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An extension of the properties of inverse α-cuts to fuzzy multisets

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Abstract. This paper describes inverse α-Cuts and their properties in fuzzy multisets. In particular, it is shown that certain properties of inverse α-Cuts that hold in fuzzy sets do not hold in fuzzy multisets.

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1. Introduction

Zadeh [13] introduced the notion of α-cut (or α-level set) of a fuzzy set. Ever since, it has been used for various purposes (see e.g., [3, 4, 5, 6, 7]). In addition, Sun and Han [10] introduced the notion of an inverse α-cut of a fuzzy set and described its properties. Also, Singh *et al.* [9] studied analogues of properties of α-cuts and Decomposition theorems of fuzzy sets in fuzzy multisets.

In this paper, α-cuts and their properties in fuzzy sets and fuzzy multiset are briefly presented in section 2. In section 3, inverse α-cuts and their properties in fuzzy sets are described. Finally, in section 4, inverse α-cuts and their properties in fuzzy multisets are studied.

2. α-cuts and their properties in fuzzy sets and fuzzy multisets

Definition 2.1. (α-cuts in fuzzy sets [13])
Let X be a non-empty set and $F(X)$ the set of all fuzzy sets of X. Let $A \in F(X)$ and $\alpha \in [0, 1]$. Then the non-fuzzy set (or crisp set)

$$^{\alpha}A = \{x \in X | \mu_A((x) \geq \alpha)\}$$

is called the α-*cut* (or α-*level* set) of A.

If the weak inequality $\geq$ is replaced by the strict inequality $>$, then it is called the *strong α-cut,* denoted by ${}^{\alpha+}A$. That is,

$$ {}^{\alpha+}A = \{x \in X|\mu_A((x) > \alpha)\}. $$

Proposition 2.2. *For $\alpha \in [0, \ 1]$, the following hold:*

i. ${}^{\alpha^+}A \subseteq {}^{\alpha}A$
ii. $\alpha_1 \leq \alpha_2 \Rightarrow {}^{\alpha_2}A \subseteq {}^{\alpha_1}A$
iii. $A \subseteq B \Rightarrow {}^{\alpha}A \subseteq {}^{\alpha}B$
iv. $A = B \Rightarrow {}^{\alpha}A = {}^{\alpha}B$
v. ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$
vi. ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$
vii. ${}^{\alpha}A' = \left({}^{(1-\alpha)^+}A\right)'$, *where A' is the complement of A.*

Proof. See [5] for proofs. □

Definition 2.3. (Fuzzy multisets) Yagar [12] defined fuzzy multisets as follows: Assume X is a set of elements. Then a fuzzy bag (fuzzy multiset) A drawn from X can be characterized by a function $Count.Mem_A$ such that

$$ Count.Mem_A : X \rightarrow Q; $$

where Q is the set of all crisp bags (multisets) drawn from the unit interval.
Syropoulos [11] defined fuzzy multisets in a relatively more elaborate form as follows: A fuzzy multiset A in some universe set X is a multiset in $X \times [0, 1]$. In other words, a fuzzy multiset is a multiset of pairs, where the first part of each pair is an element of X and the second part is the degree to which the first part belongs to the fuzzy multiset A. That is,

$$ A : X \times I \rightarrow \mathbb{N}, $$

where $I = [0, 1]$ and $\mathbb{N}$ is the set of positive integers including 0.

Definition 2.4. (α-cuts in fuzzy multisets [8, 9])
The α-cut ($\alpha \in (0, 1]$) of a fuzzy multiset A, denoted by ${}^{\alpha}[A]$**,** is a crisp multiset defined as follows:

$$ \mu_A^j(x) < \alpha \ \Rightarrow \ cont\,{}^{\alpha}[A] = 0 $$

$$ \mu_A^j(x) \geq \alpha\,, \mu_A^{j+1}(x) < \alpha \ \Rightarrow \ cont\,{}^{\alpha}[A] = j, \ j = 1, 2, 3, \ldots, \ L(x). $$

Moreover, the strong α-cut ($\alpha \in [0, 1)$), denoted by ${}^{\alpha}\,]A[$**,** is a crisp multiset defined as follows:

$$ \mu_A^j(x) \leq \alpha \ \Rightarrow \ cont\,{}^{\alpha}[A] = 0 $$

$$ \mu_A^j(x) > \alpha\,, \mu_A^{j+1}(x) \leq \alpha \ \Rightarrow \ cont\,{}^{\alpha}[A] = j, \ j = 1, 2, 3, \ldots, \ L(x). $$

For $x \in X$, the membership sequence of a fuzzy set A is defined as a (monotonic) decreasing sequence of the elements of *count* $A(x)$, denoted by

$$ (\mu_A^1(x), \ \mu_A^2(x), \ \ldots, \ \mu_A^p(x)), $$

where $\mu_A^1(x) \geq \mu_A^2(x) \geq \ \ldots \ \geq \mu_A^p(x)$.

