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Generalized ideal limit point and cluster point of double sequences in intuitionistic fuzzy 2-normed linear spaces

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Abstract. The aim of this paper is to introduced and studied the notion of (λ, τ)-ideal convergence of double sequences in intuitionistic fuzzy 2-normed space as a variant of the notion of ideal convergence of double sequences. Also $I_{\lambda,\tau}-$limit points and $I_{\lambda,\tau}$-cluster points have been defined and the relation between them are established. Furthermore, Cauchy and $I_{\lambda,\tau}$-Cauchy of double sequences are introduced and studied, where $\lambda = (\lambda_m)$ and $\tau = (\tau_n)$ be two non-decreasing sequences of positive numbers tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$, $\tau_{n+1} \leq \tau_n + 1, \tau_1 = 1$.

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1. Introduction

The concept of ideal convergence is a generalization of statistical convergence, and any concept involving ideal convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, motion planning in robotics and traffic control, speech analysis, bioinfromatics and DNA analysis.

In 1900, A. Pringsheim [57] introduced the notion of double sequence and presented the definition of convergence of a double sequence. Robinson [58] studied the divergent of double sequences and series. Later on Hamilton [11, 12] introduced the transformation of multiple sequences. We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [65] independently. A lot of developments have been made in this areas after the works of Šalát [60] and Fridy [9]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Mursaleen and Edely [53] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences. Besides this, Mursaleen [51] presented a generalization of statistical convergence with the help of λ-summability method and called it λ-statistical convergence. Mursaleen and Edely [47] introduced the notion of generalized statistical convergence of single sequences. Kumar and Mursaleen [41] presented a generalization of statistical convergence of double sequences on intuitionistic fuzzy normed spaces. In [64] Savas and Mohiuddine introduced the notion of generalized statistical convergence of double sequences in probabilistic normed spaces. In [19] Hazarika and Savaş introduced (λ, μ)-statistical convergence of double sequences in n-normed spaces. Recently, Karakaya et al., [33] introduced lacunary statistical convergence of sequences of functions in intuitionistic fuzzy normed space. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability, (see [4]).

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

The notion of I-convergence was introduced by Kostyrko et al., [35] as a generalization of statistical convergence which is based on the structure of the ideal I of subsets of natural numbers $\mathbb{N}$. Kostyrko et al., [36] gave some of basic properties of I-convergence and dealt with extermal I-limit points. Das et al., [37] introduced the concept of I-convergence of double sequences in metric space and studied some properties of this convergence. A lot of developments have been made in this areas after the works of [39] and [67]. Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X, here in our study it suffices to take I as a family of subsets of $\mathbb{N}$, positive integers, i.e. $I \subset 2^{\mathbb{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I.

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if $\phi \notin F$, $A \cap B \in F$ for each $A, B \in F$, and any superset of an element of F is in F. An ideal I is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal I. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko et al.

[35]. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be *I-convergent* to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$. In this case we write $I - \lim x_k = \ell$.

Let $A \subset \mathbb{N}$ and $d_n(A) = \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k}$, for $n \in \mathbb{N}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. If $\lim_{n\to\infty} d_n(A)$ exists, then it is called as the *logarithmic density* of A. $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$ is an ideal.

Let $T = (t_{nk})$ be a *regular non-negative matrix*. For $A \subset \mathbb{N}$, define $d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{nk}\chi_A(k)$, for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} d_T^{(n)}(A) = d_T(A)$ exists, then $d_T(A)$ is called as *T-density* of A. Clearly $I_{d_T} = \{A \subset \mathbb{N} : d_T(A) = 0\}$ is an ideal.

Note 1: I_δ and I_d are particular cases of I_{d_T}.

(i) Asymptotic density, for

$$t_{nk} = \begin{cases} \frac{1}{n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Logarithmic density, for

$$t_{nk} = \begin{cases} \frac{k^{-1}}{s_n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset }\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

The existing literature on ideal convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [40] and intutionistic fuzzy normed spaces [34, 38, 45, 49, 50]. Further details on ideal convergence we refer to [1, 2, 3, 5, 7, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 42, 43, 46, 54, 55, 59, 61, 62, 68, 69, 70, 71, 72, 73, 74, 75] and many others.

Now we recall some notations and basic definitions that we are going to used in this paper.

Definition 1.1 ([66])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions are satisfied

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

A binary operation $o : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if the following conditions are satisfied

1. o is associative and commutative,

2. o is continuous,
3. $ao0 = a$ for all $a \in [0,1]$,
4. $aob \leq cod$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2 ([10]). Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $||.,.|| : X \times X \to \mathbb{R}$ which satisfies,
1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, ||.,.||)$ is called a 2-normed space. As an example of a 2-normed space, we take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$ = the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula: $\|x, y\| = |x_1 y_2 - x_2 y_1|$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

The concept of intuitionistic fuzzy 2-normed linear space introduced and studied by Mursaleen and Lohani [52] as follows.

