathcal{C} \circledcirc \mathcal{B}$.*

For a non-empty subset G of X, the cubic characteristic set of G in X is defined to be a structure

$$\chi_G = \{\langle x, \widetilde{\mu}_{\chi_G}(x), f_{\chi_G}(x)\rangle : x \in X\}$$

which is briefly denoted by $\chi_G = \langle \widetilde{\mu}_{\chi_G}, f_{\chi_G}\rangle$

$$\widetilde{\mu}_{\chi_G}(x) = \begin{cases} [1,1] & \text{if } x \in G, \\ [0,0] & \text{otherwise} \end{cases} \quad \text{and } f_{\chi_G}(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 2.3 ([4]). *For non-empty subsets A and B of a semigroup S, we have*

(1) $\chi_A \circledcirc \chi_B = \chi_{AB}$, *i.e.,* $\langle \widetilde{\mu}_{\chi_A} \widetilde{\circ}\, \widetilde{\mu}_{\chi_B}, f_{\chi_A} \circ f_{\chi_B}\rangle = \langle \widetilde{\mu}_{AB}, f_{AB}\rangle$,
(2) $\chi_A \sqcap \chi_B = \chi_{A\cap B}$, *i.e.,* $\left\langle \widetilde{\mu}_{\chi_A} \widetilde{\cap}\, \widetilde{\mu}_{\chi_B}, f_{\chi_A} \vee f_{\chi_B}\right\rangle = \langle \widetilde{\mu}_{A\cap B}, f_{A\vee B}\rangle$,
(3)$\chi_A \sqcup \chi_B = \chi_{A\cup B}$, *i.e.,* $\left\langle \widetilde{\mu}_{\chi_A} \widetilde{\cup}\, \widetilde{\mu}_{\chi_B}, f_{\chi_A} \wedge f_{\chi_B}\right\rangle = \langle \widetilde{\mu}_{A\cup B}, f_{A\wedge B}\rangle$.

Definition 2.4 ([4]). A cubic set $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ of a semigroup S is called a *cubic subsemigroup* of S if

(1) $\widetilde{\mu}_A(xy) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$ for all $x, y \in S$.
(2) $f_A(xy) \leq \max\{f_A(x), f_A(y)\}$ for all $x, y \in S$.

Lemma 2.5 ([4]). *A cubic set $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ in a semigroup S is a cubic subsemigroup of S if and only if $\mathcal{A} \circledcirc \mathcal{A} \sqsubseteq \mathcal{A}$.*

Lemma 2.6 ([4]). *A non-empty subset A of a semigroup S is a subsemigroup of S if and only if the cubic characteristic set $\chi_A = \langle \widetilde{\mu}_A, f_A \rangle$ of A in S is a cubic subsemigroup of S.*

3. Cubic bi-ideals

Definition 3.1. A cubic subsemigroup $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ of S is called a *cubic bi-ideal* of S if

(1) $\widetilde{\mu}_A(xyz) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ for all $x, y, z \in S$.
(2) $f_A(xyz) \leq \max\{f_A(x), f_A(z)\}$ for all $x, y, z \in S$.

Theorem 3.2. *A non-empty subset A of a semigroup S is a cubic bi-ideal of S if and only if the cubic characteristic function $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ of A is a cubic bi-ideal of S.*

Proof. Suppose that A is a bi-ideal of S. Then by Lemma 2.6, $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic subsemigroup of S. Let $x, y, z \in S$. If $x, z \in A$, then by hypothesis, $xyz \in A$ and $\widetilde{\mu}_{\chi_A}(xyz) = [1, 1]$ and $f_{\chi_A}(xyz) = 0$. Thus $\widetilde{\mu}_{\chi_A}(xyz) = [1, 1] \succeq \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f_{\chi_A}(xyz) = 0 \leq \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$. If $x \notin A$ and $z \notin A$ then

Case (i) If $xyz \notin A$, then $\widetilde{\mu}_{\chi_A}(xyz) \succeq [0, 0] = \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f(xyz) \leq 1 = \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$.

Case (ii) If $xyz \in A$, then $\widetilde{\mu}_{\chi_A}(xyz) = [1, 1] \succeq [0, 0] = \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f_{\chi_A}(xyz) = 0 \leq 1 = \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$.

Thus $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic bi-ideal of S.

Conversely, let χ_A be a cubic bi-ideal of S. We have to prove that A is a bi-ideal of S. By Lemma 2.6, A is a subsemigroup of S.

Let $x, z \in A$, then $\widetilde{\mu}_{\chi_A}(x) = [1, 1] = \widetilde{\mu}_{\chi_A}(z)$ and $f_{\chi_A}(x) = 0 = f_{\chi_A}(z)$. Thus $\widetilde{\mu}_{\chi_A}(xyz) \succeq \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\} = [1, 1]$ and $f_{\chi_A}(xyz) \leq \max\{f_{\chi_A}(x), f_{\chi_A}(z)\} = 0$. So $\widetilde{\mu}_{\chi_A}(xyz) = [1, 1]$ and $f_{\chi_A}(xyz) = 0 \Longrightarrow xyz \in A$ for all $x, y, z \in S$. Hence A is a bi-ideal of S. □

Let $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ be a cubic set in S. For any $r \in [0, 1]$ and $[s, t] \in \mathrm{D}[0, 1]$, then the cubic level set $U(\mathcal{A}; [s, t], r)$ of $\mathcal{A}$ is defined by

$U(\mathcal{A}; [s, t], r) = \{x \in S : \widetilde{\mu}_A(x) \succeq [s, t], f_A(x) \leq r\}$.

Theorem 3.3. *Let S be a semigroup and $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ be a cubic subset of S. Then the following are equivalent*

(1) $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ is a cubic bi-ideal of S.
(2) Each non-empty cubic level set $U(\mathcal{A}; [s, t], r)$ is a bi-ideal of S.

Proof. Straightforward. □

Example 3.4. Consider a semigroup $S = \{1, 2, 3, 4, 5\}$ with the following Cayley table

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	4	1
3	1	5	3	3	5
4	1	2	4	4	2
5	1	5	1	3	1

Bi-ideals of S are $\{1\}, \{1, 3\}, \{1, 3, 4\}$ and S. Define a cubic set $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ in S as follows.

S	$\widetilde{\mu}_A(x)$	$f_A(x)$
1	$[0.8, 0.9]$	0.3
2	$[0.2, 0.3]$	0.7
3	$[0.6, 0.7]$	0.4
4	$[0.4, 0.5]$	0.5
5	$[0.2, 0.3]$	0.7

and

$$P(\widetilde{\mu}_A(x); [s, t]) = \begin{cases} S & \text{if } [0, 0] \prec [s, t] \preceq [0.2, 0.3] \\ \{1, 3, 4\} & \text{if } [0.2, 0.3] \prec [s, t] \preceq [0.4, 0.5] \\ \{1, 3\} & \text{if } [0.4, 0.5] \prec [s, t] \preceq [0.6, 0.7] \\ \{1\} & \text{if } [0.6, 0.7] \prec [s, t] \preceq [0.8, 0.9] \\ \varnothing & \text{if } [0.8, 0.9] \prec [s, t] \preceq [1, 1] \end{cases}$$

and

$$N(f_A(x); r) = \begin{cases} S & \text{if } 0.7 \leq r < 1 \\ \{1, 3, 4\} & \text{if } 0.5 \leq r < 0.7 \\ \{1, 3\} & \text{if } 0.4 \leq r < 0.5 \\ \{1\} & \text{if } 0.3 \leq r < 0.4 \\ \varnothing & \text{if } 0 \leq r < 0.3. \end{cases}$$

Then $U(\mathcal{A}; [s, t], r) = P(\widetilde{\mu}_A(x); [s, t]) \cap N(f_A(x); r)$ is a bi-ideal of S and by Theorem 3.3, $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ is a cubic bi-ideal of S.

