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Common fixed point theorem of fuzzy c-distance on fuzzy cone metric spaces

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Abstract. In this paper, a common fixed point theorem is proved by using the idea of fuzzy c-distance in fuzzy cone metric space. This theorem extends the contractive condition of common fixed point theorem (which is proved in an earlier paper) from constant real numbers to some control functions and the theorem is justified by a suitable example.

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1. Introduction

In last few years different types of generalized metric spaces have been developed by different authors in different approach. Some generalized metric spaces are D-metric space [7], Cone metric space [13] etc. The idea of cone metric space was introduced by H.Long-Guang et al.[13]. The definition of cone normed linear space is introduced by T.K.Samanta et al.[15] and M.Eshaghi Gordji et al. [8]. In earlier papers [2, 3], the author introduced the idea of fuzzy cone metric space as well as fuzzy cone normed linear space and studied some basic results. The study of common fixed points of mappings satisfying certain contractive conditions is now a vigorous research activity. Different authors developed more results regarding common fixed point theorem by using different types of contractive conditions for noncommuting mappings in metric spaces as well as in cone metric spaces (for references please see [1, 5, 6, 10, 11]). Recently Shenghua Wang et al. [16] have been developed a distance called c-distance on a cone metric space and prove a new common fixed point theorem by using this concept.

In this paper, following the idea of c-distance on a metric space [16], an idea of fuzzy c-distance in fuzzy cone metric space is introduced in [4] and by using this concept, one common fixed point theorem is established in such space.There is an advantage to use fuzzy c-distance to establish common fixed point theorem, since it is not required that contraction mapping be weakly compatible.

The organization of the paper is as follows:
Section 1, comprises some preliminary results which are used in this paper.
In Section 2, some properties of fuzzy c-distance are studied. One common fixed point theorem is established in Section 3.

2. Preliminaries

A fuzzy number is a mapping $x : R \to [0, 1]$ over the set R of all reals.
A fuzzy number x is convex if $x(t) \geq min\ (x(s), x(r))$ where $s \leq t \leq r$.
If there exists $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy number x is called non-negative if $x(t) = 0, \forall t < 0$.
Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0, 1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0, 1]$.

Fuzzy real number $\bar{0}$ is defined as $\bar{0}(t) = 1$ if $t = 0$ and $\bar{0}(t) = 0$ otherwise.

According to Mizumoto and Tanaka [14] , the arithmetic operations $\oplus$, $\ominus$ on $E \times E$ are defined by

$$(x \oplus y)(t) = Sup_{s\in R} min\ \{x(s), y(t-s)\}, \quad t \in R$$
$$(x \ominus y)(t) = Sup_{s\in R} min\ \{x(s), y(s-t)\}, \quad t \in R$$

Proposition 2.1 ([14]). *Let* $\eta, \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha], \alpha \in (0, 1]$. *Then*

$$[\eta \bigoplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha, b^1_\alpha + b^2_\alpha]$$
$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha, b^1_\alpha - a^2_\alpha]$$
$$[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha, b^1_\alpha b^2_\alpha]$$

Definition 2.2 ([12]). A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\to\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]\ \forall \alpha \in (0, 1]$.

Note 2.3 ([12]). *If* $\eta, \delta \in G(R^*(I))$ *then* $\eta \oplus \delta \in G(R^*(I))$.

Note 2.4 ([12]). *For any scalar t, the fuzzy real number* $t\eta$ *is defined as* $t\eta(s) = 0$ *if t=0 otherwise* $t\eta(s) = \eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below.

Definition 2.5 ([9]). Let X be a vector space over R.
Let $|| \ || : X \to R^*(I)$ and let the mappings

$L, U\ : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfy
$L(0\ ,\ 0) = 0$ and $U(1\ ,\ 1) = 1$.
Write
$[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ for $x \in X,\ 0 < \alpha \leq 1$ and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0 \in (0\ ,\ 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||_\alpha^2 < \infty$
(B) $\inf ||x||_\alpha^1 > 0$.

