Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 6, (June 2015), pp. 925–940
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On intuitionistic Q-fuzzy sets in ternary semirings

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Received 25 August 2014; Revised 8 December 2014; Accepted 23 December 2014

Abstract. In this paper we introduce the notions of Q-fuzzy right ideal, anti Q-fuzzy right ideal, intuitionistic Q-fuzzy right ideal, Q-fuzzy right k-ideal, anti Q-fuzzy right k-ideal, intuitionistic Q-fuzzy right k-ideal, Q-fuzzy bi-ideal, anti Q-fuzzy bi-ideal and intuitionistic Q-fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We characterize regular ternary semiring through intuitionistic Q-fuzzy right ideal. We introduce normal intuitionistic Q-fuzzy ideals in ternary semirings.

2010 AMS Classification: 08A72, 16Y60, 12K10, 17A40

Keywords: Q-fuzzy right ideal, Anti Q-fuzzy right ideal, Intuitionistic Q-Fuzzy right ideal, Q-fuzzy right k-ideal, Anti Q-fuzzy right k-ideal, Intuitionistic Q-fuzzy right k-ideal, Q-fuzzy bi-ideal, Anti Q-fuzzy bi-ideal, Intuitionistic Q-fuzzy bi-ideal, Intrinsic product, Normal intuitionistic Q-fuzzy ideal, Intuitionistic Q-fuzzy maximal ideal.

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1. Introduction

The notion of ternary algebraic system was introduced by Lehmer [16] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [18] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [4] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [4, 5, 6, 7, 8, 9, 10, 11]. The theory of fuzzy sets was first studied by Zadeh [19] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Atanassov introduced intuitionistic fuzzy sets which constitute a generalization of the notion of fuzzy sets [2, 3]. The degree of membership of an element is given in a fuzzy set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. Osman Kazanci, Sultan yamark and Serife yilmaz in [14] have introduced the notion of intuionistic Q-fuzzification of N-subgroups (subnear-rings) in a near-ring and investigated some related properties. S.Lekkoksung [17] studied intuitionistic Q-fuzzy K-ideals of semirings. K.H.Kim [15] studied intuitionistic Q-fuzzy ideals. Kavikumar et al.[12] and [13] studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings. A.H. Mohammed [1] studied Anti fuzzy k-ideals of ternary semirings. In this paper we introduce the notions of Q-fuzzy right ideal, anti Q-fuzzy right ideal, intuitionistic Q-fuzzy right ideal, Q-fuzzy right k-ideal, anti Q-fuzzy right k-ideal, intuitionistic Q-fuzzy right k-ideal, Q-fuzzy bi-ideal, anti Q-fuzzy bi-ideal and intuitionistic Q-fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We characterize regular ternary semiring through intuitionistic Q-fuzzy right ideal. We introduce normal intuitionistic Q-fuzzy ideals in ternary semirings.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a + b)cd = acd + bcd$,
(iii) $a(b + c)d = abd + acd$ and
(iv) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in S$.

Definition 2.2. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Definition 2.3. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.4. An additive subsemigroup I of S is called a left (resp. right, lateral) ideal of S if $s_1s_2i \in I$ (resp. $is_1s_2 \in I$, $s_1is_2 \in I$) for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.

It is obvious that every ideal of a ternary semiring with zero contains a zero element.

Definition 2.5. A left (right, lateral) ideal I is called a left (right, lateral) k-ideal of a ternary semiring S if $x + y, y \in I$ implies $x \in I$.

Definition 2.6. An additive subsemigroup $(B, +)$ of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.

Definition 2.7. Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.

Definition 2.8. A ternary semiring S is said to be regular if for each $a \in S$, there exists $x \in S$ such that $a = axa$.

Lemma 2.9 ([10]). *If a ternary semiring S is regular if and only if $RML = R \cap M \cap L$ for all right ideal R, lateral ideal M and left ideal L in the ternary semiring S.*

Definition 2.10. Let X be a non-empty set and Q be a non-empty set. A Q-fuzzy subset f of X is a mapping $f : X \times Q \to [0, 1]$. The complement of a Q-fuzzy subset f of a set X is denoted by f^c and defined as $f^c(x, q) = 1 - f(x, q)$, for all $x \in X$, $q \in Q$.

Definition 2.11. Let f, g and h be any three Q-fuzzy subsets of a ternary semiring S. Then $f \cap g$, $f \cup g$, $f + g$, $f \circ g \circ h$ are Q-fuzzy subsets of S defined by for all $x, y, z, u, v, w \in S, q \in Q$

$$(f \cap g)(x, q) = min\{f(x, q),\ g(x, q)\}$$

$$(f \cup g)(x, q) = max\{f(x, q),\ g(x, q)\}$$

$$(f + g)(x, q) = \begin{cases} sup\{min\{f(y, q), g(z, q)\}\} & if \quad x = y + z \\ 0 \qquad otherwise \end{cases}$$

$$(f \circ g \circ h)(x, q) = \begin{cases} sup\{min\{f(u, q), g(v, q), h(w, q)\}\} & if \quad x = uvw, \\ 0 \qquad otherwise \end{cases}$$

Definition 2.12. An upper level set of a Q-fuzzy subset f denoted by $U(f; t)$ is defined as $U(f; t) = \{x \in X/f(x, q) \geq t$, for all $q \in Q\}$ and a lower level set of a Q-fuzzy subset f denoted by $L(f; t)$ is defined as $L(f; t) = \{x \in X/f(x, q) \leq t$, for all $q \in Q\}$, for all $t \in [0, 1]$.

Definition 2.13 ([17]). An intuitionistic Q-fuzzy subset (IQFS for short) defined on non-empty sets S and Q as objects of the form
$A = \{< x, q, f_A(x, q), g_A(x, q) > /x \in S, q \in Q\}$, where the function $f : S \times Q \to [0, 1]$ and $g : S \times Q \to [0, 1]$ denote the degree of membership (namely $f_A(x, q)$) and the degree of non-membership (namely $g_A(x, q)$) for each element $x \in S$, $q \in Q$ to the set A, respectively, and $0 \leq f_A(x, q) + g_A(x, q) \leq 1$ for each $x \in S$, $q \in Q$. Obviously, every Q-fuzzy subset f we can have $A = \{< x, q, f_A(x, q), g_A(x, q) > /x \in S, q \in Q\}$. For the sake of simplicity, we shall use the symbol $A = (f_A, g_A)$ for the intuitionistic Q-fuzzy subset $A = \{< x, q, f_A(x, q), g_A(x, q) > /x \in S, q \in Q\}$. Obviously for an IQFS $A = (f_A, g_A)$ in S, when $g_A(x, q) = 1 - f_A(x, q)$, for every $x \in S$, $q \in Q$, the IQFS A is a Q-fuzzy subset.

3. Intuitionistic Q-fuzzy ideals

Definition 3.1. A Q-fuzzy subset f of a ternary semiring S is said to be a Q-fuzzy right (left, lateral) ideal of S if
1. $f(x + y, q) \geq min\{f(x, q), f(y, q)\}$
2. $f(xyz, q) \geq f(x, q)$ $(f(xyz, q) \geq f(z, q),\ f(xyz, q) \geq f(y, q))$ for all $x, y, z \in S, q \in Q$.

Definition 3.2. A Q-fuzzy subset f of a ternary semiring S is said to be an anti Q-fuzzy right (left, lateral) ideal of S if
1. $f(x+y,q) \leq max\{f(x,q), f(y,q)\}$
2. $f(xyz,q) \leq f(x,q)$ $(f(xyz,q) \leq f(z,q),\ f(xyz,q) \leq f(y,q))$ for all $x, y, z \in S, q \in Q$.