Moreover, for defining an operation between two fuzzy multisets A and B, the lengths of the membership sequences $\mu_A^1(x), \ \mu_A^2(x), \cdots, \mu_A^p(x)$ and $\mu_B^1(x), \ \mu_B^2(x)$,

$\cdots$, $\mu_B^{p'}(x)$ need to be set equal. Lengths $L(x;\ A)$ and $L(x;\ A,\ B)$ are, respectively, defined as

$L(x; A) = \max\{j : \mu_A^j(x) \neq 0\}$; and

$$L(x;\ A,\ B) = \max\{L(x;\ A),\ L(x;\ B)\}.$$

For brevity, $L\ (x)$ for $L\ (x;\ A)$ or $L\ (x;\ A,\ B)$ is also used if no confusion arises.

Proposition 2.5. *Let* $A,\ B \in$ FM(X)*, the class of all finite fuzzy multisets of* X*. The following results hold for* $\alpha,\ \beta\ \in (0,\ 1]$*:*

i. $]A[_\alpha \subseteq [A]_\alpha$
ii. $\alpha\ \leq\ \beta$ *implies* $[A]_\alpha\ \supseteq\ [A]_\beta$ *and* $]A[_\alpha\ \supseteq\ ]A[_\beta$
The total ordering of values of α *in* $[0,1]$ *is inversely preserved by set inclusion of the corresponding* α*-cuts as well as strong* α*-cuts. Equivalently, for any fuzzy set* A *and pair* $\alpha_1, \alpha_2 \in [0,1]$ *of distinct values such that* $\alpha_1 < \alpha_2$*, we have:*
$[A]_{\alpha_1} \cap [A]_{\alpha_2}\ =\ [A]_{\alpha_2},\quad [A]_{\alpha_1} \cup [A]_{\alpha_2}\ =\ [A]_{\alpha_1}$ *, and similarly for strong* α*-cuts.*
iii. $[A \cup B]_\alpha\ =\ [A]_\alpha \cup\ [B]_\alpha$ *and* $[A \cap B]_\alpha\ =\ [A]_\alpha \cap [B]_\alpha$
iv. $]A \cup B[_\alpha\ =\ ]A[_\alpha\ \cup\ ]B[_\alpha$ *and* $]A \cap B[_\alpha\ =\ ]A[_\alpha\ \cap\ ]B[_\alpha$
That is, all α*-cuts and strong* α*-cuts of any fuzzy multiset form two distinct families of nested crisp multisets.*
v. $[A \oplus B]_\alpha = [A]_\alpha \oplus\ [B]_\alpha$ *and* $]A \oplus B[_\alpha\ =\ ]A[_\alpha\ \oplus\ ]B[_\alpha$

Proof. See [9] for proofs. □

There exist a multitude of mathematical concepts such as *boundedness, convexity, connectedness, arc-wise connectedness, star-shaped, having holes,* etc., defined in terms of α - cuts (see e.g., [1, 2, 8], for details). Moreover, the collection of fuzzy multisets, akin to that of ordinary fuzzy sets on a universe, forms a distributive lattice, though not a complemented one (see [2], for details).

3. Inverse α-cuts and their properties in fuzzy sets

Definition 3.1. (Inverse α-cuts [10])
Let $A\ \in F(X)$ and $\alpha \in\ [0,1]$. Then the non-fuzzy set

$$^{\alpha}A^{-1} = \{x\ \in X|\ \mu_A(x) < \alpha\}$$

is called an *inverse* α*-cut* (or *inverse* α*-level set*) of A.