The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$;
(ii) $||rx|| = |r|||x||,\ x \in X,\ r \in R$;
(iii) for all $x, y \in X$,
(a) whenever $s \leq ||x||_1^1,\ t \leq ||y||_1^1$ and $s + t \leq ||x + y||_1^1$,
$||x + y||(s + t) \geq L(||x||(s)\ ,\ ||y||(t))$,
(b) whenever $s \geq ||x||_1^1,\ t \geq ||y||_1^1$ and $s + t \geq ||x + y||_1^1$,
$||x + y||(s + t) \leq U(||x||(s)\ ,\ ||y||(t))$

Remark 2.6 ([9]). Felbin proved that,
if $L = \bigwedge(\text{Min})$ and $U = \bigvee(\text{Max})$ then the triangle inequality (iii) in the Definition 1.3 is equivalent to
$||x + y|| \preceq ||x|| \bigoplus ||y||$.

Further $||\ ||_\alpha^i;\ i = 1, 2$ are crisp norms on X for each $\alpha \in (0\ ,\ 1]$. In that case we simply denote $(X\ ,\ ||\ ||)$.

Definition 2.7 ([2]). Let $(E, ||\ ||)$ be a fuzzy real Banach space (Felben sense) where $||\ || : E \to R^*(I)$.
Denote the range of $||\ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.8 ([2]). A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim\limits_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.9 ([2]). A subset P of $E^*(I)$ is called a fuzzy cone if

(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;
(ii) $a, b \in R,\ a, b \geq 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$;
(iii) $\eta \in P$ and $-\eta \in P\ \Rightarrow \eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta \ll \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $\eta, \delta \in E^*(I)$,
with $\bar{0} \leq \eta \leq \delta$ implies $\eta \preceq K\delta$. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{\eta_n\}$ is a sequence such that $\eta_1 \leq \eta_2 \leq .... \leq \eta_n \leq .... \leq \eta$ for some $\eta \in E^*(I)$, then there is $\delta \in E^*(I)$ such that $\eta_n \to \delta$ as $n \to \infty$.
Equivalently, the fuzzy cone P is regular if every decreasing sequence which is

bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that E is a fuzzy real Banach (Felbin sense) space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.10 ([2])**.** Let X be a nonempty set. Suppose the mapping $d: \ X \times X \rightarrow E^*(I)$ satisfies

(Fd1) $\bar{0} \leq d(x,y)\ \forall x,y \in X$ and $d(x,y) = \bar{0}$ iff $x = y$;
(Fd2) $d(x,y) = d(y,x) \ \ \forall x,y \in X$;
(Fd3) $d(x,y) \leq d(x,z) \oplus d(z,y) \ \ \forall x,y,z \in X$.

Then d is called a fuzzy cone metric and (X,d) is called a fuzzy cone metric space.

Definition 2.11 ([2])**.** Let (X,d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} \ll ||c||$ there is a positive integer N such that for all $n > N, \ d(x_n,x) \ll ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim\limits_{n\rightarrow\infty} x_n = x$.

Lemma 2.12 ([2])**.** *Let (X,d) be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent then its limit is unique.*

Definition 2.13 ([2])**.** Let (X,d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall m,n > N, \ d(x_n,x_m) \ll ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.14 ([2])**.** Let (X,d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Definition 2.15 ([1])**.** Let f and g be self mappings defined on a set X. If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Proposition 2.16 ([1])**.** *Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence $w = f(x) = g(x)$, then w is the unique common fixed point of f and g.*

Definition 2.17 ([4])**.** Let $(X \ , \ d)$ be a fuzzy cone metric space. Then the mapping $Q : \ X \times X \rightarrow E^*(I)$ is called a c-fuzzy distance on X if the following conditions hold:

(Q1) $\bar{0} \leq Q(x,y) \ \ \forall x,y \in X$;
(Q2) $Q(x,z) \leq Q(x,y) \oplus Q(y,z) \ \ \forall x,y,z \in X$;
(Q3) $\forall x \in X$, if $Q(x,y_n) \leq \eta$ for some $\eta = \eta(x) \in P, \ n \geq 1$,
then $Q(x,y) \leq \eta$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(Q4) $\forall c \in E$ with $\bar{0} \ll ||c||, \ \exists e \in E$ with $\bar{0} \ll ||e||$ such that $Q(z,x) \ll ||e||$ and $Q(z,y) \ll ||e||$ imply $d(x,y) \ll ||c||$.

