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The average chance simulation of busy time in random fuzzy queuing system with multiple servers

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Abstract. In this paper, we consider a queuing system with multiple servers that its interarrival times and service times are presented by random fuzzy variables. Here a new theorem concerning the average chance of event "all of k-servers are busy at time t" is proved. We simulate of average chance by fuzzy simulation method and solve some application examples.

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1. Introduction

Queuing systems with multiple servers are important for analysis of computer systems and networks. Zheng and Buckley [13] used both fuzzy optimization and normal simulation methods to solve fuzzy web planning model problems, which are queuing system problems for designing web servers. Zhang and Phillis [12] consider the problem of optimal control of queuing systems with heterogeneous servers using fuzzy logic techniques. The system objective is to assign customers dynamically to idle servers in order to minimize the average cost of holding customers. Jain, Maheshwari and Baghel [2] develop queuing model for the performance prediction of flexible manufacturing systems (FMSs) with a multiple discrete material-handling devices (MHD). Chen [1] proposed a procedure for constructing the membership functions of the performance measures in finite-capacity queuing systems with the arrival rate and service rate being fuzzy numbers. Kreimer [3] studied a real-time multiserver system with homogeneous servers (such as unmanned air vehicles or machine controllers) and several nonidentical channels (such as surveillance regions or assembly lines) working under maximum load regime. Rahmati [11] proposed a novel multi-objective location model within multi-server queuing framework, in which facilities behave as M/M/m queues. Yang and Chang [15] investigated the

F-policy queue using fuzzy parameters, in which the arrival rate, service rate, and start-up rate are all fuzzy numbers. The F-policy deals with the control of arrivals in a queuing system, in which the server requires a start-up time before allowing customers to enter. Yamashiro and Yuasa [14] studied M/M/2 and M/M/3 machine repair problems in which the number of repairmen changes depending on the number of failed machines in the system.

Here, we simulate the average chance of event random fuzzy queuing system is busy at time t when the queuing system has k-servers. We estimate the average chance of event "all of k-servers are busy at time t" in which all servers work independently and interarrival times and service times are random fuzzy variables.

In section 2, we state some concepts of fuzzy set theory, fuzzy variable, random fuzzy variable. In section 3, we describe a queuing system with multiple servers with random fuzzy interarrival times and service times and estimate the average chance of event "all of k-servers are busy at time t". In section 4, is considered the fuzzy simulation method and section 5, provided some application examples.

2. Definitions and preliminaries

Credibility theory introduced by Liu [6] in 2004 and refined by Liu [8] in 2015, is a branch of mathematics for studying the behavior of fuzzy phenomena. Many peoples studied about credibility theory. In [10] solve the multiple objective minimum cost flow problem with fuzzy data using credibility approach. The emphasis in this section is mainly on credibility measure, credibility space, fuzzy variable, membership function, credibility distribution, expected value, random fuzzy variable and its expected value, independence, identical distribution.

Let Θ be a nonempty set, and P the power set of Θ (i.e., the largest σ- algebra over Θ). Each element in P is called an *event*. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms: **Axiom 1.** (*Normality*) $Cr\{\Theta\} = 1$.

Axiom 2. (*Monotonicity*)$Cr\{A\} \leq Cr\{B\}$ for $A \subset B$.

Axiom 3. (*Self-Duality*) $Cr\{A\} + Cr\{A^c\} = 1$ for any event A.

Axiom 4. (*Maximality*) $C\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

Definition 2.1. (Liu and Liu [7]) The set function Cr is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms. Then the triplet (Θ, P, Cr) is called a credibility space.

Product credibility measure may be defined in multiple ways. We accept the following axiom.

Axiom 5. (*Product Credibility Axiom*) Let Θ_k be nonempty sets on which Cr_k are credibility measures, $k = 1, 2, ..., n$, respectively, and $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$. Then

$$Cr\{(\theta_1, \theta_2, ..., \theta_n)\} = Cr_1\{\theta_1\} \wedge Cr_2\{\theta_2\} \wedge ... \wedge Cr_n\{\theta_n\} \tag{2.1}$$

for each $(\theta_1, \theta_2, ..., \theta_n) \in \Theta$.
Let $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$ be credibility spaces, $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$ and $Cr_1 \wedge Cr_2 \wedge ... \wedge Cr_n$. Then (Θ, P, Cr) is called the product credibility space of $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$.