If the strict inequality is replaced by the weak inequality $\leq$, then it is called a *weak inverse* α*-cut* of A, denoted by $^{\alpha^-}A^{-1}$. That is,

$$^{\alpha^-}A^{-1} = \{x\ \in X|\ \mu_A(x) \leq \alpha\}.$$

Proposition 3.2. *Let* $A, B\ \in\ F(X)$ *and* $\alpha, \beta\ \in\ [0,1]$*. The following properties hold:*

i. $^{\alpha}A^{-1} \subseteq\ ^{\alpha^-}A^{-1}$
ii. $\alpha \leq \beta$ *implies* $^{\alpha}A^{-1} \subseteq\ ^{\beta}A^{-1}$ *and* $^{\alpha^-}A^{-1} \subseteq\ ^{\beta^-}A^{-1}$
iii. $^{\alpha}(A \cap B)^{-1} =\ ^{\alpha}A^{-1} \cup {^{\alpha}B^{-1}}$, $^{\alpha}(A \cup B)^{-1} =\ ^{\alpha}A^{-1} \cap {^{\alpha}B^{-1}}$, *and*
$^{\alpha}A^{-1} \cap^{\alpha} B^{-1} \subseteq\ ^{\alpha}(A \cap B)^{-1}$

iv. ${}^{\alpha^-}(A \cap B)^{-1} = {}^{\alpha^-}A^{-1} \cup {}^{\alpha^-}B^{-1}$, ${}^{\alpha^-}(A \cup B)^{-1} = {}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1}$, *and* ${}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1} \subseteq {}^{\alpha^-}(A \cap B)^{-1}$

v. ${}^{\alpha}((A^{-1})') = {}^{(1-\alpha)^-}(A^{-1})'$

vi. ${}^{1}A^{-1} = X$

vii. $A \subseteq B$ iff ${}^{\alpha}B^{-1} \subseteq {}^{\alpha}A^{-1}$; $A \subseteq B$ iff ${}^{\alpha^-}B^{-1} \subseteq {}^{\alpha^-}A^{-1}$

viii. $A = B$ iff ${}^{\alpha}B^{-1} = {}^{\alpha}A^{-1}$; $A = B$ iff ${}^{\alpha^-}B^{-1} = {}^{\alpha^-}A^{-1}$

Proof. i. $x \in {}^{\alpha}A^{-1} \Longrightarrow \mu_A(x) < \alpha \Longrightarrow \mu_A(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}A^{-1}$, hence ${}^{\alpha}A^{-1} \subseteq {}^{\alpha^-}A^{-1}$.

ii. Part I

$x \in {}^{\alpha}A^{-1} \Longrightarrow \mu_A(x) < \alpha \Longrightarrow \mu_A(x) < \beta$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}A^{-1}$, hence ${}^{\alpha}A^{-1} \subseteq {}^{\beta}A^{-1}$.

Part II

$x \in {}^{\alpha^-}A^{-1} \Longrightarrow \mu_A(x) \leq \alpha \Longrightarrow \mu_A(x) \leq \beta$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta^-}A^{-1}$, hence ${}^{\alpha^-}A^{-1} \subseteq {}^{\beta^-}A^{-1}$.

iii. Part I

$x \in {}^{\alpha}(A \cap B)^{-1} \Longrightarrow \mu_{(A\cap B)}(x) < \alpha \Longrightarrow \min\{\mu_A(x), \mu_B(x)\} < \alpha \Longrightarrow \mu_A(x) < \alpha$ or $\mu_B(x) < \alpha \Longrightarrow x \in ({}^{\alpha}A^{-1} \cup {}^{\alpha}B^{-1})$, hence ${}^{\alpha}(A \cap B)^{-1} \subseteq {}^{\alpha}A^{-1} \cup {}^{\alpha}B^{-1}$.

Conversely, $x \in {}^{\alpha}A^{-1} \cup {}^{\alpha}B^{-1} \Longrightarrow x \in {}^{\alpha}A^{-1}$ or $x \in {}^{\alpha}B^{-1} \Longrightarrow \mu_A(x) < \alpha$ or $\mu_B(x) < \alpha \Longrightarrow \min\{\mu_A(x), \mu_B(x)\} < \alpha \Longrightarrow \mu_{(A\cap B)}(x) < \alpha \Longrightarrow x \in {}^{\alpha}(A \cap B)^{-1}$, hence ${}^{\alpha}A^{-1} \cup {}^{\alpha}B^{-1} \subseteq {}^{\alpha}(A \cap B)^{-1}$.

Thus, ${}^{\alpha}(A \cap B)^{-1} = {}^{\alpha}A^{-1} \cup {}^{\alpha}B^{-1}$.