Definition 1.3. The five-tuple $(X, \mu, \nu, *, o)$ is said to be intuitionistic fuzzy 2-normed linear space (for short, IF2NLS) if X linear space, $*$ is a continuous t-norm, o is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions for every $x, y, z \in X$ and $s, t > 0$:
1. $\mu(x, y; t) + \nu(x, y; t) \leq 1$,
2. $\mu(x, y; t) > 0$,
3. $\mu(x, y; t) = 1$ if and only if x and y are linearly dependent,
4. $\mu(\alpha x, y; t) = \mu\left(x, y; \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
5. $\mu(x, y; t) * \mu(x, z; s) \leq \mu(x, y + z; t + s)$,
6. $\mu(x, y; .) : (0, \infty) \to [0, 1]$ is continuous,
7. $\lim_{t \to \infty} \mu(x, y; t) = 1$ and $\lim_{t \to 0} \mu(x, y; t) = 0$,
8. $\mu(x, y; t) = \mu(y, x; t)$,
9. $\nu(x, y; t) < 1$,
10. $\nu(x, y; t) = 0$ if and only if x and y are linearly dependent,
11. $\nu(\alpha x, y; t) = \nu\left(x, y; \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
12. $\nu(x, y; t)\, o \nu(x, z; s) \geq \nu(x, y + z; t + s)$,
13. $\nu(x, y; .) : (0, \infty) \to [0, 1]$ is continuous,
14. $\lim_{t \to \infty} \nu(x, y; t) = 0$ and $\lim_{t \to 0} \nu(x, y; t) = 1$,
15. $\nu(x, y; t) = \nu(y, x; t)$.

In this case (μ, ν) is called an intuitionistic fuzzy 2-norm on X and we denote it by $(\mu, \nu)_2$.

Definition 1.4. Let $(X, \mu, \nu, *, o)$ be an intuitionistic fuzzy 2-normed linear space, and let $r \in (0, 1)$ and $x \in X$. The set

$$B(x, r; t) = \{y \in X : \mu(y - x, r; t) > 1 - r \text{ and } \nu(y - x, r; t) < r, \text{ for all } z \in X\}$$

is called open ball with center x and radius r with respect to t.

Definition 1.5. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $J_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V,\lambda)-$summable to number L [44] if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V, \lambda)-$summability reduces to $(C, 1)$-summability. For details we refer to [48, 56] and many others.

Mursaleen [51] defined $\lambda-$statistically convergent sequence as follows: A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in J_n : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

Let S_λ denotes the set of all $\lambda-$statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.

Definition 1.6 ([63])**.** Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A sequence $x = (x_k)$ is said to be $I - [V, \lambda]$-convergent to a number L if, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} |x_k - L| \geq \varepsilon \right\} \in I.$$

In this case we write $I - [V, \lambda] - \lim x = L$.

Recently, in [6] Esi and Hazarika introduced the concept of λ-ideal convergence in intuitionistic fuzzy 2-normed spaces.

2. Ideal convergence of double sequences in IF2NLS

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [57]. A double sequence $x = (x_{k,l})$ has a *Pringsheim* limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "*$P -$ convergent*".

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m, n)$ denotes the number of (i, j) in K such that $i \leq m$ and $j \leq n$, (see [53]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty,\infty} \frac{|K(m,n)|}{mn}$. In case, the sequence $(\frac{K(m,n)}{mn})$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined by $P - \lim_{m,n\to\infty,\infty} \frac{|K(m,n)|}{mn} = \delta_2(K)$.

Example 2.1. Let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n\to\infty,\infty} \frac{|K(m,n)|}{mn} \leq P - \lim_{m,n\to\infty,\infty} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 3j) : i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{3}$.

Mursaleen and Edely [53], defined the statistical analogue for double sequences $x = (x_{k,l})$ as follows. A real double sequence $x = (x_{k,l})$ is said to be P-statistically convergent to ℓ provided that for each $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{rs} |\{(k,l) : k < r \text{ and } l < s, |x_{k,l} - \ell| \geq \varepsilon\}| = 0.$$

In this case we write S_2-$\lim_{k,l} x_{k,l} = \ell$ and denote the set of all statistical convergent double sequences by S_2. It is clear that a convergent double sequence is also S_2-convergent but the converse is not true, in general. Also S_2-convergent double sequence need not be bounded. For example we consider the $x = (x_{k,l})$ defined by

$$x_{k,l} = \begin{cases} kl, & \text{if } k,l = m^2, m = 1,2,3,...; \\ 1, & \text{otherwise.} \end{cases}$$

is S_2-$\lim_{k,l} x_{k,l} = 1$. But it is neither convergent nor bounded.