Lemma 3.5. *A cubic subset* $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ *of a semigroup* S *is a cubic bi-ideal of* S *if and only if* $\mathcal{A} \circledcirc \mathcal{A} \sqsubseteq \mathcal{A}$ *and* $\mathcal{A} \circledcirc \chi_S \circledcirc \mathcal{A} \sqsubseteq \mathcal{A}$, *where* χ_S *is the cubic characteristic function of* S.

Proof. Let $\mathcal{A}$ be a cubic bi-ideal of S, by Definition 3.1, $\mathcal{A}$ is a cubic subsemigroup. We have $\mathcal{A}ⓒ\mathcal{A} \sqsubseteq \mathcal{A}$ by Lemma 2.5. Now

$$\begin{aligned}
(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(x) &= \underset{x=yz}{\text{rsup}}\left[\text{rmin}\left\{(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S})(y), \widetilde{\mu}_A(z)\right\}\right] \\
&= \underset{x=yz}{\text{rsup}}\left[\text{rmin}\left\{\underset{y=pq}{\text{rsup}}\left\{\text{rmin}\left\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S}(q)\right\}\right\}, \widetilde{\mu}_A(z)\right\}\right] \\
&= \underset{y=pqz}{\text{rsup}}\left[\text{rmin}\left\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S}(q), \widetilde{\mu}_A(z)\right\}\right] \\
&= \underset{y=pqz}{\text{rsup}}\left[\text{rmin}\left\{\widetilde{\mu}_A(p), [1,1], \widetilde{\mu}_A(z)\right\}\right] \\
&= \underset{y=pqz}{\text{rsup}}\left[\text{rmin}\left\{\widetilde{\mu}_A(p), \widetilde{\mu}_A(z)\right\}\right] \\
&\preceq \underset{y=pqz}{\text{rsup}}\left[\widetilde{\mu}_A(pqz)\right] \\
&= \widetilde{\mu}_A(x),
\end{aligned}$$

so we have $(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(x) \preceq \widetilde{\mu}_A(x)$ and

$$\begin{aligned}
(f_A \circ f_{\chi_S} \circ f_A)(x) &= \bigwedge_{x=yz} [\max\{(f_A \circ f_{\chi_S})(y), f_A(z)\}] \\
&= \bigwedge_{x=yz}\left[\max\left\{\bigwedge_{y=pq}\{\max\{(f_A(p), f_{\chi_S}(q))\}\}, f_A(z)\right\}\right] \\
&= \bigwedge_{x=yz}\left[\max\left\{\bigwedge_{y=pq}\{\max\{f_A(p), 0\}\}, f_A(z)\right\}\right] \\
&= \bigwedge_{x=pqz} [\max\{f_A(p), f_A(z)\}] \\
&\geq \bigwedge_{x=pqz} [f_A(pqz)] \\
&= f_A(x),
\end{aligned}$$

so $(f_A \circ f_S \circ f_A)(x) \geq f_A(x)$. Hence $\mathcal{A}ⓒ\chi_Sⓒ\mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, assume that $\mathcal{A}ⓒ\chi_Sⓒ\mathcal{A} \sqsubseteq \mathcal{A}$ and $\mathcal{A}ⓒ\mathcal{A} \sqsubseteq \mathcal{A}$. By Lemma 2.5, $\mathcal{A}$ is a cubic subsemigroup of S,

$$\begin{aligned}
\widetilde{\mu}_A(a) &\succeq (\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(a) \\
&= \underset{a=yz}{\text{rsup}}[\text{rmin}\{(\widetilde{\mu}_A \widetilde{\circ}\, \widetilde{\mu}_{\chi_S})(y), \widetilde{\mu}_A(z)\}] \\
&= \underset{a=yz}{\text{rsup}}\left[\text{rmin}\left\{\underset{y=pq}{\text{rsup}}\{\text{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S}(q)\}\}, \widetilde{\mu}_A(z)\right\}\right] \\
&= \underset{a=yz}{\text{rsup}}\ \underset{y=pq}{\text{rsup}}[\text{rmin}\{\widetilde{\mu}_A(p), [1,1], \widetilde{\mu}_A(z)\}] \\
&= \underset{a=pqz}{\text{rsup}}[\text{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_A(z)\}]
\end{aligned}$$

and

$$\begin{aligned} f_A(a) &\leq (f_A \circ f_{\chi_S} \circ f_A)(a) \\ &= \bigwedge_{a=yz} [\max\{(f_A \circ f_{\chi_S})(y), f_A(z)\}] \\ &= \bigwedge_{a=yz} \left[\max\left\{\bigwedge_{y=pq} \{\max\{f_A(p), f_{\chi_S}(q)\}\}, f_A(z)\right\}\right] \\ &= \bigwedge_{a=yz} \bigwedge_{y=pq} [\max\{f_A(p), 0, f_A(z)\}] \\ &= \bigwedge_{a=pqz} [\max\{f_A(p), f_A(z)\}]. \end{aligned}$$

So $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ is a cubic bi-ideal of S. □

Lemma 3.6. *The intersection of any family of cubic bi-ideals of a semigroup S is a cubic bi-ideal of S.*

Proof. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of cubic bi-ideals of S. We have to prove that $\sqcap \mathcal{A}_i = \left\langle \underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}, \bigvee_{i\in I} f_{A_i} \right\rangle$ is a cubic bi-ideal of S. Let $x, y \in S$, then

$$\begin{aligned} &\left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(xy) = \underset{i\in I}{\mathrm{rinf}} \{\widetilde{\mu}_{A_i}(xy) : i \in I \text{ and } x \in S\} \\ &\succeq \underset{i\in I}{\mathrm{rinf}} \{\mathrm{rmin}\{\widetilde{\mu}_{A_i}(x), \widetilde{\mu}_{A_i}(y)\} : i \in I \text{ and } x, y \in S\} \\ &= \mathrm{rmin}\left\{\underset{i\in I}{\mathrm{rinf}} \{\widetilde{\mu}_{A_i}(x), \widetilde{\mu}_{A_i}(y)\} : i \in I \text{ and } x, y \in S\right\} \\ &= \mathrm{rmin}\left\{\underset{i\in I}{\mathrm{rinf}} \{\widetilde{\mu}_{A_i}(x) : i \in I \text{ and } x \in S\}, \underset{i\in I}{\mathrm{rinf}} \{\widetilde{\mu}_{A_i}(y) : i \in I \text{ and } y \in S\}\right\} \\ &= \mathrm{rmin}\left\{\left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(x), \left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(y)\right\} \end{aligned}$$

and

$$\begin{aligned} &\left(\bigvee_{i\in I} f_{A_i}\right)(xy) = \sup\{f_{A_i}(xy) : i \in I \text{ and } x, y \in S\} \\ &\leq \sup\{\max\{f_{A_i}(x), f_{A_i}(y)\} : i \in I \text{ and } x, y \in S\} \\ &= \max\{\sup\{f_{A_i}(x), f_{A_i}(y)\} : i \in I \text{ and } x, y \in S\} \\ &= \max\{\sup\{f_{A_i}(x) : i \in I \text{ and } x \in S\}, \sup\{f_{A_i}(y) : i \in I \text{ and } y \in S\}\} \\ &= \max\left\{\left(\bigvee_{i\in I} \widetilde{\mu}_{A_i}\right)(x), \left(\bigvee_{i\in I} \widetilde{\mu}_{A_i}\right)(y)\right\}. \end{aligned}$$