Definition 2.2. A fuzzy variable is a measurable function from a credibility space (Θ, P, Cr) to the set of real numbers.

Definition 2.3. Let ξ be a fuzzy variable defined on the credibility space (Θ, P, Cr). Then its membership function is derived from the credibility measure by

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, x \in \Re. \tag{2.2}$$

Definition 2.4. Let ξ be a fuzzy variable defined on the credibility space $(\Theta$, $P(\Theta)$, $Cr)$, and $\alpha \in (0, 1]$. Then

$$\xi'_\alpha = \inf\{r|\mu_\xi(r) \geq \alpha\}, \quad \xi''_\alpha = \sup\{r|\mu_\xi(r) \geq \alpha\}, \tag{2.3}$$

are called α-pessimistic value and α-optimistic value of ξ, respectively.

There are many ways to define an expected value operator for fuzzy variables. The most general definition of expected value operator of fuzzy variable was given by Liu and Liu [4]. This definition is applicable not only to continuous fuzzy variables but also for discrete ones.

Definition 2.5. Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr, \tag{2.4}$$

provided that at least one of the above two integrals is finite.

In particular, if the fuzzy variable ξ is positive (i.e. $Cr\{\xi \leq 0\} = 0$), then

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr. \tag{2.5}$$

Proposition 2.6. *Let ξ be a fuzzy variable defined on the credibility space* $(\Theta$, $P(\Theta)$, $Cr)$. *Then we have*

$$E[\xi] = \frac{1}{2}\int_0^1 [\xi'_\alpha + \xi''_\alpha]d\alpha. \tag{2.6}$$

Proof. Let ξ is normalized, i.e., there exists a real number r_0 such that $\mu_\xi(r_0) = 1$ and if $r_0 > 0$, then the equation (2.5) can be rewritten as

$$\begin{aligned} E[\xi] &= \frac{1}{2}[r_0 + \int_{r_0}^{+\infty} Cr(\xi \geq r)dr + r_0 - \int_{-\infty}^{r_0} Cr(\xi \leq r)dr] \\ &= \frac{1}{2}\int_0^1 (\xi'_\alpha + \xi''_\alpha)d\alpha, \end{aligned}$$

the same result can be obtained when $r_0 < 0$. □

Definition 2.7. A random fuzzy variable is a function from the credibility space (Θ, P, Cr) to the set of random variables.

Definition 2.8. The expected value of a random fuzzy variable ξ is defined by
(2.7)
$$E[\xi] = \int_0^{\infty} Cr\{\theta \in \Theta | E[\xi(\theta)] \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}Cr\{\theta \in \Theta | E[\xi(\theta)] \leq r\}dr,$$

Proposition 2.9. *Let ξ be a random fuzzy variable defined on (Θ, P, Cr). Then, for any $\theta \in \Theta$, $E[\xi(\theta)]$ is a fuzzy variable provided that $E[\xi(\theta)]$ is finite for fixed $\theta \in \Theta$.*

Definition 2.10. The random fuzzy variables ξ and η are said to be identically distributed if

$$\sup_{Cr\{A\} \geq \alpha} = \inf_{\theta \in A}\{Pr\{\xi(\theta) \in B\}\} = \sup_{Cr\{A\} \geq \alpha} = \inf_{\theta \in A}\{Pr\{\eta(\theta) \in B\}\} \tag{2.8}$$

for any $\alpha \in (0, 1]$ and Borel set B of real numbers.

Definition 2.11. The random fuzzy variables $\xi_i, i = 1, ..., n$ are said to be independent if
(1) $\xi_i(\theta), i = 1, ..., n$ are independent random variables for each $\theta \in \Theta$.
(2) $E[\xi_i(.)], i = 1, ..., n$ are independent fuzzy variables.

Definition 2.12. Let ξ be a random fuzzy variable on the possibility space (Θ, $P(\Theta)$, Pos). Then the average chance of random fuzzy event $\xi \leq 0$ is defined as

$$Ch\{\xi \leq 0\} = \int_0^1 Cr\{\theta \in \Theta | Pr\{\xi(\theta) \leq 0\} \geq p\}dp. \tag{2.9}$$

3. Random fuzzy queuing system with k-servers

Consider a stochastic queuing system with k-server, $RF/RF/k/FCFS/\infty/\infty$, where RF denotes that interarrival times and service times are random fuzzy variables, the queue discipline is first come, first served (FCFS) and the size of source population is infinite. The interarrival times of customers arriving at the server are independent and identically distributed random fuzzy variables, $\xi_i \sim EXP(\lambda_i)$, where λ_i are fuzzy variables defined on the credibility space $(\Theta_i, P(\Theta_i), Cr_i), i = 1, 2, ...,$ and the service times are independent and identically distributed random fuzzy variables, $\eta_i \sim EXP(\mu_i)$, where μ_i are fuzzy variables defined on credibility space $(\Gamma_i, P(\Gamma_i), Cr'_i), i = 1, 2, ...$ ξ_i and η_i are independent.