Example 3.3. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q\}$. Let the Q-fuzzy subset f of S be defined by

$$f(x,q) = \begin{cases} 0.7 & if \quad x \in \langle -2 \rangle \\ 0.2 & otherwise, \end{cases}$$

Then f is a Q-fuzzy right ideal of S.

$$f(x,q) = \begin{cases} 0.3 & if \quad x \in \langle -2 \rangle \\ 0.8 & otherwise, \end{cases}$$

Then f is an anti Q-fuzzy right ideal of S.

Definition 3.4. An intuitionistic Q-fuzzy subset $A = (f_A, g_A)$ in S is called an intuitionistic Q-fuzzy right (left, lateral) ideal of S if
1. $f_A(x+y,q) \geq min\{f_A(x,q), f_A(y,q)\}$
2. $f_A(xyz,q) \geq f_A(x,q)$ $(f_A(xyz,q) \geq f_A(z,q), f_A(xyz,q) \geq f_A(y,q))$
3. $g_A(x+y,q) \leq max\{g_A(x,q), g_A(y,q)\}$
4. $g_A(xyz,q) \leq g_A(x,q)$ $(g_A(xyz,q) \leq g_A(z,q),\ g_A(xyz,q) \leq g_A(y,q))$ for all $x, y, z \in S, q \in Q$.

Definition 3.5. An intuitionistic Q-fuzzy subset $A = (f_A, g_A)$ is called an intuitionistic Q-fuzzy ideal of S if it is intuitionistic Q-fuzzy right, intuitionistic Q-fuzzy lateral and intuitionistic Q-fuzzy left ideal of S.

Example 3.6. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q_1, q_2\}$. Let the Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x,q_1) = \begin{cases} 0.7 & if \quad x \in \langle -3 \rangle \\ 0.1 & otherwise, \end{cases} \qquad f_A(x,q_2) = \begin{cases} 0.8 & if \quad x \in \langle -3 \rangle \\ 0.3 & otherwise, \end{cases}$$

$$g_A(x,q_1) = \begin{cases} 0.2 & if \quad x \in \langle -3 \rangle \\ 0.8 & otherwise, \end{cases} \qquad g_A(x,q_2) = \begin{cases} 0.1 & if \quad x \in \langle -3 \rangle \\ 0.7 & otherwise. \end{cases}$$

Then $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy ideal of S.

Example 3.7. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q_1, q_2\}$. Let the Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x,q_1) = \begin{cases} 0.7 & if \quad x \in \langle -3 \rangle \\ 0.1 & otherwise, \end{cases} \qquad f_A(x,q_2) = \begin{cases} 0.8 & if \quad x = \{-2, -3\} \\ 0.3 & otherwise, \end{cases}$$

$$g_A(x,q_1) = \begin{cases} 0.2 & if \quad x \in \langle -3 \rangle \\ 0.8 & otherwise, \end{cases} \qquad g_A(x,q_2) = \begin{cases} 0.1 & if \quad x \in \langle -3 \rangle \\ 0.7 & otherwise. \end{cases}$$

Then $A = (f_A, g_A)$ is not an intuitionistic Q-fuzzy ideal of S, since
$f_A([-2 + (-3)], q_2) = 0.3 \not\geq \min\{f_A(-2, q_2), f_A(-3, q_2)\} = 0.8$.

Definition 3.8. Let $A = (f_A, g_A)$ be an intuitionistic Q-fuzzy subset of S and let $s, t \in [0, 1]$. Then the set $S_A^{(s,t)} = \{x \in S/\ f_A(x, q) \geq s, g_A(x, q) \leq t, q \in Q\}$ is called a (s, t)-level set of $A = (f_A, g_A)$.
The set $\{(s, t) \in Im(f_A) \times Im(g_A)/\ s + t \leq 1\}$ is called image of $A = (f_A, g_A)$.
Clearly $S_A^{(s,t)} = U(f_A; s) \cap L(g_A; t)$, where $U(f_A; s)$ and $L(g_A; t)$ are upper and lower level subsets of f_A and g_A respectively.

Theorem 3.9. *If a Q-fuzzy subset f is a Q-fuzzy right (left, lateral) ideal of a ternary semiring S if and only if f^c is an anti Q-fuzzy right (left, lateral) ideal of S.*

Proof. Let f be a Q-fuzzy right ideal of a ternary semiring S. Let $x, y, z \in S$, $q \in Q$.
Then $f(x + y, q) \geq min\{f(x, q), f(y, q)\}$
$\Rightarrow -f(x + y, q) \leq -min\{f(x, q), f(y, q)\}$
$\Rightarrow 1 - f(x + y, q) \leq 1 - min\{f(x, q), f(y, q)\} = max\{1 - f(x, q), 1 - f(y, q)\}$
$\Rightarrow 1 - f(x + y, q) \leq max\{1 - f(x, q), 1 - f(y, q)\}$
$\Rightarrow f^c(x + y, q) \leq max\{f^c(x, q), f^c(y, q)\}$
and $f(xyz, q) \geq f(x, q)$
$\Rightarrow -f(xyz, q) \leq -f(x, q)$
$\Rightarrow 1 - f(xyz, q) \leq 1 - f(x, q)$
$\Rightarrow f^c(xyz, q) \leq f^c(x, q)$.
Thus f^c is an anti Q-fuzzy right ideal of S. By similar argument, we can prove the converse part. □

Theorem 3.10. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right (left, lateral) ideal of S if and only if any level set $S_A^{(s,t)}$ is a right (left, lateral) ideal of S for $s, t \in [0, 1]$ whenever nonempty.*

Proof. Let $A = (f_A, g_A)$ be an intuitionistic Q-fuzzy right ideal in S and let $q \in Q$. Let $x, y \in S_A^{(s,t)}$. Then $f_A(x+y, q) \geq min\{f_A(x, q), f_A(y, q)\} \geq s$ and $g_A(x+y, q) \leq max\{g_A(x, q), g_A(y, q)\} \leq t$. So $x + y \in S_A^{(s,t)}$. Now let $x \in S_A^{(s,t)}$ and $y, z \in S$ then $f_A(x, q) \geq s$ and $g_A(x, q) \leq t$. Since A is an intuitionistic Q-fuzzy right ideal, $f_A(xyz, q) \geq f_A(x, q) \geq s$ and $g_A(xyz, q) \leq g_A(x, q) \leq t$. Which implies $xyz \in S_A^{(s,t)}$. Hence $S_A^{(s,t)}$ is a right ideal. Conversely let $S_A^{(s,t)}$ be a right ideal of S, for any $s, t \in [0, 1]$ with $s + t \leq 1$. Let $x, y \in S$ and $q \in Q$ such that $f_A(x, q) = \alpha_1, f_A(y, q) = \alpha_2$ and $g_A(x, q) = \beta_1, g_A(y, q) = \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$. Then $\alpha_1 + \beta_1 \leq 1, \alpha_2 + \beta_2 \leq 1$. Let $\alpha = min\{\alpha_1, \alpha_2\}$ and $\beta = max\{\beta_1, \beta_2\}$ then $x, y \in S_A^{(\alpha,\beta)}$. Since $S_A^{(\alpha,\beta)}$ be a right ideal of S then $x + y \in S_A^{(\alpha,\beta)}$ that means $f_A(x + y, q) \geq \alpha = min\{f_A(x, q), f_A(y, q)\}$, $g_A(x + y, q) \leq \beta = max\{g_A(x, q), g_A(y, q)\}$. Again let $x, y, z \in S$, $q \in Q$, $c_1, c_2 \in [0, 1]$ and $f_A(x, q) = c_1$, $g_A(x, q) = c_2$ with $c_1 + c_2 \leq 1$ then $x \in S_A^{(c_1,c_2)}$ implies $xyz \in S_A^{(c_1,c_2)}$ that means $f_A(xyz, q) \geq c_1 = f_A(x, q)$ and