In a similar way, for $x, y, z \in S$, we get

$$\left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(xyz) \succeq \mathrm{rmin}\left\{\left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(x), \left(\underset{i\in I}{\mathrm{rinf}} \widetilde{\mu}_{A_i}\right)(z)\right\}$$

and

$$\left(\bigvee_{i\in I} f_{A_i}\right)(xyz) \leq \max\left\{\left(\bigvee_{i\in I} f_{A_i}\right)(x), \left(\bigvee_{i\in I} f_{A_i}\right)(z)\right\}.$$

Therefore $\sqcap \mathcal{A}_i = \left\langle \operatorname{rinf}_{i\in I} \widetilde{\mu}_{A_i}, \bigvee_{i\in I} f_{A_i} \right\rangle$ is a cubic bi-ideal of S. □

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be any cubic bi-ideals of a semigroup S. Then $\mathcal{A}©\mathcal{B}$ is also a cubic bi-ideal of S.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cubic bi-ideals of S, it follows from Lemma 3.5 that

$$\begin{aligned}(\mathcal{A}©\mathcal{B})\,©\,(\mathcal{A}©\mathcal{B}) &= \mathcal{A}©\,[\mathcal{B}©\,(\mathcal{A}©\mathcal{B})] \\ &\sqsubseteq \mathcal{A}©\,(\mathcal{B}©\chi_S©\mathcal{B}) \\ &\sqsubseteq \mathcal{A}©\mathcal{B}.\end{aligned}$$

Then by Lemma 3.5, $\mathcal{A}©\mathcal{B}$ is a cubic subsemigroup of S. And we have

$$\begin{aligned}(\mathcal{A}©\mathcal{B})\,©\chi_S©\,(\mathcal{A}©\mathcal{B}) &= \mathcal{A}©\,[\mathcal{B}©\,(\chi_S©\mathcal{A})\,©\mathcal{B}] \\ &\sqsubseteq \mathcal{A}©\,(\mathcal{B}©\chi_S©\mathcal{B}) \\ &\sqsubseteq \mathcal{A}©\mathcal{B}.\end{aligned}$$

Hence, $\mathcal{A}©\mathcal{B}$ is a cubic bi-ideal of S, by Lemma 3.5. □

Definition 3.8. The cubic subset $a_{(\widetilde{t},s)}$ of S defined by

$$a_{(\widetilde{t},s)}(x) = \begin{cases} \left(\widetilde{t}, s\right) & \text{if } x = a \text{ for all } x \in S \\ \left(\widetilde{0}, 1\right) & \text{otherwise,} \end{cases}$$

is called a *cubic point* of S where $a \in S$, $s \in (0,1]$ and $\widetilde{t} \in D[0,1]$. Let $\mathcal{A}$ be any cubic subset of S, it is clear that $\mathcal{A} = {}_{(\widetilde{t},s)\in\mathcal{A}}\, a_{(\widetilde{t},s)}$.

Definition 3.9. Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a cubic subset of a semigroup S. A non-empty intersection of all cubic bi-ideals of S containing $\mathcal{A}$ is also a cubic bi-ideal of S, and is called the *cubic bi-ideal of S generated by $\mathcal{A}$*, denoted by $B(\mathcal{A})$.

Theorem 3.10. *The cubic bi-ideal generated by a cubic point $a_{(\widetilde{t},s)}$ is defined at each $x \in S$ by*

$$\begin{aligned} B\left(a_{(\widetilde{t},s)}\right)(x) &= \left\langle \widetilde{\mu}_{B(a_{\widetilde{t}})}(x), f_{B(a_s)}(x) \right\rangle, \\ \widetilde{\mu}_{B(a_{\widetilde{t}})}(x) &= \begin{cases} \widetilde{t} & \textit{if } x \in B(a) \\ \widetilde{0} & \textit{otherwise,} \end{cases} \\ f_{B(a_s)}(x) &= \begin{cases} s & \textit{if } x \in B(a) \\ 1 & \textit{otherwise,} \end{cases} \end{aligned}$$

where $B(a) = \{a\} \cup aS^1a$ is a bi-ideal of S generated by $a \in S$.

Proof. Let $x, y, z \in S$. If $y, z \in B(a)$ is such that $x = yz$, then

$$\begin{aligned} B\left(a_{(\widetilde{t},s)}\right) \copyright B\left(a_{(\widetilde{t},s)}\right)(x) &= \left\langle \widetilde{\mu}_{B(a_{\widetilde{t}})} \widetilde{\circ} \widetilde{\mu}_{B(a_{\widetilde{t}})}(x), f_{B(a_s)} \circ f_{B(a_s)}(x) \right\rangle \\ (\widetilde{\mu}_{B(a_{\widetilde{t}})} \widetilde{\circ} \widetilde{\mu}_{B(a_{\widetilde{t}})})(x) &= \operatorname*{rsup}_{x=yz} \left[\operatorname{rmin}\left\{ \widetilde{\mu}_{B(a_{\widetilde{t}})}(y), \widetilde{\mu}_{B(a_{\widetilde{t}})}(z) \right\} \right] \\ &= \operatorname*{rsup}_{x=yz} \left[\operatorname{rmin}\{\widetilde{t}, \widetilde{t}\} \right] \\ &= \widetilde{t} \\ &\preceq \widetilde{\mu}_{B(a_{\widetilde{t}})} \end{aligned}$$

and

$$\begin{aligned} (f_{B(a_s)} \circ f_{B(a_s)})(x) &= \bigwedge_{x=yz} \max\{f_{B(a_s)}(y), f_{B(a_s)}(z)\}\} \\ &= \bigwedge_{x=yz} \max\{s, s\} \\ &= s \\ &\geq f_{B(a_s)}(x). \end{aligned}$$

If $y, z \notin B(a)$, then $\operatorname{rmin}\{\widetilde{\mu}_{B(a_{\widetilde{t}})}(y), \widetilde{\mu}_{B(a_{\widetilde{t}})}(z)\} = [0, 0] \preceq \widetilde{\mu}_{B(a_{\widetilde{t}})}(x)$ and $\max\{f_{B(a_s)}(y), f_{B(a_s)}(z)\} = 1 \geq f_{B(a_s)}(x)$. Thus $B\left(a_{(\widetilde{t},s)}\right) \copyright B\left(a_{(\widetilde{t},s)}\right) \sqsubseteq B\left(a_{(\widetilde{t},s)}\right)$.