Here, we discuss on the busy period and idle period. The busy period and idle period are one of the most important of problems is mentioned, nowadays. We have obtained the average chance of event "random fuzzy queuing system is busy at time t" when the queuing system has k-servers. In [9] is consider this problem when we have a queuing system with one server.

Set $P(t)$=$Pr\{all\ of\ k-servers\ are\ busy\ at\ time\ t\}$, and $P_i(t)$=$Pr\{the\ ith\ server\ is\ busy\ at\ time\ t\}$, then $P(t)$ and $P_i(t)$ are fuzzy variable and P'_{α_0} and P''_{α_0} are the α_0-pessimistic values and the α_0-optimistic values $P(t)$ and $E[\frac{\lambda}{\mu}] < 1$. The following lemma proved in [9].

Lemma 3.1. *Assume that, in a random fuzzy queuing system $RF/\ RF/\ k/\ FCFS/\ \infty/\ \infty$, the fuzzy variable λ has the same α_0-pessimistic values and the α_0-optimistic* *values λ_i and the fuzzy variable μ has the same α_0-pessimistic values and the α_0-optimistic values μ_i and are continuous at the point $\alpha_0, \alpha_0 \in [0,1]$. Also, let k-servers work independently, then we have*

$$\lim_{t\to\infty} P'_{i\alpha}(t) = \frac{\lambda'_\alpha}{\mu''_\alpha}, \tag{3.1}$$

and

$$\lim_{t\to\infty} P''_{i\alpha}(t) = \frac{\lambda''_\alpha}{\mu'_\alpha}. \tag{3.2}$$

Theorem 3.2. *Assume that, in a random fuzzy queuing system $RF/\ RF/\ k/\ FCFS/\ K/\ \infty$, the distributions $\xi_i(\theta)$ and $\eta_i(\gamma)$ are nonlattice, and fuzzy variables λ_i and μ_i, $i = 1, 2...$, are continuous at point $\alpha, \alpha \in (0,1]$. Also, let k-servers work independently, then we have*

$$\lim_{t\to\infty} Ch\{all\ of\ k - server\ are\ busy\ at\ time\ t\} = (E[\frac{\lambda}{\mu}])^k. \tag{3.3}$$

Proof. From Definition 2.10 and Proposition 2.6, it follows for ith server, $i = 1, 2, ..., k$, that

$$\begin{aligned} &Ch\{the\ ith\ server\ is\ busy\ at\ time\ t\} \\ &= \int_0^1 Cr\{\theta \in \Theta | P_i(t)(\theta) \geq p\}dp \\ &= \int_0^\infty Cr\{\theta \in \Theta | P_i(t)(\theta) \geq p\}dp \\ E[P_i(t)] &= \frac{1}{2}\int_0^1 (P'_{i\alpha}(t) + P''_{i\alpha}(t))dp \end{aligned}$$

It follows from the definition of limit that there exist two real numbers t_1 and t_2 with $t_1 \geq 0$ and $t_2 \geq 0$ such that for any $t \geq t_1$ and $t \geq t_2$

$$0 \leq P'_{i\alpha}(t) \leq \frac{\lambda'_\alpha}{\mu''_\alpha},$$

$$0 \leq P''_{i\alpha}(t) \leq \frac{\lambda''_\alpha}{\mu'_\alpha}.$$

Therefore, we have for any $t \geq \max(t_1, t_2)$

$$0 \leq P'_{i\alpha}(t) + P''_{i\alpha}(t) \leq 2 + \frac{\lambda'_\alpha}{\mu''_\alpha} + \frac{\lambda''_\alpha}{\mu'_\alpha}.$$