Definition 2.2. Let $I \subset 2^{\mathbb{N}\times\mathbb{N}}$. A double sequence $(x_{k,l})$ in a normed space $(X, ||.||)$ is said to be *I-convergent* to some $\ell \in X$ with respect to the norm if for each $\varepsilon > 0$ such that the set $\{(k,l) \in \mathbb{N} \times \mathbb{N} : ||x_{k,l} - \ell|| \geq \varepsilon\}$ belong to I.

Let $\lambda = (\lambda_m)$ and $\tau = (\tau_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{m+1} \leq \cdot\lambda_m + 1,\ \lambda_1 = 1;\ \tau_{n+1} \leq \tau_n + 1,\ \tau_1 = 1$. Let $\gamma = \gamma_{mn} = \lambda_m \tau_n$, $J_m = [m - \lambda_m + 1, m]$, $J_n = [n - \tau_n + 1, n]$ and $J_{mn} = J_m \times J_n$.

We define

$$T_{mn}(x) = \frac{1}{\gamma_{mn}} \sum_{(k,l) \in J_{mn}} x_{k,l}.$$

Throughout the paper, we shall denote by I and γ are admissible ideal of subsets of $\mathbb{N} \times \mathbb{N}$ and $\gamma = \gamma_{mn} = \lambda_m \tau_n$ sequence as defined above, respectively, unless otherwise stated.

We now obtain our main results.

Definition 2.3. Let $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ and $(X, \mu, \nu, *, o)$ be an IF2NLS. A double sequence $x = (x_{k,l})$ in X is said to be $I_{\lambda,\tau}-$convergent to $L \in X$ with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$ if, for every $\varepsilon \in (0,1)$ and $t > 0$, the set

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} :\ \mu(T_{mn}(x) - L, y; t) \leq 1 - \varepsilon \text{ or } \nu(T_{mn}(x) - L, y; t) \geq \varepsilon\ \} \in I.$$

L is called the $I_{\lambda,\tau}-$limit of the sequence $x = (x_{k,l})$ and we write $I_{\lambda,\tau}^{(\mu,\nu)_2} - \lim x = L$.

Example 2.4. Let $(\mathbb{R}^2, |.|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$ and $aob = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider $\mu(x,t) = \frac{t}{t + \|x,y\|}$ and $\nu(x,t) = \frac{\|x,y\|}{t+\|x,y\|}$. Then $(\mathbb{R}^2, \mu, \nu, *, o)$ is an IF2NLS. If we take $I = \{A \subset \mathbb{N} \times \mathbb{N} :\ \delta_2(A) = 0\}$, where $\delta_2(A)$ denotes the double natural density of the set A, then I is a non-trivial admissible ideal. Define a sequence $x = (x_{k,l})$ as follows:

$$x_{k,l} = \begin{cases} (1,0), & \text{if } k,l = i^2, i \in \mathbb{N} \\ (0,0), & \text{otherwise.} \end{cases}$$

Then for every $\varepsilon \in (0,1)$ and for any $t>0$, the set

$$K(\varepsilon,t)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu(T_{mn}(x),y;t)\leq 1-\varepsilon \text{ or } \nu(T_{mn}(x),y;t)\geq\varepsilon\ \}$$

will be a finite set. Hence, $\delta_2(K(\varepsilon,t))=0$ and consequently $K(\varepsilon,t)\in I$, i.e., $I_{\lambda,\tau}^{(\mu,\nu)}-\lim x=0$.

Lemma 2.5. *Let $(X,\mu,\nu,*,o)$ be an IF2NLS and $x=(x_{k,l})$ be a double sequence in X. Then, for every $\varepsilon>0$ and $t>0$ the following statements are equivalent:*

(i) $I_{\lambda,\tau}^{(\mu,\nu)_2}-\lim x=L$,

(ii) $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu(T_{mn}(x)-L,y;t)\leq 1-\varepsilon\ \}\in I$ *and* $\{(m,n)\in\mathbb{N}:\ \nu(T_{mn}(x)-L,y;t)\geq\varepsilon\ \}\in I$,

(iii) $\left\{\begin{array}{c}(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu(T_{mn}(x)-L,y;t)>1-\varepsilon\\ \text{and } \nu(T_{mn}(x)-L,y;t)<\varepsilon\end{array}\right\}\in F(I)$,

(iv) $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu(T_{mn}(x)-L,y;t)>1-\varepsilon\ \}\in F(I)$ *and* $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \nu(T_{mn}(x)-L,y;t)<\varepsilon\ \}\in F(I)$,

(v) $I_{\lambda,\tau}-\lim\mu(x_{k,l}-L,y;t)=1$ *and* $I_{\lambda,\tau}-\lim\nu(x_{k,l}-L,y;t)=0$.