3. Some results of fuzzy c-distance on fuzzy cone metric space

Observation 3.1. If Q be a fuzzy c-distance on a fuzzy cone metric space (X,d) then

(A) $Q(x, y) = Q(y, x)$ does not necessarily hold $\forall x, y \in X$.
(B) $Q(x, y) = \bar{0}$ is not necessarily equivalent to $x = y$ $\forall x, y \in X$.

Proof. To justify the above results we consider the following Example. □

Example 3.2. Let E=R (set of real numbers).
Define $||\ || : E \to R^*(I)$ by

$$||x||(t) = \begin{cases} \frac{|x|}{t} & \text{if } t \geq |x|, x \neq \theta \\ 1 & \text{if } t = |x| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $(E, ||\ ||)$ is a complete fuzzy normed linear space (Felbin's sense) where $[||x||]_\alpha = [|x|, \frac{|x|}{\alpha}]$ $\forall \alpha \in (0, 1]$ (please see Example 3.4[2]).
Let $X = [0, \infty)$ and $P = \{x \in E : ||x|| \succeq \bar{0}\}$.
Define a mapping $d : X \times X \to E^*(I)$ by $d(x, y) = ||x - y||$ $\forall x, y \in X$.
If we chose the ordering $\leq$ of E w.r.t. P as $\preceq$ then (X, d) is a fuzzy cone metric space with normal cone P.
Now define a mapping $Q : X \times X \to E^*(I)$ by $Q(x, y) = ||y||$ $\forall x, y \in X$. Then Q is a fuzzy c-distance on (X, d).
In fact, (Q1)-(Q2) are obvious. Let $\{y_n\}$ be a sequence in X converging to a point $y \in X$.
We have, $x \in X,\ Q(x, y_n) \preceq \eta(x),\ \eta \in P$ implies $||y_n|| \preceq \eta(x)$.
Now $||y|| \preceq ||y - y_n|| \oplus ||y_n||$
$\Rightarrow ||y||^i_\alpha \leq ||y - y_n||^i_\alpha + ||y_n||^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$
$\Rightarrow ||y||^i_\alpha \leq \lim_{n\to\infty} ||y_n||^i_\alpha \leq \eta^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$
$\Rightarrow ||y|| \preceq \eta$
$\Rightarrow Q(x, y) \preceq \eta$ $\forall x \in X$.
So (Q3) holds.
Let $||e|| \succ \bar{0}$ be given where $e \in E$. Set $||c|| = \frac{||e||}{2}$.
If $Q(z, x) = ||x|| \prec ||c||$ and $Q(z, y) = ||y|| \prec ||c||$ then
$d(x, y) = ||x - y|| \preceq ||x|| \oplus ||y|| \prec 2||c|| = ||e||$.
Thus (Q4) holds and hence Q is a fuzzy c-distance.
Choose $x_0, y_0 \in X$ where $x_0 \neq y_0$. So $|x_0| \neq |y_0|$ and hence $||x_0|| \neq ||y_0||$.
Now, $Q(x, y_0) = ||y_0||$ $\forall x \in X$. So in particular $Q(x_0, y_0) = ||y_0||$.
Similarly $Q(y_0, x_0) = ||x_0||$. So $Q(x_0, y_0) \neq Q(y_0, x_0)$.
Since $x_0, y_0 \in X$ are arbitrary, thus $Q(x, y) = Q(y, x)$ does necessarily hold $\forall x, y \in X$.
So (A) is proved.
Again $Q(x, y) = \bar{0} = ||\theta||$ $\forall x, y \in X$
$\Rightarrow y = \theta$ $\forall x \in X$
$\Rightarrow x \neq y$ $\forall x \in X (x \neq \theta)$ (3.2.1).
Next suppose $x_0 = y_0 \in X$.
If $x_0 = y_0 = \theta$ then $Q(x_0, y_0) = ||y_0|| = \bar{0}$.
If $x_0 = y_0 \neq \theta$ then $Q(x_0, y_0) = ||y_0|| \neq \bar{0}$.
Thus $x = y$ $\forall x, y \in X$ does not imply $Q(x, y) = \bar{0}$ (3.2.2).
From (3.2.1) and (3.2.2) (B) is proved.