Part II

$x \in {}^{\alpha}(A\cup B)^{-1} \Longrightarrow \max\{\mu_A(x), \mu_B(x)\} < \alpha \Longrightarrow \mu_A(x) < \alpha$ and $\mu_B(x) < \alpha \Longrightarrow x \in {}^{\alpha}A^{-1} \cap {}^{\alpha}B^{-1}$, hence ${}^{\alpha}(A \cup B)^{-1} \subseteq {}^{\alpha}A^{-1} \cap {}^{\alpha}B^{-1}$.

Conversely, $x \in {}^{\alpha}A^{-1} \cap {}^{\alpha}B^{-1} \Longrightarrow x \in {}^{\alpha}A^{-1}$ and $x \in {}^{\alpha}B^{-1} \Longrightarrow \mu_A(x) < \alpha$ and $\mu_B(x) < \alpha \Longrightarrow \max\{\mu_A(x), \mu_B(x)\} < \alpha, \Longrightarrow \mu_{(A\cup B)}(x) < \alpha \Longrightarrow x \in {}^{\alpha}(A \cup B)^{-1}$, hence ${}^{\alpha}A^{-1} \cap {}^{\alpha}B^{-1} \subseteq {}^{\alpha}(A \cup B)^{-1}$.

Thus, ${}^{\alpha}(A \cup B)^{-1} = {}^{\alpha}A^{-1} \cap {}^{\alpha}B^{-1}$.

Part III

The proof follows from the converse of part II.

iv. Part I

$x \in {}^{\alpha^-}(A \cap B)^{-1} \Longrightarrow \mu_{(A\cap B)}(x) \leq \alpha \Longrightarrow \min\{\mu_A(x), \mu_B(x)\} \leq \alpha \Longrightarrow \mu_A(x) \leq \alpha$ or $\mu_B(x) \leq \alpha \Longrightarrow x \in ({}^{\alpha^-}A^{-1} \cup {}^{\alpha^-}B^{-1})$, hence ${}^{\alpha^-}(A \cap B)^{-1} \subseteq {}^{\alpha^-}A^{-1} \cup {}^{\alpha^-}B^{-1}$.

Conversely, $x \in {}^{\alpha^-}A^{-1} \cup {}^{\alpha^-}B^{-1} \Longrightarrow x \in {}^{\alpha^-}A^{-1}$ or $x \in {}^{\alpha^-}B^{-1} \Longrightarrow \mu_A(x) \leq \alpha$ or $\mu_B(x) \leq \alpha \Longrightarrow \min\{\mu_A(x), \mu_B(x)\} \leq \alpha \Longrightarrow \mu_{(A\cap B)}(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}(A \cap B)^{-1}$, hence ${}^{\alpha^-}A^{-1} \cup {}^{\alpha^-}B^{-1} \subseteq {}^{\alpha^-}(A \cap B)^{-1}$.

Therefore, ${}^{\alpha^-}(A \cap B)^{-1} = {}^{\alpha^-}A^{-1} \cup {}^{\alpha}B^{-1}$.

Part II

$x \in {}^{\alpha^-}(A\cup B)^{-1} \Longrightarrow \max\{\mu_A(x), \mu_B(x)\} \leq \alpha \Longrightarrow \mu_A(x) \leq \alpha$ and $\mu_B(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1}$, hence ${}^{\alpha^-}(A \cup B)^{-1} \subseteq {}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1}$.

Conversely, $x \in {}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1} \Longrightarrow x \in {}^{\alpha^-}A^{-1}$ and $x \in {}^{\alpha^-}B^{-1} \Longrightarrow \mu_A(x) \leq \alpha$ and $\mu_B(x) \leq \alpha \Longrightarrow \max\{\mu_A(x), \mu_B(x)\} \leq \alpha \Longrightarrow \mu_{(A\cup B)}(x) \leq \alpha$

$\Longrightarrow x \in {}^{\alpha^-}(A \cup B)^{-1}$, hence ${}^{\alpha^-}A^{-1} \cap {}^{\alpha^-}B^{-1} \subseteq {}^{\alpha^-}(A \cup B)^{-1}$. Therefore, ${}^{\alpha^-}(A \cup B)^{-1} = {}^{\alpha^-}A^{-1} \cap {}^{\alpha}B^{-1}$.
Part III
The proof follows from the converse of part II.