Theorem 2.6. *Let $(X,\mu,\nu,*,o)$ be an IF2NLS and if a double sequence $x=(x_{k,l})$ in X is $I_{\lambda,\tau}$-convergent to $L\in X$ with respect to the intuitionistic fuzzy norm $(\mu,\nu)_2$, then $I_{\lambda,\tau}^{(\mu,\nu)_2}-\lim x$ is unique.*

Proof. Suppose that $I_{\lambda,\tau}^{(\mu,\nu)_2}-\lim x=L_1$ and $I_{\lambda,\tau}^{(\mu,\nu)}-\lim x=L_2\ (L_1\neq L_2)$. Given $\varepsilon\in(0,1)$ and choose $\beta\in(0,1)$ such that $(1-\beta)*(1-\beta)>1-\varepsilon$ and $\beta o\beta<\varepsilon$. Then for any $t>0$, define the following sets:

$$K_{\mu,1}(\beta,t)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu\left(T_{mn}(x)-L_1,y;\frac{t}{2}\right)\leq 1-\beta\ \right\},$$

$$K_{\mu,2}(\beta,t)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \mu\left(T_{mn}(x)-L_2,y;\frac{t}{2}\right)\leq 1-\beta\ \right\},$$

$$K_{\nu,1}(\beta,t)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \nu\left(T_{mn}(x)-L_1,y;\frac{t}{2}\right)\geq \beta\ \right\},$$

$$K_{\nu,2}(\beta,t)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \nu\left(T_{mn}(x)-L_2,y;\frac{t}{2}\right)\geq \beta\ \right\}.$$

Since $I_{\lambda,\tau}^{(\mu,\nu)}-\lim x=L_1$, using Lemma 2.5 , we have $K_{\mu,1}(\beta,t)\in I$ and $K_{\nu,1}(\beta,t)\in I$ for all $t>0$. Also, using $I_{\lambda,\tau}^{(\mu,\nu)}-\lim x=L_2$, we get $K_{\mu,2}(\beta,t)\in I$ and $K_{\nu,2}(\beta,t)\in I$ for all $t>0$. Now let

$$K_{\mu,\nu}(\beta,t)=[K_{\mu,1}(\beta,t)\cup K_{\mu,2}(\beta,t)]\cap[K_{\nu,1}(\beta,t)\cup K_{\nu,2}(\beta,t)].$$

Then $K_{\mu,\nu}(\beta,t)\in I$. This implies that its complement $K_{\mu,\nu}^C(\beta,t)$ is a non-empty set in $F(I)$. Now if $(m,n)\in K_{\mu,\nu}^C(\beta,t)$, first let us consider the case $(m,n)\in\left[K_{\mu,1}^C(\beta,t)\cap K_{\mu,2}^C(\beta,t)\right]$. Then we have

$$\mu\left(T_{mn}(x)-L_1,y;\frac{t}{2}\right)>1-\beta \text{ and } \mu\left(T_{mn}(x)-L_2,y;\frac{t}{2}\right)>1-\beta.$$

Now, we choose $u, v \in \mathbb{N}$ such that

$$\mu\left(x_{u,v} - L_1, y; \frac{t}{2}\right) > \mu\left(T_{mn}(x) - L_1, y; \frac{t}{2}\right) > 1 - \beta$$

and

$$\mu\left(x_{u,v} - L_2, y; \frac{t}{2}\right) > \mu\left(T_{mn}(x) - L_2, y; \frac{t}{2}\right) > 1 - \beta$$

e.g., consider $\max\left\{\mu\left(x_{k,l} - L_1, y; \frac{t}{2}\right), \mu\left(x_{k,l} - L_2, y; \frac{t}{2}\right) : (k,l) \in J_{mn}\right\}$ and choose that k, l as u, v for which the maximum occurs. Then we have

$$\mu\left(L_1 - L_2, y; t\right) \geq \mu\left(x_{u,v} - L_1, y; \frac{t}{2}\right) * \mu\left(x_{u,v} - L_2, y; \frac{t}{2}\right)$$
$$> (1-\beta) * (1-\beta) > 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu\left(L_1 - L_2, y; t\right) = 1$ for all $t > 0$, which implies that $L_1 = L_2$. On the other hand, if $(m, n) \in \left[K^C_{\nu,1}(\beta, t) \cap K^C_{\nu,2}(\beta, t)\right]$, then using a similar technique, it can be proved that $\nu\left(L_1 - L_2, y; t\right) < \varepsilon$ for all $t > 0$ and arbitrary $\varepsilon > 0$. Thus we obtain $L_1 = L_2$. Therefore, we conclude that $I^{(\mu,\nu)_2}_{\lambda,\tau} - \lim x$ is unique. □

Here, we introduce the notion of (λ, τ)-convergence in an IF2NLS and discuss some properties.