Proposition 3.3. *Let $(X, d, \leq')$ be a fuzzy cone metric space and Q be a fuzzy c-distance on X and $\{x_n\}$ be a sequence in X. Suppose that $\{||u_n||\}$ is a sequence in P ($u_n \in E$) converging to $\bar{0}$. If $Q(x_n, x_m) \leq ||u_n||$ $\forall m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. Let $c \in E$ with $\bar{0} \ll ||c||$.
Then $\exists$ a positive δ such that $||c|| \ominus ||x|| \in intP$ for any $x \in E$ with $||x|| \prec \bar{\delta}$.
For, $\bar{0} \ll ||c|| \Rightarrow ||c|| \ominus \bar{0} = ||c|| \in intP$.
Now, $||x|| = ||c|| \ominus (||c|| \ominus ||x||) \prec \bar{\delta}$
$\Rightarrow ||c|| \ominus ||x|| \in intP$.
Since $\{||u_n||\}$ to $\bar{0}$, there exists a positive integer N such that $||u_n|| \prec \bar{\delta}$ $\forall n \geq N$
$\Rightarrow ||u_n|| = ||c|| \ominus (||c|| \ominus ||u_n||) \prec \bar{\delta}$ $\forall n \geq N$
$\Rightarrow ||c|| \ominus ||u_n|| \in intP$ $\forall n \geq N$
$\Rightarrow ||u_n|| \ll ||c||$ $\forall n \geq N$.
By the hypothesis, $Q(x_n, x_m) \leq ||u_n|| \ll ||c||$ $\forall m > n$ with $n \geq N$.
This implies that $Q(x_n, x_{n+1}) \leq ||u_n|| \ll ||c||$ and $Q(x_n, x_{m+1}) \leq ||u_n|| \ll ||c||$ $\forall m > n$ with $n \geq N$.
From (Q4), for $e \in E$ with $||e|| = ||c||$, it follows that
$d(x_{n+1}, x_{m+1}) \ll ||c||$ $\forall m > n$ with $n \geq N$.
This implies that $\{x_n\}$ is a Cauchy sequence in X. □

4. Common fixed point theorem in fuzzy cone metric spaces

In this Section a common fixed point theorem is proved by using the idea of fuzzy c-distance in fuzzy cone metric space.