Since, $E[\frac{\lambda}{\mu}]$ is finite, than $2 + \frac{\lambda'_\alpha}{\mu''_\alpha} + \frac{\lambda''_\alpha}{\mu'_\alpha}$ is an integrable function of α. It follows from Fatou's lemma that

$$\liminf_{t\to\infty} \int_0^1 (P'_{i\alpha}(t) + P''_{i\alpha}(t))d\alpha \geq \int_0^1 \liminf_{t\to\infty} (P'_{i\alpha}(t) + P''_{i\alpha}(t))d\alpha,$$

and

$$\limsup_{t\to\infty} \int_0^1 (P'_{i\alpha}(t) + P''_{i\alpha}(t))d\alpha \leq \int_0^1 \limsup_{t\to\infty} (P'_{i\alpha}(t) + P''_{i\alpha}(t))d\alpha.$$

Since $\lambda'_\alpha, \lambda''_\alpha, \mu'_\alpha, \mu''_\alpha$ are almost surely continuous at point α, then we have from Lemma 1,

$$\lim_{t\to\infty} Ch\{the\ ith\ server\ is\ busy\ at\ time\ t\}$$

$$= \frac{1}{2}\lim_{t\to\infty}\int_0^1 (P'_{i\alpha}(t) + P''_{i\alpha}(t))dp = \frac{1}{2}\int_0^1 \lim_{t\to\infty}(P'_{i\alpha}(t) + P''_{i\alpha}(t))dp$$

$$= \frac{1}{2}\int_0^1 (\frac{\lambda'_\alpha}{\mu''_\alpha} + \frac{\lambda''_\alpha}{\mu'_\alpha})d\alpha$$

$$= E[\frac{\lambda}{\mu}].$$

Now, for k servers that work independently and identically distributed, we have

$$\lim_{t\to\infty} Ch\{all\ of\ k - server\ are\ busy\ at\ time\ t\}$$

$$= \prod_{i=1}^{k} E[\frac{\lambda}{\mu}] = (E[\frac{\lambda}{\mu}])^k,$$

the proof is completed• □

4. Fuzzy simulation approach

In order to evaluate the expected value of a fuzzy variable, Liu and Liu [5] designed a fuzzy simulation for both discrete and continuous fuzzy variables.
(i) Discrete fuzzy vector: assume that f is a function, and $\xi = (\xi_1, ..., \xi_m)$ is discrete fuzzy vector whose joint credibility distribution function is defined by

$$\mu_\xi(u) = \begin{cases} \mu_1, & u = u_1 \\ \mu_2, & u = u_2 \\ ... & \\ \mu_n, & u = u_n \end{cases} \tag{4.1}$$

where $\mu_u = \min_{1\le i\le m} \mu^{(i)}(u_i)$ and $u = (u_1, ..., u_m) \in \Re^m$ and $\mu^{(i)}$ are credibility distribution function of ξ_i for $i = 1, 2, ..., m$.
Let $a_i = f(u_i)$. Without losing of generality, we assume that $a_1 \le a_2 \le ... \le a_n$, then the expected value is given by

$$E[f(\xi)] = \sum_{i=1}^{n} a_i p_i, \tag{4.2}$$

where

$$p_i = \frac{1}{2}(\vee_{j=i}^{n}\mu_j - \vee_{j=i+1}^{n+1}\mu_j) + \frac{1}{2}(\vee_{j=1}^{i}\mu_j - \vee_{j=0}^{i-1}\mu_j), \tag{4.3}$$

where $(\mu_0 = \mu_{n+1} = 0)$ for $i = 1, 2, ..., n$.
(ii) Continuous fuzzy vector: assume that ξ is a continuous fuzzy vector with a credibility distribution function μ. In this case, we can estimate the expected value by formula (4.2).

5. Experiential results

Now, we consider some examples. We present some practical applications of this model to show that how using fuzzy simulation method is estimated the average chance.

Example 5.1. Consider a big market with 10 server (k=10). Let the interarrival times of customer are fuzzy variables with exponential distribution with parameter $\lambda = (1/2/3)$ in minutes and service times are fuzzy variables with exponential distribution with parameter $\mu = (3/4/5)$ in minutes for 10-server. We want to calculate the average chance of "all of 10-server are busy at time t". By Theorem 3.1, for estimating the $E([\frac{\lambda}{\mu}])^k$, we use simulation method in section 4. The simulation results are shown in Table 1 and Figure 1.
Table 1 and Figure 1 show that the average chance of all of 10-servers are busy at time t after 30000 times is equal 0.0018 and it remains at 0.0018, level. It equals with real solution.