Hence by Lemma 2.5, $B\left(a_{(\widetilde{t},s)}\right)$ is a cubic subsemigroup of S.

Let $x, y, z \in S$. If $x, z \in B(a)$ then $xyz \in B(a)$ and so

$$B\left(a_{(\widetilde{t},s)}\right)(xyz) = \left\langle \widetilde{\mu}_{B(a_{\widetilde{t}})}(xyz),\ f_{B(a_s)}(xyz) \right\rangle$$

where

$$\widetilde{\mu}_{B(a_{\widetilde{t}})}(xyz) = \widetilde{t} = \operatorname{rmin}\{\widetilde{\mu}_{B(a_{\widetilde{t}})}(x), \widetilde{\mu}_{B(a_{\widetilde{t}})}(z)\}$$

and

$$f_{B(a_s)}(xyz) = s = \max\{f_{B(a_s)}(x), f_{B(a_s)}(z)\}.$$

If $x \notin B(a)$ and $z \notin B(a)$, then $\operatorname{rmin}\{\widetilde{\mu}_{B(a_{\widetilde{t}})}(x), \widetilde{\mu}_{B(a_{\widetilde{t}})}(z)\} = [0, 0] \preceq \widetilde{\mu}_{B(a_{\widetilde{t}})}(xyz)$ and $\max\{f_{B(a_s)}(x), f_{B(a_s)}(z)\} = 1 \geq f_{B(a_s)}(xyz)$. Thus $B\left(a_{(\widetilde{t},s)}\right)$ is a cubic bi-ideal of S. Clearly $a_{(\widetilde{t},s)} \in B\left(a_{(\widetilde{t},s)}\right)$.

Let $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ be a cubic bi-ideal of S containing $a_{(\widetilde{t},s)}$ and let $x \in S$. If $x \notin B(a)$, then

$$\widetilde{\mu}_{B(a_{\widetilde{t}})}(x) = [0, 0] \preceq \widetilde{\mu}_C(x) \text{ and } f_{B(a_s)}(x) = 1 \geq f_C(x).$$

If $x \in B(a) = \{a\} \cup aS^1 a$, then $B\left(a_{(\widetilde{t},s)}\right)(x) = (\widetilde{t}, s)$. If $x = a$, then $a_{(\widetilde{t},s)}(x) = (\widetilde{t}, s) \in \mathcal{C}(x)$. If $x = aya$ for some $y \in S^1$, then

$$\begin{aligned} \mathcal{C}(x) &= \langle \widetilde{\mu}_C(x), f_C(x) \rangle \\ &= \langle \widetilde{\mu}_C(aya), f_C(aya) \rangle, \\ \widetilde{\mu}_C(aya) &\succeq \mathrm{rmin}\{\widetilde{\mu}_C(a),\ \widetilde{\mu}_C(a)\} \\ &\succeq \mathrm{rmin}\{a_{\widetilde{t}}(a), a_{\widetilde{t}}(a)\} \\ &= \widetilde{t} \end{aligned}$$

and

$$\begin{aligned} f_C(aya) &\leq \max\{f_C(a), f_C(a)\} \\ &\leq \max\{a_s(a), a_s(a)\} \\ &= s, \\ B\left(a_{(\widetilde{t},s)}\right)(x) &= (\widetilde{t}, s) \in \mathcal{C}(x). \end{aligned}$$

Hence

$$B\left(a_{(\widetilde{t},s)}\right)(x) \sqsubseteq \mathcal{C}(x).$$

Thus $B\left(a_{(\widetilde{t},s)}\right)(x)$ is the smallest cubic bi-ideal of S containing $a_{(\widetilde{t},s)}$. □

4. Prime cubic bi-ideals

Definition 4.1. A cubic bi-ideal $\mathcal{A} = \left\langle \widetilde{\mu}_A, f_A \right\rangle$ of a semigroup S is called a *prime cubic bi-ideal* if for any cubic bi-ideals $\mathcal{B} = \left\langle \widetilde{\mu}_B, f_B \right\rangle$ and $\mathcal{C} = \left\langle \widetilde{\mu}_C, f_C \right\rangle$ of S, $\mathcal{B} © \mathcal{C} \sqsubseteq \mathcal{A}$ implies either $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$.

Theorem 4.2. *A non-empty subset A of a semigroup S is a prime bi-ideal of S if and only if the cubic characteristic function χ_A of A is a prime cubic bi-ideal of S.*

Proof. Let A be a bi-ideal of S. Then by Theorem 3.2, χ_A is a cubic bi-ideal of S. Let $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ be any cubic bi-ideals of S such that $\mathcal{B} © \mathcal{C} \sqsubseteq \chi_A$. If $\widetilde{\mu}_B \npreceq \widetilde{\mu}_{\chi_A}$ and $f_B \ngeq f_{\chi_A}$, then there exists a cubic point $x_{(\widetilde{t},s)} \in \mathcal{B}$ such that $x_{(\widetilde{t},s)} \notin \chi_A$. For any $y_{(\widetilde{r},p)} \in \mathcal{C}$,

$$B\left(x_{(\widetilde{t},s)}\right) © B\left(y_{(\widetilde{r},p)}\right) \sqsubseteq \mathcal{B} © \mathcal{C} \sqsubseteq \chi_A, \tag{1}$$

we have for all $z \in S$

$$B\left(x_{(\widetilde{t},s)}\right) © B\left(y_{(\widetilde{r},p)}\right)(z) = \left\langle (\widetilde{\mu}_{B(x_{\widetilde{t}})} \widetilde{\circ} \mu_{B(y_{\widetilde{r}})})(z), (f_{B(x_s)} \circ f_{B(a_p)})(z) \right\rangle \tag{2}$$

$$(\widetilde{\mu}_{B(x_{\widetilde{t}})} \widetilde{\circ} \mu_{B(y_{\widetilde{r}})})(z) = \begin{cases} \mathrm{rmin}\{\widetilde{t}, \widetilde{r}\} & \text{if } z \in B(x)B(y) \\ \widetilde{0} & \text{otherwise} \end{cases}$$

and

$$(f_{B(x_s)} \circ f_{B(a_p)})(z) = \begin{cases} \max\{s, p\} & \text{if } z \in B(x)B(y) \\ 1 & \text{otherwise.} \end{cases}$$

Thus by (1) and (2)

$$B(x)B(y) \subseteq A.$$

Now from hypothesis it follows that

$$B(x) \subseteq A \text{ or } B(y) \subseteq A.$$

Since $x_{(\widetilde{t},s)} \notin \chi_A$, we have $\mathcal{C} = \bigcup_{y_{(\widetilde{r},p)} \in \mathcal{C}} y_{(\widetilde{r},p)} \sqsubseteq \chi_A$, thus χ_A is a prime cubic bi-ideal of S.

Conversely, let B and C be bi-ideals of S such that $BC \subseteq A$. Then χ_B and χ_C are cubic bi-ideals of S. By Lemma 2.3, $\chi_B \circledcirc \chi_C = \chi_{BC} \sqsubseteq \chi_A$. This implies that $\chi_B \sqsubseteq \chi_A$ or $\chi_C \sqsubseteq \chi_A$ by hypothesis. Thus $B \subseteq A$ or $C \subseteq A$. Hence A is a prime bi-ideal of S. □

Definition 4.3. A cubic bi-ideal $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of a semigroup S is called *strongly prime cubic bi-ideal*, if $(\mathcal{B} \circledcirc \mathcal{C}) \sqcap (\mathcal{C} \circledcirc \mathcal{B}) \sqsubseteq \mathcal{A}$ implies that either $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$, for any cubic bi-ideals $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ of S.