Theorem 4.1. *Let $(X, d, \leq')$ be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Let $Q: X \times X \to E^*(I)$ be a fuzzy c-distance on X and S, $T: X \to X$ be two mappings such that $T(X) \subset S(X)$ and $S(X)$ be a complete subspace of X. Suppose that there exist mappings $\beta, \gamma, \delta, \mu : X \to [0, 1]$ such that the following conditions hold:*
1. $\beta(Tx) \leq \beta(Sx)$, $\gamma(Tx) \leq \gamma(Sx)$, $\delta(Tx) \leq \delta(Sx)$ and $\mu(Tx) \leq \mu(Sx)$ $\forall x \in X$;
2. $(\beta + \gamma + \delta + \mu)(x) < 1$ $\forall x \in X$;
3. $Q(Tx, Ty) \leq' \beta(Sx)Q(Sx, Sy) \oplus \gamma(Sx)Q(Sx, Ty) \oplus \delta(Sx)Q(Sx, Tx)$
$\oplus\ \mu(Sx)Q(Sy, Ty)$ $\forall x, y \in X$;
4. For some $\alpha_0 \in (0, 1]$,
$inf\{Q^1_{\alpha_0}(Tx, y) + Q^1_{\alpha_0}(Sx, y) + Q^1_{\alpha_0}(Sx, Tx) : x \in X\} > 0$, $\forall y \in X$ with $y \neq Ty$ or $y \neq Sy$. Then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $T(X) \subset S(X)$, $\exists x_1 \in X$ such that $Tx_0 = Sx_1$.
By induction, we construct a sequence $\{x_n\}$ in X such that $Sx_0 = Tx_{n-1}$, for $n \geq 1$
From (1) and (3) we have,
$Q(Sx_n, Sx_{n+1}) = Q(Tx_{n-1}, Tx_n)$
$\leq' \beta(Sx_{n-1})Q(Sx_{n-1}, Sx_n) \oplus \gamma(Sx_{n-1})Q(Sx_{n-1}, Tx_n) \oplus \delta(Sx_{n-1})Q(Sx_{n-1}, Tx_{n-1})$
$\oplus \mu(Sx_{n-1})Q(Sx_n, Tx_n)$
$= \beta(Tx_{n-2})Q(Sx_{n-1}, Sx_n) \oplus \gamma(Tx_{n-2})Q(Sx_{n-1}, Sx_{n+1}) \oplus \delta(Tx_{n-2})Q(Sx_{n-1}, Sx_n)$
$\oplus \mu(Tx_{n-2})Q(Sx_n, Sx_{n+1})$