v. $x \in {}^{\alpha}((A^{-1})') \Longrightarrow 1 - \mu_A(x) = \mu_{A'}(x) < \alpha \Longrightarrow \mu_A(x) > 1 - \alpha \Longrightarrow x \notin {}^{(1-\alpha)^-}A^{-1} \Longrightarrow x \in {}^{(1-\alpha)^-}(A^{-1})'$ hence ${}^{\alpha}((A^{-1})') \subseteq {}^{(1-\alpha)^-}(A^{-1})'$.
Conversely, $x \in {}^{(1-\alpha)^-}(A^{-1})', \Longrightarrow x \notin {}^{(1-\alpha)^-}A^{-1} \Longrightarrow \mu_A(x) > 1 - \alpha$ and $1 - \mu_A(x) < \alpha \Longrightarrow \mu_{A'}(x) < \alpha \Longrightarrow x \in {}^{\alpha}((A^{-1})') \Longrightarrow {}^{(1-\alpha)^-}(A^{-1})' \subseteq^{\alpha} ((A^{-1})')$, hence ${}^{\alpha}((A^{-1})') = {}^{(1-\alpha)^-}(A^{-1})'$.

vi. The proof follows from the definition.

vii. For any $x \in X$, $A \subseteq B$ if and only if $\mu_A \leq \mu_B$, if $\mu_B < \alpha \Longrightarrow \mu_A < \alpha$ if and only if $x \in {}^{\alpha}B^{-1}$ and $x \in {}^{\alpha}A^{-1}$ if and only if ${}^{\alpha}B^{-1} \subseteq {}^{\alpha}A^{-1}$.
The proof of the second part follows similarly.

viii. The proof follows from the definition.

□

Remark 3.3. Property (ii) stresses that the set sequences $\{{}^{\alpha^-}A^{-1} | \, \alpha \in [0,1]\}$ and $\{{}^{\alpha}A^{-1} | \, \alpha \in [0,1]\}$ of inverse α-cuts and weak inverse α-cuts, respectively, are monotonic increasing with respect to α and thus form a nested family of sets. Further, contrary to the case with α-cuts and strong α-cuts, properties (iii) and (iv) show that the standard fuzzy set intersection and fuzzy set union are neither inverse α-cutworthy nor weak inverse α-cutworthy.

4. Inverse α-cuts in fuzzy multisets

Definition 4.1. The inverse α-cut ($\alpha \in (0,1]$) of a fuzzy multiset A, denoted by ${}^{\alpha}[A]^{-1}$, is a crisp multiset defined as follows:

$$\mu_A^j(x) \geq \alpha \Rightarrow cont\, {}^{\alpha}[A]^{-1} = 0$$

$$\mu_A^j(x) < \alpha \, , \mu_A^{j-1}(x) \geq \alpha \Rightarrow cont\, {}^{\alpha}[A]^{-1} = L(x) - j + 1, \; j = 1, 2, 3, \ldots, \; L(x).$$

Moreover the weak inverse α-cut ($\alpha \in [0,1)$), denoted by ${}^{\alpha}]A[^{-1}$, is a crisp multiset defined as follows:

$$\mu_A^j(x) > \alpha \Rightarrow cont\, {}^{\alpha}]A[^{-1} = 0$$

$$\mu_A^j(x) \leq \alpha \, , \mu_A^{j-1}(x) > \alpha \Rightarrow cont\, {}^{\alpha}]A[= L(x) - j + 1, \; j = 1, 2, 3, \ldots, \; L(x).$$

Remark 4.2. It may be observed that ${}^{\alpha}[A] \cup {}^{\alpha}[A]^{-1} \neq X$, contrary to the case in fuzzy sets where ${}^{\alpha}A \cup {}^{\alpha}A^{-1} = X$ [10].

Example 4.3. Let $X = \{x_1, x_2, x_3\}$,

$$A = \{\{0.9, 0.3, 0.3\}/x_1, \; \{0.7, 0.4\}/x_2, \; \{0.7, 0.6, 0.6\}/x_3\},$$

and $\alpha = 0.4$. Thus we have

$${}^{0.4}[A] = \{1/x_1, \; 2/x_2, \; 3/x_3\}$$

and

$${}^{0.4}[A]^{-1} = \{2/x_1\} \, .$$

Now, ${}^{0.4}[A] \cup {}^{0.4}[A]^{-1} = \{2/x_1, \; 2/x_2, \; 3/x_3\}$ whose root set is X.