$g_A(xyz,q) \leq c_2 = g_A(x,q)$. Therefore A is an intuitionistic Q-fuzzy right ideal. Similarly we can prove the left and lateral ideals. □

Corollary 3.11. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right ideal of S if and only if for every $s, t \in [0,1]$ such that $s + t \leq 1$ all nonempty $U(f_A; s)$ and $L(g_A; t)$ are right ideals of S.*

Theorem 3.12. *Let I be a non-empty subset of a ternary semiring S. Then an IQFS $A = (f_A, g_A)$ defined by*

$$f_A(x,q) = \begin{cases} s_2 & if \quad \forall x \in I \\ s_1 & otherwise \end{cases}$$

$$g_A(x,q) = \begin{cases} t_2 & if \quad \forall x \in I \\ t_1 & otherwise \end{cases}$$

where $0 \leq s_1 < s_2 \leq 1$, $0 \leq t_2 < t_1 \leq 1$ and $s_i + t_i \leq 1$ for each $i = 1,2$ is an intuitionistic Q-fuzzy right (left, lateral) ideal of S if and only if I is a right ideal (left, lateral) of S.

Proof. Let I be a right ideal of S. Let $x, y, z \in S$, $q \in Q$. If $x, y, z \in I$, then $x + y, xyz \in I$. Then $f_A(x + y, q) = s_2 \geq min\{f_A(x,q), f_A(y,q)\}$, $g_A(x + y, q) = t_2 \leq max\{g_A(x,q), g_A(y,q)\}$, $f_A(xyz,q) \geq f_A(x,q)$ and $g_A(xyz,q) \leq g_A(x,q)$. If either x or y or $z \notin I$, then also $f_A(x + y, q) \geq s_1 = min\{f_A(x,q), f_A(y,q)\}$, $g_A(x + y, q) \leq t_1 = max\{g_A(x,q), g_A(y,q)\}$, $f_A(xyz,q) \geq f_A(x,q)$ and $g_A(xyz,q) \leq g_A(x,q)$. Hence $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right ideal of S.
Conversely, let $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right ideal of S. Then $S_A^{(s_1,t_1)} = I$. So, by Theorem 3.10, I must be a right ideal of S. □

Corollary 3.13. *Let I be a non-empty subset of a ternary semiring S. Then I is a right (left, lateral) ideal of S if and only if the IQFS $A = (\chi_I, 1 - \chi_I)$ defined by*

$$\chi_I(x,q) = \begin{cases} 1 & if \quad \forall x \in I \\ 0 & otherwise \end{cases}$$

is an intuitionistic Q-fuzzy right (left, lateral) ideal of S.

Theorem 3.14. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right (left, lateral) ideal of S if and only if the Q-fuzzy subsets f_A and g_A^c are Q-fuzzy right (left, lateral) ideals of S.*

Proof. If $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right ideal of S, then clearly f_A is a Q-fuzzy right ideal of S. For all $x, y, z \in S$, $q \in Q$,
$g_A^c(x+y,q) = 1 - g_A(x+y,q) \geq 1 - max\{g_A(x,q), g_A(y,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q)\} = min\{g_A^c(x,q), g_A^c(y,q)\}$ and
$g_A^c(xyz,q) = 1 - g_A(xyz,q) \geq 1 - g_A(x,q) = g_A^c(x,q)$. Thus g_A^c is a Q-fuzzy right ideal of S. Conversely assume that f_A and g_A^c are Q-fuzzy right ideals of S, then clearly the conditions i) and ii) of definition 3.4 are satisfied. Now for all $x, y, z \in S$, $q \in Q$, $1 - g_A(x + y, q) = g_A^c(x + y, q) \geq min\{g_A^c(x,q), g_A^c(y,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q)\} = 1 - max\{g_A(x,q), g_A(y,q)\}$ which implies $-g_A(x + y, q) \geq -max\{g_A(x,q), g_A(y,q)\}$ implies $g_A(x + y, q) \leq max\{g_A(x,q), g_A(y,q)\}$ and $1 -$

$g_A(xyz,q) = g_A^c(xyz,q) \geq g_A^c(x,q) = 1 - g_A(x,q)$ which implies $-g_A(xyz,q) \geq -g_A(x,q)$ implies $g_A(xyz,q) \leq g_A(x,q)$. Therefore $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right ideal of S. □

Corollary 3.15. *Let IQFS $A = (f_A, g_A)$ be in S. Then IQFS $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right(left, lateral) ideal of S if and only if IQFS $A_1 = (f_A, f_A^c)$ and IQFS $A_2 = (g_A^c, g_A)$ are intuitionistic Q-fuzzy right (left, lateral) ideals of S.*

Proof. It is straightforward by Theorem 3.9 and Theorem 3.14. □

Definition 3.16. Let X, Y and Q be non-empty sets and let $\Phi : X \times Q \to Y \times Q$. If $A = (f_A, g_A)$ and $B = (f_B, g_B)$ are IQFSs in X and Y respectively, then the preimage of B under Φ, denoted by $\Phi^{-1}(B)$, is an IQFS in X defined by $\Phi^{-1}(B) = (\Phi^{-1}(f_B), \Phi^{-1}(g_B))$.

Definition 3.17. Let S_1, S_2 be ternary semirings and let Q be any nonempty set. A mapping $f : S_1 \times Q \to S_2 \times Q$ is said to be a homomorphism if $f(x+y,q) = f(x,q) + f(y,q)$ and $f(xyz,q) = f(x,q)f(y,q)f(z,q)$ for all $x, y, z \in S_1$, $q \in Q$.

Theorem 3.18. *Let S_1, S_2 be ternary semirings and Q be any non-empty set and let $\Phi : S_1 \times Q \to S_2 \times Q$ be a homomorphism. If $B = (f_B, g_B)$ is an intuitionistic Q-fuzzy right (left, lateral) ideal of S_2, then $\Phi^{-1}(B) = (\Phi^{-1}(f_B), \Phi^{-1}(g_B))$ where $\Phi^{-1}(f_B)(x,q) = f_B(\Phi(x,q))$ and $\Phi^{-1}(g_B)(x,q) = g_B(\Phi(x,q))$ for all $x \in S_1$, $q \in Q$, is an intuitionistic Q-fuzzy right (left, lateral) ideal of S_1.*

Proof. Assume $B = (f_B, g_B)$ is an intuitionistic Q-fuzzy right ideal of S_2, and let $x, y, z \in S_1$, $q \in Q$. Then
1. $\Phi^{-1}(f_B)(x+y,q) = f_B(\Phi(x+y,q)) = f_B(\Phi(x,q) + \Phi(y,q))$
$\geq min\{f_B(\Phi(x,q)), f_B(\Phi(y,q))\} = min\{\Phi^{-1}(f_B)(x,q), \Phi^{-1}(f_B)(y,q)\}$.
2. $\Phi^{-1}(f_B)(xyz) = f_B(\Phi(xyz,q)) = f_B(\Phi(x,q)\Phi(y,q)\Phi(z,q))$
$\geq f_B(\Phi(x,q)) = \Phi^{-1}(f_B)(x,q)$.
3. $\Phi^{-1}(g_B)(x+y,q) = g_B(\Phi(x+y,q)) = g_B(\Phi(x,q) + \Phi(y,q))$
$\leq max\{g_B(\Phi(x,q)), g_B(\Phi(y,q))\} = max\{\Phi^{-1}(g_B)(x,q), \Phi^{-1}(g_B)(y,q)\}$.
4. $\Phi^{-1}(g_B)(xyz) = g_B(\Phi(xyz,q)) = g_B(\Phi(x,q)\Phi(y,q)\Phi(z,q))$
$\leq g_B(\Phi(x,q)) = \Phi^{-1}(g_B)(x,q)$. Therefore $\Phi^{-1}(B) = (\Phi^{-1}(f_B), \Phi^{-1}(g_B))$ is an intuitionistic Q-fuzzy ideal of S_1. □

4. Intuitionistic Q-fuzzy k-ideals

Definition 4.1. A Q-fuzzy right (left, lateral) ideal f of a ternary semiring S is said to be a Q-fuzzy right (left, lateral) k-ideal of S if for all $x, y \in S, q \in Q$, $f(x,q) \geq min\{f(x+y,q), f(y,q)\}$.