$\leq' \beta(Sx_{n-2})Q(Sx_{n-1},Sx_n) \oplus \gamma(Sx_{n-2})Q(Sx_{n-1},Sx_{n+1}) \oplus \delta(Sx_{n-2})Q(Sx_{n-1},Sx_n)$
$\oplus \mu(Sx_{n-2})Q(Sx_n,Sx_{n+1})$
....
i.e. $Q(Sx_n,Sx_{n+1}) \leq' \beta(Sx_0)Q(Sx_{n-1},Sx_n) \oplus \gamma(Sx_0)Q(Sx_{n-1},Sx_{n+1})$
$\oplus \delta(Sx_0)Q(Sx_{n-1},Sx_n) \oplus \mu(Sx_0)Q(Sx_n,Sx_{n+1})$
$\leq' \beta(Sx_0)Q(Sx_{n-1},Sx_n) \oplus \gamma(Sx_0)[Q(Sx_{n-1},Sx_n) \oplus Q(Sx_n,Sx_{n+1})]$
$\oplus \delta(Sx_0)Q(Sx_{n-1},Sx_n) \oplus \mu(Sx_0)Q(Sx_n,Sx_{n+1})$
$= [\beta(Sx_0)+\gamma(Sx_0)+\delta(Sx_0)]Q(Sx_{n-1},Sx_n) \oplus [\gamma(Sx_0)+\mu(Sx_0)]Q(Sx_n,Sx_{n+1})$
$\Rightarrow Q(Sx_n,Sx_{n+1}) \leq' \frac{\beta(Sx_0)+\gamma(Sx_0)+\delta(Sx_0)}{1-\gamma(Sx_0)-\mu(Sx_0)}Q(Sx_{n-1},Sx_n)\ \forall n \geq 1$ (4.1.1).
Let $k = \frac{\beta(Sx_0)+\gamma(Sx_0)+\delta(Sx_0)}{1-\gamma(Sx_0)-\mu(Sx_0)} < 1$ by (2).
By repeating (4.1.1) we get,
$Q(Sx_n,Sx_{n+1}) \leq' k^n Q(Sx_0,Sx_1)$ (4.1.2).
Now for $m > n \geq 1$, from (4.1.2) we have,
$Q(Sx_n,Sx_m) \leq' Q(Sx_n,Sx_{n+1}) \oplus Q(Sx_{n+1},Sx_{n+2}) \oplus ..... \oplus Q(Sx_{m-1},Sx_m)$
$\leq' k^n Q(Sx_0,Sx_1) \oplus k^{n+1}Q(Sx_0,Sx_1) \oplus ..... \oplus k^{m-1}Q(Sx_0,Sx_1)$
$= \frac{k^m}{1-k}Q(Sx_0,Sx_1)$
$Q(Sx_n,Sx_m) \leq' \frac{k^m}{1-k}Q(Sx_0,Sx_1)$ (4.1.3).
Since P is a normal cone with normal constant K, we have from (4.1.3),
$Q(Sx_n,Sx_m) \preceq K\frac{k^m}{1-k}Q(Sx_0,Sx_1)$
$\Rightarrow Q^i_\alpha(Sx_n,Sx_m) \preceq K\frac{k^m}{1-k}Q^i_\alpha(Sx_0,Sx_1),\ \alpha \in (0,1], i = 1,2$
$\Rightarrow \lim_{m,n\to\infty} Q^i_\alpha(Sx_n,Sx_m) = 0\ \alpha \in (0,1], i = 1,2$
$\Rightarrow \lim_{m,n\to\infty} Q(Sx_n,Sx_m) = \bar{0}$.
Thus $\{Sx_n\}$ is a Cauchy sequence in S(X). Since S(X) is complete, $\exists$ a point $z \in S(X)$ such that
$Sx_n \to z$ as $n \to \infty$.
Again from (3.1.4) and (Q3), we have
$Q(Sx_n,z) \leq' \frac{k^n}{1-k}Q(Sx_0,Sx_1)\ \forall n \geq 1$.
Since P is a normal cone with normal constant K we have,
$Q(Sx_n,z) \preceq K(\frac{k^n}{1-k})Q(Sx_0,Sx_1)\ \forall n \geq 1$ (4.1.4)
$\Rightarrow Q^i_\alpha(Sx_n,z) \leq K(\frac{k^n}{1-k})Q^i_\alpha(Sx_0,Sx_1)\ \forall n \geq 1,\ \alpha \in (0,1],\ i = 1,2.$ (4.1.5)
If possible that $Tz \neq z$ or $Sz \neq z$.
Then by hypothesis and form (4.1.4) and (4.1.5) we have for some $\alpha_0 \in (0,1]$,
$0 < Inf\{Q^1_{\alpha_0}(Tx,z) + Q^1_{\alpha_0}(Sx,z) + Q^1_{\alpha_0}(Sx,Tx) : x \in X\}$
$\leq Inf\{Q^1_{\alpha_0}(Tx_n,z) + Q^1_{\alpha_0}(Sx_n,z) + Q^1_{\alpha_0}(Sx_n,Tx_n) : n \geq 1\}$
$= Inf\{Q^1_{\alpha_0}(Sx_{n+1},z) + Q^1_{\alpha_0}(Sx_n,z) + Q^1_{\alpha_0}(Sx_n,Sx_{n+1}) : n \geq 1\}$
$\leq Inf\{K(\frac{k^{n+1}}{1-k})Q^1_{\alpha_0}(Sx_0,Sx_1) + K(\frac{k^n}{1-k})Q^1_{\alpha_0}(Sx_0,Sx_1) + K(\frac{k^n}{1-k})Q^1_{\alpha_0}(Sx_0,Sx_1) : n \geq 1\}$
$= 0$ which is a contradiction. Thus $z = Sz = Tz$.