Definition 2.7. Let $(X, \mu, \nu, *, o)$ be an IF2NLS. A double sequence $x = (x_{k,l})$ in X is (λ, τ)-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$ if, for $\varepsilon \in (0, 1)$ and every $t > 0$, there exist $m_0, n_0, \in \mathbb{N}$ such that

$$\mu\left(T_{mn}(x) - L, y; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(x) - L, y; t\right) < \varepsilon$$

for all $m \geq m_0, n \geq n_0$. In this case, we write $(\mu, \nu)_2^{\lambda,\tau} - \lim x = L$.

Theorem 2.8. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS and let $x = (x_{k,l})$ in X. If $x = (x_{k,l})$ is (λ, τ)-convergent with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2$, then $(\mu, \nu)_2^{(\lambda,\tau)} - \lim x$ is unique.*

Proof. Suppose that $(\mu, \nu)_2^{(\lambda,\tau)} - \lim x = L_1$ and $(\mu, \nu)_2^{(\lambda,\tau)} - \lim x = L_2$ $(L_1 \neq L_2)$. Given $\varepsilon \in (0, 1)$ and choose $\beta \in (0, 1)$ such that $(1-\beta) * (1-\beta) > 1 - \varepsilon$ and $\beta o \beta < \varepsilon$. Then for any $t > 0$, there exist $m_1, n_1 \in \mathbb{N}$ such that

$$\mu\left(T_{mn}(x) - L_1, y; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(x) - L_1, y; t\right) < \varepsilon$$

for all $m \geq m_1, n \geq n_1$. Also, there exists $m_2, n_2 \in \mathbb{N}$ such that

$$\mu\left(T_{mn}(x) - L_2, y; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(x) - L_2, y; t\right) < \varepsilon$$

for all $m \geq m_2, n \geq n_2$. Now, consider $n_0 = \max\{n_1, n_2\}, m_0 = \max\{m_1, m_2\}$. Then for $m \geq m_0, n \geq n_0$, we will get $u, v \in \mathbb{N}$ such that

$$\mu\left(x_{u,v} - L_1, y; \frac{t}{2}\right) > \mu\left(T_{mn}(x) - L_1, y; \frac{t}{2}\right) > 1 - \beta$$

and

$$\mu\left(x_{u,v} - L_2, y; \frac{t}{2}\right) > \mu\left(T_{mn}(x) - L_2, y; \frac{t}{2}\right) > 1 - \beta.$$

Then, we have

$$\mu(L_1 - L_2, y; t) \geq \mu\left(x_{u,v} - L_1, y; \frac{t}{2}\right) * \mu\left(x_{u,v} - L_2, y; \frac{t}{2}\right)$$
$$> (1-\beta) * (1-\beta) > 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(L_1 - L_2, y; t) = 1$ for all $t > 0$, by using a similar technique, it can be proved that $\nu(L_1 - L_2, y; t) = 0$ for all $t > 0$, which implies that $L_1 = L_2$. $\square$

Theorem 2.9. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS and let $x = (x_{k,l})$ in X. If $(\mu,\nu)_2^{(\lambda,\tau)}-$ $\lim x = L$, then $I_{\lambda,\tau}^{(\mu,\nu)_2} - \lim x = L$.*

Proof. Let $(\mu,\nu)_2^{(\lambda,\tau)} - \lim x = L$, then for every $t > 0$ and given $\varepsilon \in (0,1)$, there exists $m_0, n_0 \in \mathbb{N}$ such that

$$\mu(T_{mn}(x) - L, y; t) > 1 - \varepsilon \text{ and } \nu(T_{mn}(x) - L, y; t) < \varepsilon$$

for all $m \geq m_0, n \geq n_0$. Therefore the set

$$B = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \ \mu(T_{mn}(x) - L, y; t) \leq 1 - \varepsilon \text{ or } \nu(T_{mn}(x) - L, y; t) \geq \varepsilon\}$$
$$\subseteq \{(1,1), (2,2), ..., (m_0 - 1, n_0 - 1)\}.$$