Definition 4.2. An anti Q-fuzzy right (left, lateral) ideal f of a ternary semiring S is said to be an anti Q-fuzzy right (left, lateral) k-ideal of S if for all $x, y \in S, q \in Q$, $f(x,q) \leq max\{f(x+y,q), f(y,q)\}$.

Definition 4.3. An intuitionistic Q-fuzzy right (left, lateral) ideal $A = (f_A, g_A)$ in S is called an intuitionistic Q-fuzzy right (left, lateral) k-ideal of S if
1. $f_A(x,q) \geq min\{f_A(x+y,q), f_A(y,q)\}$
2. $g_A(x,q) \leq max\{g_A(x+y,q), g_A(y,q)\}$
for all $x, y \in S, q \in Q$.

Example 4.4. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q_1, q_2\}$. Let the Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x,q_1) = \begin{cases} 0 & if \quad x = -1 \\ 0.7 & otherwise \end{cases} \quad f_A(x,q_2) = \begin{cases} 0.7 & if \quad x \in \langle -3 \rangle \\ 0.1 & otherwise, \end{cases}$$

$$g_A(x,q_1) = \begin{cases} 1 & if \quad x = -1 \\ 0.2 & otherwise \end{cases} \quad g_A(x,q_2) = \begin{cases} 0.2 & if \quad x \in \langle -3 \rangle \\ 0.6 & otherwise. \end{cases}$$

Then clearly $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy ideal of S. But $A = (f_A, g_A)$ is not an intuitionistic Q-fuzzy k-ideal of S,
since $f_A(-1, q_1) = 0 \ngeq \min\{f_A(-1 + (-2), q_1), f_A(-2, q_1)\} = 0.7$
and $g_A(-1, q_1) = 1 \nleq \max\{g_A(-1 + (-2), q_1), g_A(-2, q_1)\} = 0.2$.

Example 4.5. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q_1, q_2\}$. Let Q-fuzzy subsets f_A and g_A of S be defined by

$$f_A(x,q_1) = \begin{cases} 0.9 & if \quad x \in \langle -2 \rangle \\ 0.1 & otherwise \end{cases} \quad f_A(x,q_2) = \begin{cases} 0.7 & if \quad x \in \langle -2 \rangle \\ 0.4 & otherwise, \end{cases}$$

$$g_A(x,q_1) = \begin{cases} 0 & if \quad x \in \langle -2 \rangle \\ 0.6 & otherwise. \end{cases} \quad g_A(x,q_2) = \begin{cases} 0.2 & if \quad x \in \langle -2 \rangle \\ 0.5 & otherwise. \end{cases}$$

Then $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy k-ideal of S.

Theorem 4.6. *If a Q-fuzzy subset f is a Q-fuzzy right (left, lateral) k-ideal of a ternary semiring S if and only if f^c is an anti Q-fuzzy right (left, lateral) k-ideal of S.*

Proof. Let f be a Q-fuzzy right k-ideal of a ternary semiring S. By Theorem 3.9, f^c is an anti Q-fuzzy right ideal of S. Let $x, y \in S$, $q \in Q$.
Then $f(x,q) \geq min\{f(x+y,q), f(y,q)\}$
$\Rightarrow -f(x,q) \leq -min\{f(x+y,q), f(y,q)\}$
$\Rightarrow 1 - f(x,q) \leq 1 - min\{f(x+y,q), f(y,q)\}$
$\Rightarrow 1 - f(x,q) \leq max\{1 - f(x+y,q), 1 - f(y,q)\}$
$\Rightarrow f^c(x,q) \leq max\{f^c(x+y,q), f^c(y,q)\}$.
Therefore f^c is an anti Q-fuzzy right ideal of S. By similar argument, we can prove the converse part. □

Theorem 4.7. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right (left, lateral) k-ideal of S if and only if any level set $S_A^{(s,t)}$ is a right (left, lateral) k-ideal of S for $s, t \in [0,1]$ whenever nonempty.*

Proof. Let A be an intuitionistic Q-fuzzy right k-ideal of S. BY Theorem 3.10, $S_A^{(s,t)}$ is a right ideal of S. If there exists $x, y \in S, q \in Q$ such that $x + y, y \in S_A^{(s,t)}$ and $x \notin S_A^{(s,t)}$, then $min\{f_A(x+y,q), f_A(y,q)\} \geq s > f_A(x,q)$ and $max\{g_A(x+y,q), g_A(y,q)\} \leq t < g_A(x,q)$, which is a contradiction. Therefore $S_A^{(s,t)}$ is a right k-ideal in S. Conversely, if there exists $x, y \in S, q \in Q$ such that $f_A(x,q) < s =$

$min\{f_A(x+y,q), f_A(y,q)\}$ and $g_A(x,q) > t = max\{g_A(x+y,q), g_A(y,q)\}$ then $x+y, y \in S_A^{(s,t)}$ and $x \notin S_A^{(s,t)}$ which is a contradiction. Therefore by Theorem 3.10, $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right k-ideal of S. □

Corollary 4.8. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right (left, lateral) k-ideal of S if and only if for every $s, t \in [0,1]$ such that $s + t \leq 1$ all nonempty $U(f_A; s)$ and $L(g_A; t)$ are right (left, lateral) k-ideals of S.*

Theorem 4.9. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy right (left, lateral) k-ideal of S if and only if the Q-fuzzy subsets f_A and g_A^c are Q-fuzzy right (left, lateral) k-ideals of S.*

Proof. If $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right k-ideal of S, then clearly f_A is a Q-fuzzy right k-ideal of S. For all $x, y \in S, q \in Q$, $g_A^c(x,q) = 1 - g_A(x,q) \geq 1 - max\{g_A(x+y,q), g_A(y,q)\} = min\{1 - g_A(x+y,q), 1 - g_A(y,q)\} = min\{g_A^c(x+y,q), g_A^c(y,q)\}$. Thus g_A^c is a Q-fuzzy right k-ideal of S. Conversely assume that f_A and g_A^c are Q-fuzzy right k-ideals of S, then by Theorem 3.14, $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right ideal of S. Now for all $x, y \in S, q \in Q$, $1 - g_A(x,q) = g_A^c(x,q) \geq min\{g_A^c(x+y,q), g_A^c(y,q)\} = min\{1 - g_A(x+y,q), 1 - g_A(y,q)\} = 1 - max\{g_A(x+y,q), g_A(y,q)\}$ which implies $-g_A(x,q) \geq -max\{g_A(x+y,q), g_A(y,q)\}$ implies $g_A(x,q) \leq max\{g_A(x+y,q), g_A(y,q)\}$. Therefore $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right k-ideal of S. □

Corollary 4.10. *Let IQFS $A = (f_A, g_A)$ be in S. Then IQFS $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right (left, lateral) k-ideal of S if and only if IQFS $A_1 = (f_A, f_A^c)$ and IQFS $A_2 = (g_A^c, g_A)$ are intuitionistic Q-fuzzy right (left, lateral) k-ideals of S.*

Proof. It is straightforward by Theorem 4.6 and Theorem 4.9. □

5. Intuitionistic Q-fuzzy bi-ideals

Definition 5.1. A Q-fuzzy subset f of a ternary semiring S is said to be a Q-fuzzy bi-ideal of S if
1. $f_A(x+y,q) \geq min\{f_A(x,q), f_A(y,q)\}$
2. $f_A(xs_1ys_2z,q) \geq min\{f_A(x,q), f_A(y,q), f_A(z,q)\}$ for all $x, s_1, y, s_2, z \in S, q \in Q$.