Number of iterations	100	500	1000	10000	20000	30000
The average chance	0.0012	0.0014	0.0016	0.0017	0.0018	0.0018

Table 1: The average chance of random fuzzy queuing system is busy at time t with fuzzy simulation method for example 1

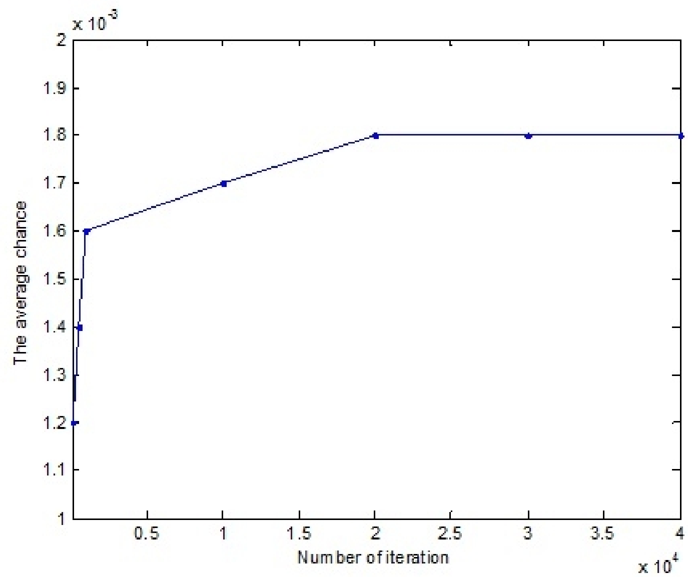

Figure 1. The convergence of fuzzy simulation method for example 1

Example 5.2. Let a bank has 5 servers that service to customers. Let the interarrival times of customer are fuzzy variables with exponential distribution with parameter $\lambda = (0.5/1/1.5)$ in minutes and service times are fuzzy variables with exponential distribution with parameter $\mu = (1/2/3)$ in minutes for 5-server. We calculate the average chance of "all of 5-server are busy at time t" by Theorem 3.1 and estimate the $(E[\frac{\lambda}{\mu}])^k$. The simulation results are shown in Table 2 and Figure 2.

Number of iterations	100	500	1000	10000	20000	30000
The average chance	0.0463	0.0497	0.0650	0.0677	0.0736	0.0736

Table 2: The average chance of random fuzzy queuing system is busy at time t with fuzzy simulation method for example 2

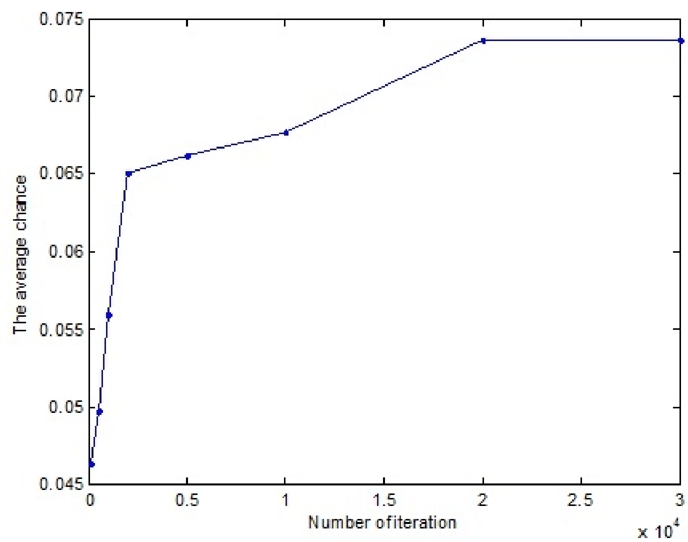

FIGURE 2. The convergence of fuzzy simulation method for example 2

Table 2 and Figure 2 show that the proposed modeling leads to the result the average chance of all of 5-servers are busy at time t after 30000 times is equal 0.0736 and it remains at 0.0736, level.

6. Conclusion

In this paper discussed random fuzzy queuing systems with multiple servers where interarrival times and service times are random fuzzy variables. Fuzzy simulation technique was designed to estimate the average chance of event "all of k-servers are busy at time t". Finally, some examples was given to illustrate the effectiveness of proposed technique. The stated problem is usually used in big markets and bank ,... that have many servers for servicing the customers. Also, we estimated the average chance without using α-cuts and simulated the results.

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