Theorem 4.4. *Let A be a non-empty subset of a semigroup S, then the following assertions are equivalent,*

(1) A is a strongly prime bi-ideal of S.

(2) Cubic characteristic function $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ of A is a strongly prime cubic bi-ideal of S.

Proof. Follows from Theorem 4.2. □

Definition 4.5. Let S be a semigroup, a cubic bi-ideal $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of a semigroup S is called *idempotent* if $\mathcal{A}^2 = \mathcal{A} \circledcirc \mathcal{A} = \mathcal{A}$.

Definition 4.6. A cubic bi-ideal $\mathcal{A}$ of a semigroup S is a *semiprime cubic bi-ideal* of S if $\mathcal{B} \circledcirc \mathcal{B} = \mathcal{B}^2 \sqsubseteq \mathcal{A}$ implies $\mathcal{B} \sqsubseteq \mathcal{A}$, for every cubic bi-ideal $\mathcal{B}$ of S.

Theorem 4.7. *A non-empty subset A of a semigroup S is a semiprime bi-ideal of S if and only if the cubic characteristic function χ_A of A is a semiprime cubic bi-ideal of S.*

Proof. Straightforward. □

Remark 4.8. A prime cubic bi-ideal of S is not necessarily strongly prime cubic bi-ideal of S. This is clear from the following example.

Example 4.9. Consider the semigroup $S = \{p, q, r\}$ with the following Cayley table.

.	p	q	r
p	p	p	p
q	p	q	q
r	p	r	r

Bi-ideals of S are $\{p\}, \{p, q\}, \{p, r\}, S$. Here $\{p\}$ is a prime bi-ideal but it is not strongly prime as $(\{p, q\}\{p, r\}) \cap (\{p, r\}\{p, q\}) = \{p\} \subseteq \{p\}$ but $\{p, q\} \subsetneq \{p\}$, and $\{p, r\} \subsetneq \{p\}$. Hence by Theorem 4.2 and 4.4, $\chi_{\{p\}}$ is a prime cubic bi-ideal of S but not a strongly prime cubic bi-ideal of S.

Lemma 4.10. *For a semigroup S, the intersection of any family of prime cubic bi-ideals of S is a semiprime cubic bi-ideal of S.*

Proof. Let $\{\mathcal{A}_i : i \in I\}$ be a family of prime cubic bi-ideals of S. Then $\underset{i\in I}{\sqcap}\mathcal{A}_i$ is a cubic bi-ideal of S, by Lemma 4.6. Let $\mathcal{B}$ be any cubic bi-ideal of S such that $\mathcal{B}©\mathcal{B} \sqsubseteq \underset{i\in I}{\sqcap}\mathcal{A}_i$, then $\mathcal{B}©\mathcal{B} \sqsubseteq \mathcal{A}_i$, for all $i \in I$. Since each $\mathcal{A}_i$ is a prime cubic bi-ideal of S, so $\mathcal{B} \sqsubseteq \mathcal{A}_i$ for all $i \in I$. Hence $\mathcal{B} \sqsubseteq \underset{i\in I}{\sqcap}\mathcal{A}_i$. Thus $\underset{i\in I}{\sqcap}\mathcal{A}_i$ is a semiprime cubic bi-ideal of S. □

Definition 4.11. A cubic bi-ideal $\mathcal{A} = (\widetilde{\mu}_A, f_A)$ of a semigroup S is said to be an *irreducible (strongly irreducible) cubic bi-ideal* of S, if for any cubic bi-ideals $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ of S, $\mathcal{B} \sqcap \mathcal{C} = \mathcal{A}$ ($\mathcal{B} \sqcap \mathcal{C} \sqsubseteq A$) implies that either $\mathcal{B} = \mathcal{A}$ or $\mathcal{C} = \mathcal{A}$ ($\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$).

Theorem 4.12. *A non-empty subset A of a semigroup S is an irreducible (strongly irreducible) bi-ideal of S if and only if the cubic characteristic function $\chi_A = \langle\widetilde{\mu}_{\chi_A}, f_{\chi_A}\rangle$ of A is an irreducible (strongly irreducible) cubic bi-ideal of S.*

Proof. Follows from Theorem 4.2. □

Every strongly irreducible cubic bi-ideal of S is an irreducible cubic bi-ideal of S but every irreducible cubic bi-ideal of S is not a strongly irreducible cubic bi-ideal of S as shown in the following example.

Example 4.13. Consider the semigroup $S = \{1, 2, 3, 4, 5, 6\}$ with the following Cayley table.

.	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	4	2	2
4	1	2	2	2	3	4
5	1	2	5	6	2	2
6	1	2	2	2	5	6

All bi-ideals of S are $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 2, 6\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and S. Here $\{1\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and S are irreducible but only $\{a\}$ and S are strongly irreducible. If $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 4, 5\}$, $C = \{1, 2, 3, 4\}$ and $D = \{1, 2, 5, 6\}$, then by Theorem 4.12, $\chi_A, \chi_B, \chi_C, \chi_D$ are irreducible cubic bi-ideals of S which are not strongly irreducible cubic bi-ideals of S.

Proposition 4.14. *Every strongly irreducible, semiprime cubic bi-ideal of a semigroup S is a strongly prime cubic bi-ideal of S.*

Proof. Let $\mathcal{A} = \langle\widetilde{\mu}_A, f_A\rangle$ be a strongly irreducible, semiprime cubic bi-ideal of S. Let $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ be any two cubic bi-ideals of S such that

$$(\mathcal{B}©\mathcal{C}) \sqcap (\mathcal{C}©\mathcal{B}) \sqsubseteq \mathcal{A}.$$

Since $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{C}$, so we have $(\mathcal{B} \sqcap \mathcal{C}) © (\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq \mathcal{B}©\mathcal{C}$ and $(\mathcal{B} \sqcap \mathcal{C}) © (\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq \mathcal{C}©\mathcal{B}$. Thus $(\mathcal{B} \sqcap \mathcal{C}) © (\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq (\mathcal{B}©\mathcal{C}) \sqcap (\mathcal{C}©\mathcal{B}) \sqsubseteq \mathcal{A}$. This implies that $(\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq \mathcal{A}$, because $\mathcal{A}$ is a semiprime cubic bi-ideal of S. Since $\mathcal{A}$ is a strongly irreducible cubic bi-ideal of S, we have $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$. Hence $\mathcal{A}$ is a strongly prime cubic bi-ideal of S. □

5. Semigroups in which each cubic bi-ideal is strongly prime

In this section, we study those semigroups in which each cubic bi-ideal is semiprime, we also observe the semigroups in which each cubic bi-ideal is strongly prime.