Definition 5.2. A Q-fuzzy subset f of a ternary semiring S is said to be an anti Q-fuzzy bi-ideal of S if
1. $f_A(x+y,q) \leq max\{f_A(x,q), f_A(y,q)\}$
2. $f_A(xs_1ys_2z,q) \leq max\{f_A(x,q), f_A(y,q), f_A(z,q)\}$ for all $x, s_1, y, s_2, z \in S, q \in Q$.

Definition 5.3. An intuitionistic Q-fuzzy subset $A = (f_A, g_A)$ in S is called an intuitionistic Q-fuzzy bi-ideal of S if
1. $f_A(x+y,q) \geq min\{f_A(x,q), f_A(y,q)\}$
2. $f_A(xs_1ys_2z,q) \geq min\{f_A(x,q), f_A(y,q), f_A(z,q)\}$
3. $g_A(x+y,q) \leq max\{g_A(x,q), g_A(y,q)\}$
4. $g_A(xs_1ys_2z,q) \leq max\{g_A(x,q), g_A(y,q), g_A(z,q)\}$ for all $x, s_1, y, s_2, z \in S, q \in Q$.

Example 5.4. Consider

$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & h \end{pmatrix} : a, b, c, d, e, h \in Z_0^- \right\}$$

and $Q = \{q_1, q_2\}$. Then S is a ternary semiring with respect to matrix addition and matrix multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix} : p, q \in Z_0^- \right\}$$

Let Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x, q_1) = \begin{cases} 0.8 & if \;\; x \in B \\ 0.1 & otherwise \end{cases} \qquad f_A(x, q_2) = \begin{cases} 0.9 & if \;\; x \in B \\ 0.3 & otherwise \end{cases}$$

$$g_A(x, q_1) = \begin{cases} 0.2 & if \;\; x \in B \\ 0.7 & otherwise \end{cases} \qquad g_A(x, q_2) = \begin{cases} 0.1 & if \;\; x \in B \\ 0.7 & otherwise \end{cases}$$

Then $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy bi-ideal of S, but not an intuitionistic Q-fuzzy ideal of S. Since $f_A(ssb, q_1) = 0.1 < f_A(b, q_1)$; $f_A(sbs, q_1) = 0.1 < f_A(b, q_1)$; $f_A(bss, q_1) = 0.1 < f_A(b, q_1)$; $g_A(ssb, q_1) = 0.7 > g_A(b, q_1)$; $g_A(sbs, q_1) = 0.7 > g_A(b, q_1)$ and $g_A(bss, q_1) = 0.7 > g_A(b, q_1)$, where

$$s = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 5.5. *If a Q-fuzzy subset f is a Q-fuzzy bi-ideal of a ternary semiring S if and only if f^c is an anti Q-fuzzy bi-ideal of S.*

Proof. Let f be a Q-fuzzy bi-ideal of a ternary semiring S. Let $x, y, z \in S$, $q \in Q$. Then

$$\begin{gathered} f(x + y, q) \geq min\{f(x, q), f(y, q)\} \\ \Rightarrow -f(x + y, q) \leq -min\{f(x, q), f(y, q)\} \\ \Rightarrow 1 - f(x + y, q) \leq 1 - min\{f(x, q), f(y, q)\} \\ \Rightarrow 1 - f(x + y, q) \leq max\{1 - f(x, q), 1 - f(y, q)\} \\ \Rightarrow f^c(x + y, q) \leq max\{f^c(x, q), f^c(y, q)\} \end{gathered}$$

and

$$\begin{gathered} f(xs_1ys_2z, q) \geq min\{f(x, q), f(y, q), f(z, q)\} \\ \Rightarrow -f(xs_1ys_2z, q) \leq -min\{f(x, q), f(y, q), f(z, q)\} \\ \Rightarrow 1 - f(xs_1ys_2z, q) \leq 1 - min\{f(x, q), f(y, q), f(z, q)\} \\ \Rightarrow 1 - f(xs_1ys_2z, q) \leq max\{1 - f(x, q), 1 - f(y, q), 1 - f(z, q)\} \\ \Rightarrow f^c(xs_1ys_2z, q) \leq max\{f^c(x, q), f^c(y, q), f^c(z, q)\}. \end{gathered}$$

Thus f^C is an anti Q-fuzzy bi-ideal of S. By similar argument, we can prove the converse part. □

Theorem 5.6. *Let $(f_i, g_i)_{i \in I}$ be a family of intuitionistic Q-fuzzy bi-ideals of S then $(\cap f_i, \cup g_i)$ is also an intuitionistic Q-fuzzy bi-ideal of S.*

Proof. Let $f = \bigcap_{i \in I} f_i$ and $g = \bigcup_{i \in I} g_i$. For any $x, y, z \in S$, $q \in Q$,
1. $f(x+y,q) = \bigcap_{i \in I} f_i(x+y,q) \geq \bigcap_{i \in I} min\{f_i(x,q), f_i(y,q)\}$
$= min\{\bigcap_{i \in I} f_i(x,q), \bigcap_{i \in I} f_i(y,q)\} = min\{f(x,q), f(y,q)\}$.
2. $f(xs_1ys_2z,q) = \bigcap_{i \in I} f_i(xs_1ys_2z,q) \geq \bigcap_{i \in I} min\{f_i(x,q), f_i(y,q), f_i(z,q)\}$
$= min\{\bigcap_{i \in I} f_i(x,q), \bigcap_{i \in I} f_i(y,q), \bigcap_{i \in I} f_i(z,q)\} = min\{f(x,q), f(y,q), f(z,q)\}$
3. $g(x+y,q) = \bigcup_{i \in I} g_i(x+y,q) \leq \bigcup_{i \in I} max\{g_i(x,q), g_i(y,q)\}$
$= max\{\bigcup_{i \in I} g_i(x,q), \bigcup_{i \in I} g_i(y,q)\} = max\{g_i(x,q), g_i(y,q)\}$
4. $g(xs_1ys_2z,q) = \bigcup_{i \in I} g_i(xs_1ys_2z,q) \leq \bigcup_{i \in I} max\{g_i(x,q), g_i(y,q), g_i(z,q)\}$
$= max\{\bigcup_{i \in I} g_i(x,q), \bigcup_{i \in I} g_i(y,q), \bigcup_{i \in I} g_i(z,q)\} = max\{g(x,q), g(y,q), g(z,q)\}$. Therefore $(\cap f_i, \cup g_i)$ is an intuitionistic Q-fuzzy bi-ideal of S. □

Theorem 5.7. *An IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy bi-ideal of S if and only if the Q-fuzzy subsets f_A and g_A^c are Q-fuzzy bi-ideals of S.*