But, with I being admissible, we have $B \in I$. Hence $I_{\lambda,\tau}^{(\mu,\nu)_2} - \lim x = L$. $\square$

Theorem 2.10. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS and let $x = (x_{k,l})$ in X. If $(\mu,\nu)_2^{\lambda,\tau} - \lim x = L$, then there exists a subsequence (x_{m_k,n_l}) of $x = (x_{k,l})$ such that $(\mu,\nu)_2 - \lim x_{m_k,n_l} = L$.*

Proof. Let $(\mu,\nu)_2^{\lambda} - \lim x = L$. Then, for every $t > 0$ and given $\varepsilon \in (0,1)$, there exist $m_0, n_0 \in \mathbb{N}$ such that

$$\mu(T_{mn}(x) - L, y; t) > 1 - \varepsilon \text{ and } \nu(T_{mn}(x) - L, y; t) < \varepsilon$$

for all $m \geq m_0, n \geq n_0$. Clearly, for each $m \geq m_0, n \geq n_o$, we can select $m_k, n_l \in J_{mn}$ such that

$$\mu(x_{m_k,n_l} - L, y; t) > \mu(T_{mn}(x) - L, y; t) > 1 - \varepsilon$$

and

$$\nu(x_{m_k,n_l} - L, y; t) < \nu(T_{mn}(x) - L, y; t) < \varepsilon.$$

It follows that $(\mu,\nu)_2 - \lim x_{m_k,n_l} = L$. $\square$

Definition 2.11. Let $(X, \mu, \nu, *, o)$ be an IF2NLS and let $x = (x_{k,l})$ be a double sequence in X. Then,

1. An element $L \in X$ is said to be $I_{\lambda,\tau}-$limit point of $x = (x_{k,l})$ if there is a set $M = \{(m_1, n_l) < (m_2, n_2) < ... < (m_k, n_l) < ...\} \subset \mathbb{N}\times\mathbb{N}$ such that the set $M' = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \ (m_k, n_l) \in J_{mn}\} \notin I$ and $(\mu,\nu)_2^{\lambda,\tau} - \lim x_{m_k,n_l} = L$.
2. An element $L \in X$ is said to be $I_{\lambda,\tau}-$cluster point of $x = (x_{k,l})$ if for every $t > 0$ and $\varepsilon \in (0,1)$, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \ \mu(T_{mn}(x) - L, y; t) > 1 - \varepsilon \text{ and } \nu(T_{mn}(x) - L, y; t) < \varepsilon\ \} \notin I.$$

Let $\Lambda_{(\mu,\nu)_2^{\lambda,\tau}}(x)$ denote the set of all $I_{\lambda,\tau}-$limit points and $\Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)$ denote the set of all $I_{\lambda,\tau}-$cluster points in X, respectively.

Theorem 2.12. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS. For each sequence $x = (x_{k,l})$ in X, then $\Lambda_{(\mu,\nu)_2^{\lambda,\tau}}(x) \subset \Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)$.*

Proof. Let $L \in \Lambda_{(\mu,\nu)_2^{\lambda,\tau}}(x)$, then there exists a set $M \subset \mathbb{N}$ such that $M^\imath \notin I$, where M and $M^\imath$ as in the Definition 2.11, satisfies $(\mu,\nu)_2^{\lambda,\tau} - \lim x_{m_k,n_l} = L$. Thus, for every $t > 0$ and $\varepsilon \in (0,1)$, there exist $m_0, n_0 \in \mathbb{N}$ such that

$$\mu\left(T_{m_k,n_l(x)} - L, y; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{m_k,n_l}(x) - L, y; t\right) < \varepsilon$$

for all $m \geq m_0, n \geq n_0$. Therefore,

$$\begin{aligned} B &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \ \mu\left(T_{mn}(x) - L, y; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(x) - L, y; t\right) < \varepsilon\} \\ &\supseteq M^\imath \setminus \{(m_1,n_1),(m_2,n_2),...,(m_{k_0},n_{l_0})\}. \end{aligned}$$

Now, with I being admissible, we must have $M^\imath \setminus \{(m_1,n_1),(m_2,n_2),...,(m_{k_0},n_{l_0})\} \notin I$ and as such $B \notin I$. Hence $L \in \Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)$. □

Theorem 2.13. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS. For each sequence $x = (x_{k,l})$ in X, the set $\Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)$ is closed set in X with respect to the usual topology induced by the intuitionistic fuzzy norm $(\mu,\nu)_2^{\lambda,\tau}$.*