Proposition 5.1. *Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a cubic bi-ideal of a semigroup S with $\mathcal{A}(a) = \langle \widetilde{\mu}_A(a), f_A(a) \rangle = (\widetilde{\alpha}, \beta)$, where $a \in S$ and $\widetilde{\alpha} \in D[0,1]$, $\beta \in (0,1]$. Then there exists an irreducible cubic bi-ideal $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ of S such that $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B}(a) = (\widetilde{\alpha}, \beta)$.*

Proof. Let $X = \{\mathcal{C} : \mathcal{C}$ is a cubic bi-ideal of $S,$ $\mathcal{C}(a) = (\widetilde{\alpha}, \beta)$ and $\mathcal{A} \sqsubseteq \mathcal{C}\}$. Then $X \neq \varnothing$, because $\mathcal{A} \in X$. The collection X is a partially ordered set under inclusion. If $Y = \{\mathcal{C}_i : \mathcal{C}_i$ is a cubic bi-ideal of $S,$ $\mathcal{C}_i(a) = (\widetilde{\alpha}, \beta)$ and $\mathcal{A} \sqsubseteq \mathcal{C}_i$ for all $i \in I\}$ is any totally orderd subset of X, then $\bigsqcup_{i\in I} \mathcal{C}_i = \left\langle \bigvee_{i\in I} \widetilde{\mu}_{C_i}, \bigwedge_{i\in I} f_{C_i} \right\rangle$ is a cubic bi-ideal of S such that $\mathcal{A} \sqsubseteq \bigsqcup_{i\in I} \mathcal{C}_i$. Indeed, if $a, b, c \in S$, then

$$\left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(ab) = \underset{i\in I}{\mathrm{rsup}}\left(\widetilde{\mu}_{C_i}(ab)\right)$$
$$\succeq \underset{i\in I}{\mathrm{rsup}}\left[\mathrm{rmin}\left\{\widetilde{\mu}_{C_i}(a), \widetilde{\mu}_{C_i}(b)\right\}\right] \ (\mathcal{C}_i = \langle \widetilde{\mu}_C, f_C \rangle \text{ is a cubic subsemigroup of } S)$$
$$= \mathrm{rmin}\left\{\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}(a), \underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}(b)\right\}$$
$$= \mathrm{rmin}\left\{\left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(a), \left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(b)\right\}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I} f_{\mathcal{C}_i}\right)(ab) &= \bigwedge_{i\in I}\left(f_{\mathcal{C}_i}(ab)\right)\\ &\leq \bigwedge_{i\in I}\left\{\max\left\{f_{\mathcal{C}_i}(a), f_{\mathcal{C}_i}(b)\right\}\right\}\\ &= \max\left\{\bigwedge_{i\in I} f_{C_i}(a), \bigwedge_{i\in I} f_{C_i}(b)\right\}\\ &= \max\left\{\left(\bigwedge_{i\in I} f_{C_i}\right)(a), \left(\bigwedge_{i\in I} f_{C_i}\right)(b)\right\}.\end{aligned}$$

This implies that $\bigsqcup_{i\in I} \mathcal{C}_i$ is a cubic subsemigroup of S. Now

$$\left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(abc) = \underset{i\in I}{\mathrm{rsup}}\left(\widetilde{\mu}_{C_i}(abc)\right)$$
$$\succeq \underset{i\in I}{\mathrm{rsup}}\left[\mathrm{rmin}\left\{\widetilde{\mu}_{C_i}(a), \widetilde{\mu}_{C_i}(c)\right\}\right] \ (\mathcal{C}_i = \langle \widetilde{\mu}_C, f_C \rangle \text{ is a cubic bi-ideal of } S)$$
$$= \mathrm{rmin}\left\{\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}(a), \underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}(c)\right\}$$
$$= \mathrm{rmin}\left\{\left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(a), \left(\underset{i\in I}{\mathrm{rsup}}\widetilde{\mu}_{C_i}\right)(c)\right\}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I} f_{C_i}\right)(abc) &= \bigwedge_{i\in I}\left(f_{C_i}(abc)\right)\\ &\leq \bigwedge_{i\in I}\left\{\max\left\{f_{C_i}(a), f_{C_i}(c)\right\}\right\}\\ &= \max\left\{\bigwedge_{i\in I} f_{C_i}(a), \bigwedge_{i\in I} f_{C_i}(c)\right\}\\ &= \max\left\{\left(\bigwedge_{i\in I} f_{C_i}\right)(a), \left(\bigwedge_{i\in I} f_{C_i}\right)(c)\right\}.\end{aligned}$$

Thus $\underset{i\in I}{\sqcup}\mathcal{C}_i$ is a cubic bi-ideal of S. Since $\mathcal{A}\sqsubseteq\mathcal{C}_i$ for all $i\in I$, we have $\mathcal{A}\sqsubseteq\underset{i\in I}{\sqcup}\mathcal{C}_i$. Also $\left(\underset{i\in I}{\sqcup}\mathcal{C}_i\right)(a)=\underset{i\in I}{\sqcup}\mathcal{C}_i(a)=(\widetilde{\alpha},\beta)$. Thus $\underset{i\in I}{\sqcup}\mathcal{C}_i\in Y$ and $\underset{i\in I}{\sqcup}\mathcal{C}_i$ is an upper bound of S. Hence, by Zorn's Lemma, there exists a cubic bi-ideal $\mathcal{B}$ of S which is maximal with respect to the property that $\mathcal{A}\sqsubseteq\mathcal{B}$ and $\mathcal{B}(a)=(\widetilde{\alpha},\beta)$. Now, let $\mathcal{B}_1$,$\mathcal{B}_2$ be any cubic bi-ideals of S such that $\mathcal{B}_1\sqcap\mathcal{B}_2=\mathcal{B}$. This implies $\mathcal{B}\sqsubseteq\mathcal{B}_1$ and $\mathcal{B}\sqsubseteq B_2$. We claim that $\mathcal{B}=\mathcal{B}_1$ or $\mathcal{B}=B_2$. Let us suppose $\mathcal{B}\neq\mathcal{B}_1$ and $\mathcal{B}\neq B_2$. So $\mathcal{B}_1(a)\neq\alpha$ and $\mathcal{B}_2(a)\neq\alpha$. Hence $(\mathcal{B}_1\sqcap\mathcal{B}_2)(a)=\mathcal{B}_1(a)\sqcap\mathcal{B}_2(a)\neq(\widetilde{\alpha},\beta)$, which is a contradiction to the fact that $\mathcal{B}_1(a)\sqcap\mathcal{B}_2(a)=\mathcal{B}(a)=(\widetilde{\alpha},\beta)$. Hence either $\mathcal{B}=\mathcal{B}_1$ or $\mathcal{B}=B_2$. Thus $\mathcal{B}$ is an irreducible cubic bi-ideal of S. □

In the next theorm we discuss those semigroups in which each cubic bi-ideal is semiprime.

Theorem 5.2. *For a semigroup S, the following are equivalent:*
(1) S is both regular and intra-regular,
(2) $\mathcal{A}©\mathcal{A}=\mathcal{A}$ for every cubic bi-ideal $\mathcal{A}$ of S,
(3) $\mathcal{B}\sqcap\mathcal{C}=(\mathcal{B}©\mathcal{C})\sqcap(\mathcal{C}©\mathcal{B})$ for all cubic bi-ideals of $\mathcal{B}$ and $\mathcal{C}$ of S,
(4) Each cubic bi-ideal of S is semiprime,
(5) Each proper cubic bi-ideal of S is the intersection of irreducible semiprime cubic bi-ideals of S which contains it.