Proof. If $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy bi-ideal of S, then clearly f_A is a Q-fuzzy bi-ideal of S. For all $x, y, z, s_1, s_2 \in S$, $q \in Q$, $g_A^c(x+y,q) = 1 - g_A(x+y,q) \geq 1 - max\{g_A(x,q), g_A(y,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q)\} = min\{g_A^c(x,q), g_A^c(y,q)\}$ and $g_A^c(xs_1ys_2z,q) = 1 - g_A(xs_1ys_2z,q) \geq 1 - max\{g_A(x,q), g_A(y,q), g_A(z,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q), 1 - g_A(z,q)\} = min\{g_A^c(x,q), g_A^c(y,q), g_A^c(z,q)\}$. Thus g_A^c is a Q-fuzzy bi-ideal of S. Conversely assume that f_A and g_A^c are Q-fuzzy bi-ideals of S, then clearly the conditions 1) and 2) of definition 5.3 are satisfied. Now for all $x, y, z, s_1, s_2 \in S$, $q \in Q$, $1 - g_A(x+y,q) = g_A^c(x+y,q) \geq min\{g_A^c(x,q), g_A^c(y,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q)\} = 1 - max\{g_A(x,q), g_A(y,q)\}$ which implies $-g_A(x+y,q) \geq -max\{g_A(x,q), g_A(y,q)\}$ implies $g_A(x+y,q) \leq max\{g_A(x,q), g_A(y,q)\}$ and $1 - g_A(xs_1ys_2z,q) = g_A^c(xs_1ys_2z,q) \geq min\{g_A^c(x,q), g_A^c(y,q), g_A^c(z,q)\} = min\{1 - g_A(x,q), 1 - g_A(y,q), 1 - g_A(z,q)\} = 1 - max\{g_A(x,q), g_A(y,q), g_A(z,q)\}$ which implies $-g_A(xs_1ys_2z,q) \geq -max\{g_A(x,q), g_A(y,q), g_A(z,q)\}$ implies $g_A(xs_1ys_2z,q) \leq max\{g_A(x,q), g_A(y,q), g_A(z,q)\}$. □

Corollary 5.8. *If an IQFS $A = (f_A, g_A)$ in S is an intuitionistic Q-fuzzy bi-ideal of S if and only if IQFS $A_1 = (f_A, f_A^c)$ and IQFS $A_2 = (g_A^c, g_A)$ are intuitionistic Q-fuzzy bi-ideals of S.*

Proof. It is straightforward by Theorem 5.5 and Theorem 5.7.

Theorem 5.9. *Let S_1, S_2 be ternary semirings and Q be any non-empty set and let $\Phi : S_1 \times Q \to S_2 \times Q$ be a homomorphism. If $B = (f_B, g_B)$ is an intuitionistic Q-fuzzy bi-ideal of S_2, then $\Phi^{-1}(B) = (\Phi^{-1}(f_B), \Phi^{-1}(g_B))$ where $\Phi^{-1}(f_B)(x,q) = f_B(\Phi(x,q))$ and $\Phi^{-1}(g_B)(x,q) = g_B(\Phi(x,q))$ for all $x \in S_1$, $q \in Q$, is an intuitionistic Q-fuzzy bi-ideal of S_1.*

Proof. Assume $B = (f_B, g_B)$ is an intuitionistic Q-fuzzy bi-ideal of S_2, and let $x, y, z, u, v \in S_1$, $q \in Q$. Then
1. $\Phi^{-1}(f_B)(x+y,q) = f_B(\Phi(x+y,q)) = f_B(\Phi(x,q) + \Phi(y,q))$

$\geq min\{f_B(\Phi(x,q)), f_B(\Phi(y,q))\} = min\{\Phi^{-1}(f_B)(x,q), \Phi^{-1}(f_B)(y,q)\}$.
2. $\Phi^{-1}(f_B)(xuyvz) = f_B(\Phi(xuyvz,q)) = f_B(\Phi(x,q)\Phi(u,q)\Phi(y,q)\Phi(v,q)\Phi(z,q))$
$\geq min\{f_B(\Phi(x,q)), f_B(\Phi(y,q)), f_B(\Phi(z,q))\}$
$= min\{\Phi^{-1}(f_B)(x,q), \Phi^{-1}(f_B)(y,q), \Phi^{-1}(f_B)(z,q)\}$.
3. $\Phi^{-1}(g_B)(x+y,q) = g_B(\Phi(x+y,q)) = g_B(\Phi(x,q)+\Phi(y,q))$
$\leq max\{g_B(\Phi(x,q)), g_B(\Phi(y,q))\} = max\{\Phi^{-1}(g_B)(x,q), \Phi^{-1}(g_B)(y,q)\}$.
4. $\Phi^{-1}(g_B)(xuyvz) = g_B(\Phi(xuyvz,q)) = g_B(\Phi(x,q)\Phi(u,q)\Phi(y,q)\Phi(v,q)\Phi(z,q))$
$\leq max\{g_B(\Phi(x,q)), g_B(\Phi(y,q)), g_B(\Phi(z,q))\}$
$= max\{\Phi^{-1}(g_B)(x,q), \Phi^{-1}(g_B)(y,q), \Phi^{-1}(g_B)(z,q)\}$. Therefore $\Phi^{-1}(B) = (\Phi^{-1}(f_B), \Phi^{-1}(g_B))$ is an intuitionistic Q-fuzzy bi-ideal of S_1. □

6. Regular Ternary semiring

Definition 6.1. Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ be the intuitionistic Q-fuzzy subsets in a ternary semiring S. The intuitionistic intrinsic product of $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ is defined to be the intuitionistic Q-fuzzy subset $A * B * C = (f_{A*B*C}, g_{A*B*C})$ in S given by

$$f_{A*B*C}(x,q) = \begin{cases} \bigvee\limits_{x=uvw} \{f_A(u,q) \wedge f_B(v,q) \wedge f_C(w,q)\}, & if\ x = uvw \\ 0 & otherwise. \end{cases}$$

$$g_{A*B*C}(x,q) = \begin{cases} \bigwedge\limits_{x=uvw} \{g_A(u,q) \vee g_B(v,q) \vee g_C(w,q)\}, & if\ x = uvw \\ 1 & otherwise. \end{cases}$$

Lemma 6.2. *If $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ are intuitionistic Q-fuzzy right, lateral and left ideals in a ternary semiring S respectively, then $A*B*C \subseteq A \cap B \cap C$.*

Proof. Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ be Q-fuzzy right, lateral and left ideals of a ternary semiring S respectively. Let $x \in S$, $q \in Q$. If $x = uvw$, then $f_A(x,q) \geq f_A(u,q)$; $f_B(x,q) \geq f_B(v,q)$; $f_C(x,q) \geq f_C(w,q)$; $g_A(x,q) \leq g_A(u,q)$; $g_B(x,q) \leq g_B(v,q)$ and $g_C(x,q) \leq g_C(w,q)$.
Now $f_A(x,q) \wedge f_B(x,q) \wedge f_C(x,q) \geq f_A(u,q) \wedge f_B(v,q) \wedge f_C(w,q)$
$\Rightarrow f_A(x,q) \wedge f_B(x,q) \wedge f_C(x,q) \geq \bigvee\limits_{x=uvw} \{f_A(u,q) \wedge f_B(v,q) \wedge f_C(w,q)\}$
$\Rightarrow f_{A\cap B\cap C}(x,q) \geq f_{A*B*C}(x,q)$
and $g_A(x,q) \vee g_B(x,q) \vee g_C(x,q) \leq g_A(u,q) \vee g_B(v,q) \vee g_C(w,q)$
$\Rightarrow g_A(x,q) \vee g_B(x,q) \vee g_C(x,q) \leq \bigwedge\limits_{x=uvw} \{g_A(u,q) \vee g_B(v,q) \vee g_C(w,q)\}$
$\Rightarrow g_{A\cap B\cap C}(x,q) \leq g_{A*B*C}(x,q)$.
If $x \neq uvw$, then $f_{A*B*C}(x,q) = 0$ and $g_{A*B*C}(x,q) = 1$.
Therefore $f_{A\cap B\cap C}(x,q) \geq f_{A*B*C}(x,q)$ and
$g_{A\cap B\cap C}(x,q) \leq g_{A*B*C}(x,q)$ for all $x \in S$, $q \in Q$.
Hence $A * B * C \subseteq A \cap B \cap C$. □