Proof. Let $y \in \overline{\Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)}$. Take $t > 0$ and $\varepsilon \in (0,1)$. Then there exists $L_0 \in \Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x) \cap B(y,\varepsilon,t)$. Choose $\delta > 0$ such that $B(L_0,\delta,t) \subset B(y,\varepsilon,t)$. We have

$$G = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \ \mu\left(T_{mn}(x) - y, z; t\right) > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(x) - y, z; t\right) < \varepsilon\}$$

$$\supseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : \ \mu\left(T_{mn}(x) - L_0, z; t\right) > 1 - \delta \text{ and } \nu\left(T_{mn}(x) - L_o, z; t\right) < \delta\} = H.$$

Thus $H \notin I$ and so $G \notin I$. Hence $y \in \Gamma_{(\mu,\nu)_2^{\lambda,\tau}}(x)$. □

Theorem 2.14. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS and let $x = (x_{k,l})$ in X. Then the following statements are equivalent:*

(1) *L is a $I_{\lambda,\tau}$-limit point of x,*

(2) *There exist two sequences y and z in X such that $x = y + z$ and $(\mu,\nu)_2^{\lambda,\tau} - \lim y = L$ and $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \ (k,l) \in J_{mn}, \ z_{k,l} \neq \theta\} \in I$, where θ is the zero element of X.*

Proof. Suppose that (1) holds. Then there exist sets M and $M^\imath$ as in Definition 2.11 such that $M^\imath \notin I$ and $(\mu,\nu)_2^{\lambda,\tau} - \lim x_{m_{k,l}} = L$. Define the sequences y and z as follows:

$$y_{k,l} = \begin{cases} x_{k,l}, & \text{if } (k,l) \in J_{mn}; (m,n) \in M^\imath \\ L, & \text{otherwise.} \end{cases}$$

and

$$z_{k,l} = \begin{cases} \theta, & \text{if } (k,l) \in J_{mn}; (m,n) \in M^\imath \\ x_{k,l} - L, & \text{otherwise.} \end{cases}$$

It sufficies to consider the case $(k,l) \in J_{mn}$ such that $(m,n) \in \mathbb{N} \setminus M^\imath$. Then for each $\varepsilon \in (0,1)$ and $t > 0$, we have $\mu\left(y_{k,l} - L, z; t\right) = 1 > 1 - \varepsilon$ and $\nu\left(y_{k,l} - L, z; t\right) = 0 < \varepsilon$. Thus, in this case,

$$\mu\left(T_{mn}(y) - L, z; t\right) = 1 > 1 - \varepsilon \text{ and } \nu\left(T_{mn}(y) - L, z; t\right) = 0 < \varepsilon.$$

Hence $(\mu,\nu)_2^{\lambda,\tau} - \lim y = L$. Now
$\{(m,n) \in \mathbb{N}\times\mathbb{N} : (k,l) \in J_{mn},\ z_{k,l} \neq \theta\} \subset \mathbb{N}\times\mathbb{N}\setminus M^{\imath}$, and so
$\{(m,n) \in \mathbb{N}\times\mathbb{N} : (k,l) \in J_{mn},\ z_{k,l} \neq \theta\} \in I$.

Now, suppose that (2) holds. Let $M^{\imath} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : (k,l) \in J_{mn},\ z_{k,l} = \theta\}$. Then, clearly $M^{\imath} \in F(I)$ and so it is an infinite set. Construct the set

$$M = \{(m_1,n_1) < (m_2,n_2) < ... < (m_k,n_l) < ...\} \subset \mathbb{N}\times\mathbb{N}$$

such that $m_k, n_l \in J_{mn}$ and $z_{m_k,n_l} = \theta$. Since $x_{m_k,n_l} = y_{m_k,n_l}$ and $(\mu,\nu)_2^{\lambda,\tau} - \lim y = L$ we obtain $(\mu,\nu)_2^{\lambda,\tau} - \lim x_{m_k,n_l} = L$. This completes the proof. ☐

Theorem 2.15. *Let $(X,\mu,\nu,*,o)$ be an IF2NLS and $x = (x_{k,l})$ be a double sequence in X. Let I be a non-trivial ideal in $\mathbb{N}\times\mathbb{N}$. If there is a $I_{\lambda,\tau}^{(\mu,\nu)_2}$-convergent sequence $y = (y_{k,l})$ in X such that $\{(k,l) \in \mathbb{N}\times\mathbb{N} : y_{k,l} \neq x_{k,l}\} \in I$ then x is also $I_{\lambda,\tau}^{(\mu,\nu)_2}$-convergent.*