Theorem 4.1 is justified by the following Example. □

Example 4.2. Let $E = R$ and $|| \; || : E \to R^*(I)$ defined by

$$||x||(t) = \begin{cases} \frac{|x|}{t} & \text{if } t \geq |x|, x \neq \theta \\ 1 & \text{if } t = |x| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = \; [|x|, \frac{|x|}{\alpha}] \;\; \forall \alpha \in (0, 1]$.
It can be verified that $|| \; ||$ satisfies all the conditions in Definition 1.5 and hence $(E, || \; ||)$ is a fuzzy normed linear space (Felbin's sense).
If we choose $\preceq$ as the ordering in E and define $P = \{\eta \in E^*(I) : \eta \succeq \bar{0}\}$ then P is a cone on E.
Again since for $x, y \in E$, $\;||x|| \preceq ||y|| \Leftrightarrow |x| \leq |y|$, thus P is a normal cone with normal constant 1.
Take $X = [0, 1)$ and define $d : X \times X \to E^*(I)$ by $d(x, y) = ||x - y|| \; \forall x, y \in X$. Then d is a fuzzy cone metric and (X, d) is a fuzzy cone metric space.
Define $Q : X \times X \to E^*(I)$ by $Q(x, y) = 2d(x, y) \; \forall x, y \in X$. Then Q is a c-fuzzy distance.
(Q1) and (Q2) are obvious.
For (Q3), let $\{y_n\}$ be a sequence in X such that $y_n \to y \in X$.
Now for $x \in X, \; n \geq 1$,
$Q(x, y_n) \preceq ||u||$ for some $||u|| = ||u(x)|| \in P$
$\Rightarrow 2d(x, y_n) \preceq ||u||$
$\Rightarrow 2d^i_\alpha(x, y_n) \leq ||u||^i_\alpha$ for $\alpha \in (0, 1], \; i = 1, 2$
$\Rightarrow 2d^i_\alpha(x, y) \leq ||u||^i_\alpha$ for $\alpha \in (0, 1], \; i = 1, 2$
$\Rightarrow 2d(x, y) \preceq ||u||$
$\Rightarrow Q(x, y) \preceq ||u||$
So (Q3) holds.
(Q4). Let $c \in E$ with $\bar{0} \ll ||c||$ and put $||e|| = ||c||$.
Now for $Q(z, x) \ll ||e||$ and $Q(z, y) \ll ||e||$ we have,
$d(x, y) \preceq d(z, x) \oplus d(z, y) = \frac{1}{2}Q(z, x) \oplus \frac{1}{2}Q(z, y) \ll \frac{||e||}{2} \oplus \frac{||e||}{2} = ||e|| = ||c||$.
Thus $d(x, y) \ll ||c||$. Hence Q is a c-fuzzy distance.
Let $S, T : \; X \to X$ defined by $S(x) = x$ and $T(x) = \frac{x^2}{32} \; \forall x \in X$.
Take mappings $\beta, \gamma, \delta, \mu : X \to [0, 1]$ by
$\beta(x) = \gamma(x) = \frac{x+1}{32}, \;\; \delta(x) = \frac{2x+3}{32}$ and $\mu(x) = \frac{3x+2}{32} \;\; \forall x \in X$.
Now,
(1) $\beta(Tx) = \gamma(Tx) = \frac{1}{32}(\frac{x^2}{32} + 1) \leq \frac{1}{32}(x^2 + 1) \leq \frac{x+1}{32} = \beta(Sx) = \gamma(Sx) \; \forall x \in X$.
(2) $\delta(Tx) = \delta(\frac{x^2}{32}) = \frac{2\frac{x^2}{32}+3}{32} \leq \frac{1}{32}(2x^2 + 3) \leq \frac{2x+3}{32} = \delta(Sx)$.
(3) $\mu(Tx) = \mu(\frac{x^2}{32}) = \frac{1}{32}(\frac{3\frac{x^2}{32}+2}{32}) \leq \frac{3x^2+2}{32} \leq \frac{3x+2}{32} = \mu(Sx) \; \forall x \in X$.
(4) $(\beta + 2\gamma + \delta + \mu)(x) = \frac{x+1}{32} + \frac{2(x+1)}{32} + \frac{2x+3}{32} + \frac{3x+2}{32} = \frac{8x+8}{32} = \frac{x+1}{4}$.
So $(\beta + 2\gamma + \delta + \mu)(x) = \frac{x+1}{4} < 1 \;\; \forall x \in X$.
(5) $Q^1_\alpha(Tx, Ty) = Q^1_\alpha(\frac{x^2}{32}, \frac{y^2}{32}) = 2d^1_\alpha(\frac{x^2}{32}, \frac{y^2}{32}) = 2|\frac{x^2}{32} - \frac{y^2}{32}| = \frac{1}{16}|x^2 - y^2| = \frac{(x+y)}{4}\frac{|x-y|}{4}$.
i.e. $Q^1_\alpha(Tx, Ty) \leq \frac{(x+1)}{4}\frac{|x-y|}{4}$.
Similarly $Q^2_\alpha(Tx, Ty) \leq \frac{1}{\alpha}\frac{(x+1)}{4}\frac{|x-y|}{4} \; \forall x \in X, \; \alpha \in (0, 1]$.
Now,