Theorem 6.3. *A ternary semiring S is regular if and only if $A*B*C = A\cap B\cap C$ for every intuitionistic Q-fuzzy right ideal A, intuitionistic Q-fuzzy lateral ideal B and intuitionistic Q-fuzzy left ideal C in the ternary semiring S.*

Proof. Let S be a regular ternary semiring. Let A, B and C be intuitionistic Q-fuzzy right ideal, intuitionistic Q-fuzzy lateral ideal and intuitionistic Q-fuzzy left ideal in S respectively. Then $f_A(xyz,q) \geq f_A(x,q)$; $f_B(xyz,q) \geq f_B(y,q)$; $f_C(xyz,q) \geq f_C(z,q)$; $g_A(xyz,q) \leq g_A(x,q)$; $g_B(xyz,q) \leq g_B(y,q)$ and $g_C(xyz,q) \leq g_C(z,q)$ for all $x,y,z \in S$, $q \in Q$. Let $x \in S$. Since S is a regular ternary semiring, there exists $a \in S$ such that $x = xax = xaxax$.

$f_{A*B*C}(x,q) = \bigvee_{x=uvw} \{f_A(u,q) \wedge f_B(v,q) \wedge f_C(w,q)\}$

$\geq \{f_A(x,q) \wedge f_B(axa) \wedge f_C(x,q)\}$

$\geq \{f_A(x,q) \wedge f_B(x,q) \wedge f_C(x,q)\}$

$\Rightarrow f_{A*B*C}(x,q) \geq f_{A\cap B\cap C}(x,q)$

$g_{A*B*C}(x,q) = \bigwedge_{x=uvw} \{g_A(u,q) \vee g_B(v,q) \vee g_C(w,q)\}$

$\leq \{g_A(x,q) \vee g_B(axa) \vee g_C(x,q)\}$

$\leq \{g_A(x,q) \vee g_B(x,q) \vee g_C(x,q)\}$

$\Rightarrow g_{A*B*C}(x,q) \leq g_{A\cap B\cap C}(x,q)$.

Therefore $A \cap B \cap C \subseteq A * B * C$. By lemma 6.2, $A * B * C \subseteq A \cap B \cap C$. Hence $A * B * C = A \cap B \cap C$ for all intuitionistic Q-fuzzy right ideal A, intuitionistic Q-fuzzy lateral ideal B and intuitionistic Q-fuzzy left ideal C of S.

Conversely, let $x \in S$, $q \in Q$. Let R, M and L be the right, lateral and left ideal of S respectively. By corollary 3.13, $A_1 = (\chi_R, 1-\chi_R)$, $A_2 = (\chi_M, 1-\chi_M)$ and $A_3 = (\chi_L, 1-\chi_L)$ are intuitionistic Q-fuzzy right ideal, intuitionistic Q-fuzzy lateral ideal and intuitionistic Q-fuzzy left ideal of S respectively. Clearly $RML \subseteq R\cap M\cap L$. Let $x \in R\cap M\cap L$, then $1 = \chi_R(x,q)\wedge\chi_M(x,q)\wedge\chi_L(x,q)$. Since $A*B*C = A\cap B\cap C$ implies $1 = \chi_R(x,q) \wedge \chi_M(x,q) \wedge \chi_L(x,q) = \bigvee_{x=uvw} \{\chi_R(u,q) \wedge \chi_M(v,q) \wedge \chi_L(w,q)\}$. Then $x = uvw$, for some $u \in R$, for some $v \in M$, for some $w \in L$. Thus $RML = R \cap M \cap L$. By lemma 2.9, S is regular. □

7. Normal intuitionistic Q-fuzzy right ideals

Definition 7.1. An intuitionistic Q-fuzzy right (left, lateral) ideal $A = (f_A, g_A)$ of a ternary semiring S is said to be normal if $A(0,q) = (1,0)$ that means $f_A(0,q) = 1$, $g_A(0,q) = 0$ for all $q \in Q$. Denote by NIQFRI(S) (NIQFLI(S), NIQFMI(S)) the set of all normal intuitionistic Q-fuzzy right (left, lateral) ideals of S. Note that NIQFRI(S) (NIQFLI(S), NIQFMI(S)) is a poset under set inclusion.

Example 7.2. Consider the ternary semiring $S = Z^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q\}$. Let Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x,q) = \begin{cases} 1 \quad if \quad x = 0 \\ 0.4 \qquad otherwise \end{cases}$$

$$g_A(x,q) = \begin{cases} 0 \quad if \quad x = 0 \\ 0.6 \qquad otherwise \end{cases}$$

Then $A = (f_A, g_A)$ is a normal intuitionistic Q-fuzzy ideal of S.

Theorem 7.3. *Given an intuitionistic Q-fuzzy right (left, lateral) ideal $A = (f_A, g_A)$ of a ternary semiring S. Let $f_A^+(x,q) = f_A(x,q) + 1 - f_A(0,q)$ and $g_A^+(x,q) = g_A(x,q) - g_A(0,q)$, for all $x \in S$, $q \in Q$. Then $A^+ = (f_A^+, g_A^+)$ is a normal intuitionistic Q-fuzzy right (left, lateral) ideal containing $A = (f_A, g_A)$ of S.*

Proof. For any $x, y, z \in S$ and $q \in Q$,
1. $f_A^+(x+y,q) = f_A(x+y,q) + 1 - f_A(0,q) \geq min\{f_A(x,q), f_A(y,q)\} + 1 - f_A(0,q)$
$= min\{f_A(x,q) + 1 - f_A(0,q), f_A(y,q) + 1 - f_A(0,q)\} = min\{f_A^+(x), f_A^+(y)\}$
2. $f_A^+(xyz,q) = f_A(xyz,q) + 1 - f_A(0,q) \geq f_A(x,q) + 1 - f_A(0,q)$
$= f_A^+(x)$
3. $g_A^+(x+y,q) = g_A(x+y,q) - g_A(0,q) \leq max\{g_A(x,q), g_A(y,q)\} - g_A(0,q)$
$= max\{g_A(x,q) - g_A(0,q), g_A(y,q) - g_A(0,q)\} = max\{g_A^+(x), g_A^+(y)\}$
4. $g_A^+(xyz,q) = g_A(xyz,q) - g_A(0,q) \leq g_A(x,q) - g_A(0,q)$
$= g_A^+(x)$. Hence A^+ is an intuitionistic Q-fuzzy right ideal of S. Again we have $f_A^+(0,q) = f_A(0,q) + 1 - f_A(0,q) = 1$ and $g_A^+(0,q) = g_A(0,q) - g_A(0,q) = 0$. Hence A^+ is a normal intuitionistic Q-fuzzy right ideal of S and by definition $A \subseteq A^+$. □

Corollary 7.4. *Let A and A^+ be as in the Theorem 7.3. Then*
i) for any $x \in S$, $q \in Q$, $A^+(x,q) = (1,0)$ implies $A(x,q) = (1,0)$ and
ii) A is a normal intuitionistic Q-fuzzy right ideal of S if and only if $A^+ = A$.

Remark 7.5. If $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy right (left, lateral) ideal of S, then $(A^+)^+ = A^+$. In particular, if A is normal, then $(A^+)^+ = A^+ = A$.