Proof. Suppose that $\{(k,l) \in \mathbb{N}\times\mathbb{N} : y_{k,l} \neq x_{k,l}\} \in I$ and $I_{\lambda,\tau}^{(\mu,\nu)_2} - y = \ell$. Then for every $\varepsilon \in (0,1)$ and $t > 0$, the set

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(T_{mn}(y) - L, y; t) \leq 1-\varepsilon \text{ or } \nu(T_{mn}(y) - L, y; t) \geq \varepsilon\} \in I.$$

For every $\varepsilon \in (0,1)$ and $t > 0$, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(T_{mn}(x) - L, y; t) \leq 1-\varepsilon \text{ or } \nu(T_{mn}(x) - L, y; t) \geq \varepsilon\}$$
$$\subseteq \{(k,l) \in \mathbb{N}\times\mathbb{N} : y_{k,l} \neq x_{k,l}\} \cup$$
$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(T_{mn}(y) - L, y; t) \leq 1-\varepsilon \text{ or } \nu(T_{mn}(y) - L, y; t) \geq \varepsilon\}.$$

As both the sets of right-hand side of above relation is in I, therefore we have that

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(T_{mn}(x) - L, y; t) \leq 1-\varepsilon \text{ or } \nu(T_{mn}(x) - L, y; t) \geq \varepsilon\} \in I.$$

This completes the proof of the theorem. ☐

Definition 2.16. Let $(X,\mu,\nu,*,o)$ be an IF2NLS. A sequence $x = (x_{k,l})$ in X is said to be Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu,\nu)_2^{\lambda,\tau}$ if, for every $t > 0$ and $\varepsilon \in (0,1)$, there exist $m_0, n_0 \in \mathbb{N}$ satisfying

$$\mu(T_{mn}(x) - T_{uv}(x), z; t) > 1-\varepsilon \text{ and } \nu(T_{mn}(x) - T_{uv}(x), z; t) < \varepsilon$$

for all $m, u \geq m_o, n, v \geq n_0$.

Definition 2.17. Let $(X,\mu,\nu,*,o)$ be an IF2NLS. A double sequence $x = (x_{k,l})$ in X is said to be $I_{\lambda,\tau}$-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu,\nu)_2^{\lambda,\tau}$ if, for every $t > 0$ and $\varepsilon \in (0,1)$, there exists $u, v \in \mathbb{N}$ satisfying

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu(T_{mn}(x) - T_{vu}(x), z; t) > 1-\varepsilon$$
$$\text{and } \nu(T_{mn}(x) - T_{uv}(x), z; t) < \varepsilon\} \in F(I)$$

Definition 2.18. Let $(X, \mu, \nu, *, o)$ be an IF2NLS. A double sequence $x = (x_{k,l})$ in X is said to be $I^*_{\lambda,\tau}$-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2^{\lambda,\tau}$ if, there exists a set $M = \{(m_1, n_1) < (m_2, n_2) < ... < (m_k, n_l)\} \subset \mathbb{N} \times \mathbb{N}$ such that the set $M^\imath = \{(m, n) \in \mathbb{N} \times \mathbb{N} : (m_k, n_l) \in J_{mn}\} \in F(I)$ and the subsequence (x_{m_k,n_l}) of $x = (x_{k,l})$ is a Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2^{\lambda,\tau}$.

The proof of the following result is straight forward from the definitions.

Theorem 2.19. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS. If a double sequence $x = (x_{k,l})$ in X is Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2^{\lambda,\tau}$, then it is $I_{\lambda,\tau}$-Cauchy sequence with respect to the same norm.*

We formulate the following two results with out the proofs, which can be easily established.

Theorem 2.20. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS. If a double sequence $x = (x_{k,l})$ in X is Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2^{\lambda,\tau}$, then there is a subsequence of $x = (x_{k,l})$ which is ordinary Cauchy sequence with respect to the same norm.*

Theorem 2.21. *Let $(X, \mu, \nu, *, o)$ be an IF2NLS. If a double sequence $x = (x_{k,l})$ in X is $I^*_{\lambda,\tau}$-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)_2^{\lambda,\tau}$, then it is $I_{\lambda,\tau}$-Cauchy sequence as well.*

3. Conclusion

The present work contains not only an improvement and a generalization of the works of [41], Section 2 of the paper [41] as it has been presented in more general setting, i.e. in I-convergence which is more general than the statistical case, but also an investigation of some futher results in I-convergence. So that one may expect it to be more useful tool in the field of metric space theory in modeling various problems occurring in many areas of science, computer science, information theory, dynamical systems, biological science, geographic information systems, population modeling, and motion planning in robotics. It seems that an investigation of the present work taking "nets" instead of "sequences" could be done using the properties of "nets" instead of using the properties of "sequences" in different abstract spaces.

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