Proof. (1)⇒(2) Let $\mathcal{A}=\langle\widetilde{\mu}_A, f_A\rangle$ be a cubic bi-ideal of S. Then by Lemma 2.5, $\mathcal{A}©\mathcal{A}\sqsubseteq\mathcal{A}$. Now let $a\in S$. Since S is both regular and intraregular, there exist x,y,z in S such $a=axa$ and $a=ya^2z$. Then $a=axaxa=ax(ya^2z)xa=(axya)(azxa)$. Thus we have

$$\begin{aligned}(\widetilde{\mu}_A\widetilde{\circ}\widetilde{\mu}_A)(a) &= \underset{a=(axya)(azxa)}{\mathrm{rsup}}\left[\mathrm{rmin}\left\{\widetilde{\mu}_A(axya), \widetilde{\mu}_A(azxa)\right\}\right]\\ &\succeq \underset{a=(axya)(azxa)}{\mathrm{rsup}}\left[\mathrm{rmin}\left\{\widetilde{\mu}_A(a), \widetilde{\mu}_A(a)\right\}\right]\\ &= \widetilde{\mu}_A(a),\end{aligned}$$

and

$$\begin{aligned}(f_A \circ f_A)(a) &= \bigwedge_{a=(axya)(azxa)} [\max\{f_A(axya), f_A(azxa)\}] \\ &\leq \bigwedge_{a=(axya)(azxa)} [\max\{f_A(a), f_A(a)\}] \\ &= f_A(a).\end{aligned}$$

This implies that $\mathcal{A} \sqsubseteq \mathcal{A} © \mathcal{A}$. Hence $\mathcal{A} © \mathcal{A} = \mathcal{A}$.

(2)⇒(3) Let $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ be any two cubic bi-ideals of S. Then by Lemma 3.6, $\mathcal{B} \sqcap \mathcal{C}$ is also a cubic bi-ideal of S. Thus by hypothesis, we have $\mathcal{B} \sqcap \mathcal{C} = (\mathcal{B} \sqcap \mathcal{C}) © (\mathcal{B} \sqcap \mathcal{C})$. Since $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{C}$, then $(\mathcal{B} \sqcap \mathcal{C}) © (\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq \mathcal{B} © \mathcal{C}$. This implies $(\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq (\mathcal{B} © \mathcal{C})$. Similarly $(\mathcal{B} \sqcap \mathcal{C}) \sqsubseteq (\mathcal{C} © \mathcal{B})$.

Now $\mathcal{B} © \mathcal{C}$ and $\mathcal{C} © \mathcal{B}$ are cubic bi-ideals of S by Lemma 3.7. Thus $(\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})$ is a cubic ideal of S by Lemma 3.6. Thus by hypothesis, we have

$$\begin{aligned}((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) &= ((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) © ((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) \\ &\sqsubseteq \mathcal{B} © \mathcal{C} © \mathcal{C} © \mathcal{B} \\ &= \mathcal{B} © (\mathcal{C} © \mathcal{C}) © \mathcal{B} \\ &= \mathcal{B} © \mathcal{C} © \mathcal{B} \text{ (since } \mathcal{C} © \mathcal{C} = \mathcal{C}) \\ &\sqsubseteq \mathcal{B} © \chi_S © \mathcal{B} \\ &\sqsubseteq \mathcal{B}.\end{aligned}$$

Hence $((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) \sqsubseteq \mathcal{B}$. Similarly $((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) \sqsubseteq \mathcal{C}$.

Thus $((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) \sqsubseteq \mathcal{B} \sqcap \mathcal{C}$. Therefore $((\mathcal{B} © \mathcal{C}) \sqcap (\mathcal{C} © \mathcal{B})) = \mathcal{B} \sqcap \mathcal{C}$.

(3)⇒(4) Let $\mathcal{B}$ be any cubic bi-ideal of S such that $\mathcal{A} © \mathcal{A} \sqsubseteq \mathcal{B}$ for any cubic bi-ideal $\mathcal{A}$ of S. Then by hypothesis,

$$\begin{aligned}\mathcal{A} &= \mathcal{A} \sqcap \mathcal{A} \\ &= (\mathcal{A} © \mathcal{A}) \sqcap (\mathcal{A} © \mathcal{A}) \\ &= \mathcal{A} © \mathcal{A} \\ &\sqsubseteq \mathcal{B}.\end{aligned}$$

This implies $\mathcal{A} \sqsubseteq \mathcal{B}$. Thus $\mathcal{A}$ is a semiprime cubic bi-ideal of S. Hence every cubic bi-ideal of S semiprime.

(4)⇒(5) Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a proper cubic bi-ideal of S and $\{\mathcal{A}_i : i \in I\}$ be the collection of all irreducible cubic bi-ideals of S such that $\mathcal{A} \sqsubseteq \mathcal{A}_i$ for all $i \in I$. This implies that

$$\mathcal{A} \sqsubseteq \underset{i \in I}{\sqcap} \mathcal{A}_i.$$

Let $a \in S$ and $\alpha \in (0, 1]$ be such that $\mathcal{A}(a) = (\widetilde{\alpha}, \beta)$. Then by Theorem 5.1, there exists an irreducible cubic bi-ideal $\mathcal{A}_a$ of S such that $\mathcal{A} \sqsubseteq \mathcal{A}_\alpha$ and $\mathcal{A}(a) = (\widetilde{\alpha}, \beta)$. This implies $\mathcal{A}_\alpha \in \{\mathcal{A}_i : i \in I\}$. Hence

$$\underset{i \in I}{\sqcap} \mathcal{A}_i \sqsubseteq \mathcal{A}_a$$

so $\underset{i\in I}{\sqcap}\mathcal{A}_i(a) \sqsubseteq \mathcal{A}_a(a) = \mathcal{A}(a)$ for all $a \in S$. By hypothesis, each cubic bi-ideal of S is semiprime, thus each cubic bi-ideal of S is the intersection of all irreducible semiprime cubic bi-ideals of S which contain it.

(5)⇒(2). Let $\mathcal{A}$ be a cubic bi-ideal of S. Then by Lemma 2.5, we have $\mathcal{A}©\mathcal{A} \sqsubseteq \mathcal{A}$. Also $\mathcal{A}©\mathcal{A}$ is a cubic bi-ideal of S. Thus by hypoythesis $\mathcal{A}©\mathcal{A} = \underset{i\in I}{\sqcap}\mathcal{A}_i$, where each $\mathcal{A}_i$ is an irreducible semiprime cubic bi-ideal of S, such that $\mathcal{A}©\mathcal{A} \sqsubseteq \mathcal{A}_i$ for all $i \in I$. This implies that $\mathcal{A} \sqsubseteq \mathcal{A}_i$ for all $i \in I$, because each $\mathcal{A}_i$ is a semiprime cubic bi-ideal of S. Thus $\mathcal{A} \sqsubseteq \underset{i\in I}{\sqcap}\mathcal{A}_i = \mathcal{A}©\mathcal{A}$. Hence $\mathcal{A}©\mathcal{A} = \mathcal{A}$. □

Proposition 5.3. *Let S be a regular and intra-regular semigroup. Then the following assertions for a cubic bi-ideal $\mathcal{A}$ of S are equivalent:*
(1) $\mathcal{A}$ is strongly irreducible.
(2) $\mathcal{A}$ is strongly prime.