Proposition 4.4. *Let A, B, $C \in FM(X)$. The following properties hold for $\alpha, \beta \in (0,1]$:*

i. ${}^{\alpha}[A]^{-1} \cup {}^{\alpha}[B]^{-1} = {}^{\alpha}[B]^{-1} \cup {}^{\alpha}[A]^{-1}$; ${}^{\alpha}[A]^{-1} \cap [B]^{-1} = {}^{\alpha}[B]^{-1} \cap {}^{\alpha}[A]^{-1}$

ii. ${}^{\alpha}]A[^{-1} \cup {}^{\alpha}]B[^{-1} = {}^{\alpha}]B[^{-1} \cup {}^{\alpha}]A[^{-1}$; ${}^{\alpha}]A[^{-1} \cap {}^{\alpha}]B[^{-1} = {}^{\alpha}]B[^{-1} \cap {}^{\alpha}]A[^{-1}$

iii. ${}^{\alpha}[A]^{-1} \cup \left({}^{\alpha}[B]^{-1} \cup {}^{\alpha}[C]^{-1}\right) = \left({}^{\alpha}[A]^{-1} \cup {}^{\alpha}[B]^{-1}\right) \cup {}^{\alpha}[C]^{-1}$;

$$ {}^{\alpha}[A]^{-1} \cap \left({}^{\alpha}[B]^{-1} \cap {}^{\alpha}[C]^{-1}\right) = \left({}^{\alpha}[A]^{-1} \cap {}^{\alpha}[B]^{-1}\right) \cap {}^{\alpha}[C]^{-1} $$

iv. ${}^{\alpha}]A[^{-1} \cup \left({}^{\alpha}]B[^{-1} \cup {}^{\alpha}]C[^{-1}\right) = \left({}^{\alpha}]A[^{-1} \cup {}^{\alpha}]B[^{-1}\right) \cup {}^{\alpha}]C[^{-1}$;

$$ {}^{\alpha}]A[^{-1} \cap \left({}^{\alpha}]B[^{-1} \cap {}^{\alpha}]C[^{-1}\right) = \left({}^{\alpha}]A[^{-1} \cap {}^{\alpha}]B[^{-1}\right) \cap {}^{\alpha}]C[^{-1} $$

v. ${}^{\alpha}[A]^{-1} \subseteq {}^{\alpha}]A[^{-1}$

vi. $\alpha \leq \beta$ *implies* ${}^{\alpha}[A]^{-1} \subseteq {}^{\beta}[A]^{-1`}$ *and* ${}^{\alpha}]A[^{-1} \subseteq {}^{\beta}]A[^{-1}$

Proof. The proofs of i – iv are immediate.

v. $x \in {}^{\alpha}[A]^{-1} \Longrightarrow \mu_A^j(x) < \alpha$, $j = 1, 2, \ldots, L(x) \Longrightarrow \mu_A^j(x) \leq \alpha$, $j = 1, 2, 3, \ldots, L(x) \Longrightarrow x \in {}^{\alpha}]A[^{-1}$, hence ${}^{\alpha}[A]^{-1} \subseteq {}^{\alpha}]A[^{-1}$.

vi. part I
$x \in {}^{\alpha}[A]^{-1} \Longrightarrow \mu_A^j(x) < \alpha, J = 1, 2, 3, \ldots, L(x) \Longrightarrow \mu_A^j(x) < \beta, j = 1, 2, 3, \ldots, L(x)$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}[A]^{-1}$, hence ${}^{\alpha}[A]^{-1} \subseteq {}^{\beta}[A]^{-1`}$.
Part II
$x \in {}^{\alpha}]A[^{-1} \Longrightarrow \mu_A^j(x) \leq \alpha, j = 1, 2, 3, \ldots, L(x) \Longrightarrow \mu_A^j(x) \leq \beta, j = 1, 2, 3, \ldots, L(x)$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}]A[^{-1}$, hence ${}^{\alpha}]A[^{-1} \subseteq {}^{\beta}]A[^{-1}$.

□

5. Conclusion

Inverse α-Cuts and their properties in fuzzy sets and fuzzy multisets are described. It is shown (section 4) that the ground set over which fuzzy multisets are built cannot be decomposed into its α-cuts and inverse α-cuts, unlike in fuzzy sets.

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