$Q^1_\alpha(Sx, Sy) = 2d^1_\alpha(Sx, Sy) = 2|x{-}y| \geq \frac{|x-y|}{4}$ (4.2.1)
$Q^1_\alpha(Sx, Ty) = 2d^1_\alpha(Sx, Ty) = 2|x - \frac{y^2}{32}|$
$\geq 2|x{-}\frac{y}{32}| \geq 2|x{-}y| = 8\frac{|x-y|}{4} \geq \frac{|x-y|}{4}$ (4.2.2)
$Q^1_\alpha(Sx, Tx) = 2d^1_\alpha(Sx, Tx) = 2|x - \frac{x^2}{32}| \geq 2|x - \frac{x}{32}| = \frac{2.31}{32}|x| \geq \frac{|x-y|}{4}$ (4.2.3)
$Q^1_\alpha(Sy, Ty) = 2d^1_\alpha(Sy, Ty) = 2|y - \frac{y^2}{32}| \geq 2|y - \frac{y}{32}| \geq |y| \geq \frac{|x-y|}{4}$ (4.2.4)
By using (4), (5), (4.2.1), (4.2.2), (4.2.3) and (4.2.4) we get,
$Q^1_\alpha(Tx, Ty) \leq \beta(x)Q^1_\alpha(Sx, Sy) + \gamma(x)Q^1_\alpha(Sx, Ty)$
$+\delta(x)Q^1_\alpha(Sx, Tx) + \mu(x)Q^1_\alpha(Sy, Ty)\ \forall x, y \in X,\ \alpha \in (0, 1]$ (4.2.5).
Similarly we have,
$Q^2_\alpha(Tx, Ty) \leq \beta(x)Q^2_\alpha(Sx, Sy) + \gamma(x)Q^2_\alpha(Sx, Ty)$
$+\delta(x)Q^2_\alpha(Sx, Tx){+}\mu(x)Q^2_\alpha(Sy, Ty)\ \forall x, y \in X,\ \alpha \in (0, 1]$ (4.2.6).
From (4.2.5) and (4.2.6) we get,
$Q(Tx, Ty) \preceq \beta(x)Q(Sx, Sy) {\oplus} \gamma(x)Q(Sx, Ty) {\oplus} \delta(x)Q(Sx, Tx) {\oplus} \mu(x)Q(Sy, Ty)\ \forall x, y \in X$.
Again for $\alpha \in (0, 1]$ and for any $y \neq Ty$ (i.e. $y > 0$) we get,
$Inf\{Q^1_\alpha(Tx, y) + Q^1_\alpha(Sx, y) + Q^1_\alpha(Sx, Tx) : x \in X\}$
$= Inf\{2|\frac{x^2}{32} - y| + 2|x - y| + 2|x - \frac{x^2}{32}| : x \in X\}$
$= Inf\{4|y - \frac{y^2}{32}|\} > 0.$
Thus all the conditions of the Theorem 4.1 are satisfied. So we can conclude that S and T have a common fixed point. This common fixed point is $x = 0$.

5. Conclusion

Following the concept of c-distance in cone metric space introduced by Sheughua Wang et al., in this paper, an idea of fuzzy c-distance in fuzzy cone metric space is introduced. By using this concept, common fixed point theorems for contraction mapping are established in fuzzy cone metric spaces. Generally to establish common fixed point theorems in cone metric spaces as well as in fuzzy cone metric spaces, contraction mappings should be weakly compatible. Here fuzzy c-distance is used to establish common fixed point theorems shown that it is not required that mappings are weakly compatible. I think that there is a wide scope of research to develop fixed point results in fuzzy cone metric spaces by using fuzzy c-distance.

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