Proof. (1)⇒(2) Let $\mathcal{A} = \langle\widetilde{\mu}_A, f_A\rangle$ be a strongly irreducible cubic bi-ideal of S. Suppose that $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ be any cubic bi-ideals of S such that $(B©C) \sqcap (C©B) \sqsubseteq \mathcal{A}$. Since S is both regular and intra-regular, we have by Theorem 5.2, $\mathcal{B} \sqcap \mathcal{C} = (B©\mathcal{C}) \sqcap (C©B) \sqsubseteq \mathcal{A}$. This implies that $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$, because $\mathcal{A}$ is strongly irreducible. Thus $\mathcal{A}$ is a strongly prime cubic bi-ideal of S.

(2)⇒(1) Suppose $\mathcal{A}$ is a strongly prime cubic bi-ideal of S. Let $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ be any cubic bi-ideals of S such that $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A}$. Since S is both regular and intra-regular, we have by Theorm 5.2, $(B©\mathcal{C}) \sqcap (C©B) = \mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A}$. This implies that either $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$, because $\mathcal{A}$ is a strongly prime cubic bi-ideal of S. Thus $\mathcal{A}$ is a strongly irreducible cubic bi-ideal of S. □

Theorem 5.4. *Each cubic bi-ideal of a semigroup S is strongly prime if and only if S is regular and intra-regular and the set of cubic bi-ideals of S is totally orderd by inclusion.*

Proof. Suppose that each cubic bi-ideal of S is strongly prime. Then each cubic bi-ideal of S is semiprime. By Theorem 5.2, S is both regular and intra-regular. Now, let $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ be any two cubic bi-ideals of S. Then by Theorem 5.2, $\mathcal{B} \sqcap \mathcal{C} = (\mathcal{B}©\mathcal{C}) \sqcap (\mathcal{C}©\mathcal{B})$. Since each cubic bi-ideal of S is strongly prime, so $\mathcal{B} \sqcap \mathcal{C}$ is strongly prime. Hence either $\mathcal{B} \sqsubseteq \mathcal{B} \sqcap \mathcal{C}$ or $\mathcal{C} \sqsubseteq \mathcal{B} \sqcap \mathcal{C}$. Now, if $\mathcal{B} \sqsubseteq \mathcal{B} \sqcap \mathcal{C}$, then $\mathcal{B} \sqsubseteq \mathcal{C}$ and if $\mathcal{C} \sqsubseteq \mathcal{B} \sqcap \mathcal{C}$, then $\mathcal{C} \sqsubseteq \mathcal{B}$.

Conversely, suppose that S is regular and intra-regular and the set of cubic bi-ideals of S is totally ordered by inclusion. Let $\mathcal{A} = \langle\widetilde{\mu}_A, f_A\rangle$, $\mathcal{B} = \langle\widetilde{\mu}_B, f_B\rangle$ and $\mathcal{C} = \langle\widetilde{\mu}_C, f_C\rangle$ be any cubic bi-ideals of S such that $(\mathcal{B}©\mathcal{C}) \sqcap (\mathcal{C}©\mathcal{B}) \sqsubseteq \mathcal{A}$. Since S is both regular and intra-regular, we have by Theorem 5.2,

$$\mathcal{B} \sqcap \mathcal{C} = (\mathcal{B}©\mathcal{C}) \sqcap (\mathcal{C}©\mathcal{B}) \sqsubseteq \mathcal{A}.$$

Since the set of cubic bi-ideals of S is totally ordered by inclusion, we have either $\mathcal{B} \sqsubseteq \mathcal{C}$ or $\mathcal{C} \sqsubseteq \mathcal{B}$, that is either $\mathcal{B} \sqcap \mathcal{C} = \mathcal{B}$ or $\mathcal{B} \sqcap \mathcal{C} = \mathcal{C}$. This implies that $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$. Hence $\mathcal{A}$ is a strongly prime cubic bi-ideal of S. □

Theorem 5.5. *If the set of cubic bi-ideals of a semigroup S is totally ordered under inclusion, then S is both regular and intra-regular if and only if each cubic bi-ideal of S is prime.*

Proof. Suppose S is both regular and intra-regular. Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$, $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ be any cubic bi-ideals of S such that $(\mathcal{B}ⓒ\mathcal{C}) \sqsubseteq \mathcal{A}$. Since the set of cubic bi-ideals of S is totally ordered, we have either $\mathcal{B} \sqsubseteq \mathcal{C}$ or $\mathcal{C} \sqsubseteq \mathcal{B}$. If $\mathcal{B} \sqsubseteq \mathcal{C}$, then $\mathcal{B}ⓒ\mathcal{B} \sqsubseteq \mathcal{B}ⓒ\mathcal{C} \sqsubseteq \mathcal{A}$. This implies that $\mathcal{B} \sqsubseteq \mathcal{A}$, as $\mathcal{A}$ is semiprime by Theorem 5.2. Hence $\mathcal{A}$ is a prime cubic bi-ideal of S.

Conversely, suppose that every cubic bi-ideal of S is prime, since every prime cubic bi-ideal is semiprime. By Theorem 5.2, S is both regular and intra-regular. □

Theorem 5.6. *For a semigroup S, the following assertions are equivalent.*
(1) The set of cubic bi-ideals of S is totally ordered under inclusion,
(2) Each cubic bi-ideal of S is strongly irreducible,
(3) Each cubic bi-ideal of S is irreducible.

Proof. (1)⇒(2) Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$, $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ and $\mathcal{C} = \langle \widetilde{\mu}_C, f_C \rangle$ be any cubic bi-ideals of S such that $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A}$. Since the set of cubic bi-ideals of S is totally ordered, so either $\mathcal{B} \sqsubseteq \mathcal{C}$ or $\mathcal{C} \sqsubseteq \mathcal{B}$. Thus either $\mathcal{B} \sqcap \mathcal{C} = \mathcal{B}$ or $\mathcal{B} \sqcap \mathcal{C} = \mathcal{C}$. Hence $\mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A}$ implies that $\mathcal{B} \sqsubseteq \mathcal{A}$ or $\mathcal{C} \sqsubseteq \mathcal{A}$. Therefore $\mathcal{A}$ is a strongly irreducible cubic bi-ideal of S.

(3)⇒(1) Let $\mathcal{B}$ and $\mathcal{C}$ be any cubic bi-ideals of S. Then $\mathcal{B} \sqcap \mathcal{C}$ is also a cubic bi-ideal of S. Also $\mathcal{B} \sqcap \mathcal{C} = \mathcal{B} \sqcap \mathcal{C}$. So by hypothesis, either $\mathcal{B} = \mathcal{B} \sqcap \mathcal{C}$ or $\mathcal{C} = \mathcal{B} \sqcap \mathcal{C}$, that is either $\mathcal{B} \sqsubseteq \mathcal{C}$ or $\mathcal{C} \sqsubseteq \mathcal{B}$. Hence the set of cubic bi-ideals of S is totally ordered. □

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