Theorem 7.6. *Let $A = (f_A, g_A)$ be an intuitionistic Q-fuzzy right (left, lateral) ideal of a ternary semiring S and let $\Phi : [0,1] \to [0,1]$ be an increasing function. Then an IQFS $A_\Phi = ((f_A)_\Phi, (g_A)_\Phi)$ where $(f_A)_\Phi(x,q) = \Phi(f_A(x,q))$ and $(g_A)_\Phi(x,q) = \Phi(g_A(x,q))$ for all $x \in S$, $q \in Q$ is an intuitionistic Q-fuzzy right (left, lateral) ideal of S. Moreover, if $\Phi(f_A(0,q)) = 1$ and $\Phi(g_A(0,q)) = 0$, then A_Φ is normal.*

Proof. Let $x, y, z \in S$ and $q \in Q$,
1. $(f_A)_\Phi(x+y,q) = \Phi(f_A(x+y,q)) \geq \Phi(min\{f_A(x,q), f_A(y,q)\})$
$= min\{\Phi(f_A(x,q)), \Phi(f_A(y,q))\} = min\{(f_A)_\Phi(x,q), (f_A)_\Phi(y,q)\}$
2. $(f_A)_\Phi(xyz,q) = \Phi(f_A(xyz,q)) \geq \Phi(f_A(x,q)) = (f_A)_\Phi(x,q)$
3. $(g_A)_\Phi(x+y,q) = \Phi(g_A(x+y,q)) \leq \Phi(max\{g_A(x,q), g_A(y,q)\})$
$= max\{\Phi(g_A(x,q)), \Phi(g_A(y,q))\} = max\{(g_A)_\Phi(x,q), (g_A)_\Phi(y,q)\}$
4. $(g_A)_\Phi(xyz,q) = \Phi(g_A(xyz,q)) \leq \Phi(g_A(x,q)) = (g_A)_\Phi(x,q)$.
Hence A_Φ is an intuitionistic Q-fuzzy right ideal of S. If $\Phi(f_A(0,q)) = 1$, $\Phi(g_A(0,q)) = 0$ then $(f_A)_\Phi(0,q) = 1$ and $(g_A)_\Phi(0,q) = 0$ and hence $A_\Phi = ((f_A)_\Phi, (g_A)_\Phi)$ is a normal intuitionistic Q-fuzzy right ideal of S. □

Definition 7.7. An intuitionistic Q-fuzzy ideal $A = (f_A, g_A)$ of a ternary semiring S is said to be intuitionistic Q-fuzzy maximal if it satisfies:
i) A is non-constant.
ii) A^+ is a maximal element of NIQFI(S), where NIQFI(S) denote the set of all normal intuitionistic Q-fuzzy ideal of S.

Example 7.8. Consider the ternary semiring $S = Z^-$, the set of all non positive integers with usual addition and ternary multiplication and let $Q = \{q\}$. Let Q-fuzzy subset f_A and g_A of S be defined by

$$f_A(x,q) = \begin{cases} 1 & if \quad x \in \langle -2 \rangle \\ 0 & \quad otherwise \end{cases}$$

$$g_A(x,q) = \begin{cases} 0 & if \quad x \in \langle -2 \rangle \\ 1 & \quad otherwise \end{cases}$$

Then $A = (f_A, g_A)$ is an intuitionistic Q-fuzzy maximal ideal of S.

Theorem 7.9. *Let $A = (f_A, g_A) \in NIQFRI(S)$ be non-constant such that it is maximal in the poset of NIQFRI(S) under set inclusion. Then both f_A and g_A takes only the values $(1,0)$ and $(0,1)$ respectively.*

Proof. Since A is normal intuitionistic Q-fuzzy right ideal, so $A(0,q) = (1,0)$ for all $q \in Q$. Let x_0 $(\neq 0) \in S$, $q \in Q$ be arbitrary with $f_A(x_0,q) \neq 1$. We claim that $f_A(x_0,q) = 0$. If not then there exists an element $c \in S$ such that $0 < f_A(c,q) < 1$. Let $A_c = (\sigma_A, \eta_A)$ be an intuitionistic Q-fuzzy subset of S defined by $\sigma_A(x,q) = \frac{1}{2}[f_A(x,q) + f_A(c,q)]$, $\eta_A(x,q) = \frac{1}{2}[g_A(x,q) + g_A(c,q)]$. Clearly A_c is well-defined.
Now, $\sigma_A(0,q) = \frac{1}{2}[f_A(0,q) + f_A(c,q)] \geq \frac{1}{2}[f_A(x,q) + f_A(c,q)] = \sigma_A(x,q)$,
$\eta_A(0,q) = \frac{1}{2}[g_A(0,q) + g_A(c,q)] \leq \frac{1}{2}[g_A(x,q) + g_A(c,q)] = \eta_A(x,q)$ for any $x \in S$, $q \in Q$. Again, for any $x, y, z \in S$, $q \in Q$,
1. $\sigma_A(x+y,q) = \frac{1}{2}[f_A(x+y,q) + f_A(c,q)] \geq \frac{1}{2}[min\{f_A(x,q), f_A(y,q)\} + f_A(c,q)]$
$= min\{\frac{1}{2}[f_A(x,q) + f_A(c,q)], \frac{1}{2}[f_A(y,q) + f_A(c,q)]\} = min\{\sigma_A(x,q), \sigma_A(y,q)\}$
2. $\sigma_A(xyz,q) = \frac{1}{2}[f_A(xyz,q) + f_A(c,q)] \geq \frac{1}{2}[f_A(x,q) + f_A(c,q)] = \sigma_A(x,q)$
3. $\eta_A(x+y,q) = \frac{1}{2}[g_A(x+y,q) + g_A(c,q)] \leq \frac{1}{2}[max\{g_A(x,q), g_A(y,q)\} + g_A(c,q)]$
$= max\{\frac{1}{2}[g_A(x,q) + g_A(c,q)], \frac{1}{2}[g_A(y,q) + g_A(c,q)]\} = max\{\eta_A(x,q), \eta_A(y,q)\}$
4. $\eta_A(xyz,q) = \frac{1}{2}[g_A(xyz,q) + g_A(c,q)] \leq \frac{1}{2}[g_A(x,q) + g_A(c,q)] = \eta_A(x,q)$.
Hence A_c is an intuitionistic Q-fuzzy right ideal of S. Define $A_c^+ = (\sigma_A^+, \eta_A^+)$. Then by Theorem 7.3, A_c^+ is a normal intuitionistic Q-fuzzy right ideal of S, where $\sigma_A^+(x,q) = \sigma_A(x,q) + 1 - \sigma_A(0,q) = \frac{1}{2}[f_A(x,q) + f_A(c,q)] + 1 - \frac{1}{2}[f_A(0,q) + f_A(c,q)] = \frac{1}{2}[1 + f_A(x,q)]$
and $\eta_A^+(x,q) = \eta_A(x,q) - \eta_A(0,q) = \frac{1}{2}[g_A(x,q) + g_A(c,q)] - \frac{1}{2}[g_A(0,q) + g_A(c,q)] = \frac{1}{2}[g_A(x,q)]$. Clearly $A \subseteq A_c^+$. Since $\sigma_A^+(x,q) = \frac{1}{2}[1 + f_A(x,q)] > f_A(x,q)$ and $\eta_A^+(x,q) = \frac{1}{2}[g_A(x,q)] \leq g_A(x,q)$, A is a proper subset of A_c^+. Again since $\sigma_A^+(c,q) = \frac{1}{2}[1 + f_A(c,q)] < 1 = \sigma_A^+(0,q)$. Hence A_c^+ is non-constant and A is not a maximal element of NIQFRI(S). This is a contradiction. Therefore f_A takes only two values 1 and 0. Similarly we can prove that g_A also takes the values 0 and 1. This completes the proof. □

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