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e-compactness in fuzzifying topology

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Abstract. In this paper the concepts of fuzzifying e-irresolute functions and fuzzifying e-compact spaces were characterized in terms of fuzzifying e-open sets and some of their properties are discussed.

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1. Introduction

Fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite nature discipline [4]-[6], [9], [10], [15]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [5], the kind of topologies defined by Chang [1] and Goguen [2] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family τ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [8].

On the other hand, the authors of [7], [11] proposed the terminologies I-fuzzy topologies (if the set of membership values is chosen to be the unit interval [0,1]) and L-fuzzy topologies (if the corresponding set of membership values is chosen to be lattice L). More specifically, an I-fuzzy toplogy (resp. an L-fuzzy topology) is a (resp. an L-) fuzzy family τ over $P(X)$, where $P(X)$ denotes the class of all crisp subsets of nonempty set X. They were defined and extensively studied by Höhle, ŝostak, Kubiak, Radabaugh and others [3], [7], [9], [10], [11].

In general, L-fuzzy topologies are investigated and described with algebraic and analytic methods.

In 1991, Ying [16]-[19] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In a way, fuzzifying topologies are analogous to I-fuzzy topologies, but the former appeal to some semantical experssions of Lukasiewicz logic as a basic tool, and thus can be viewed as an alternative approach to fuzzy topology. Particularly, as the author [16]-[19] indicated, by investigating fuzzifying topology we may partially answer an important question proposed by Rosser and Turquette [12] in 1952, which asked wether there are many valued theories beyond the level of predicates calculus.

Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [14, 19]. For example, Ying [19] introduced the concepts of compactness and establised a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [14] the concepts of fuzzifying e-open set and fuzzifying e-continuity were introduced and studied and also introduced and studied the concept of fuzzifying e-separation axioms.

In [13] the concepts of e-irresolute function and e-compactness for fuzzy topological spaces were introducted. In this paper we introduce and study the concept of e-irresolute function between fuzzifying topological spaces. Furthermore, we introduce and study the concept of e-compactness in the framework of fuzzifying topology. We use the finite intersection property to give a characterization of the fuzzifying e-compact spaces.

2. Preliminaries

In this section, we offer some concepts and results in fuzzifying topology, which will used in the sequel. For the details, we refer to [14], [16]-[19]. First, we display the Lukasiewicz logic and corresponding set theoretical notations used in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. By $\models^w \varphi$ (φ is feebly valid) we mean that for any valuation it always holds that $[\varphi] > 0$, and $\varphi \models^{ws} \psi$ we mean that $[\varphi] > 0$ implies $[\psi] = 1$. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) $[\alpha] = \alpha(\alpha \in [0,\ 1])$; $[\varphi \wedge \psi] :=\min([\varphi], [\psi])$; $[\varphi \to \psi] :=\min(1, 1 - [\varphi] + [\psi])$.

(2) If $\widetilde{A} \in \Im(X)$, where $\Im(X)$ is the family of all fuzzy sets of X, then

$$[x \in \widetilde{A}] := \widetilde{A}(x).$$

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x\in X}[\varphi(x)]$. In addition the following derived formulae are given,

(1) $[\neg\varphi] := [\varphi \to 0] := 1 - [\varphi]$;

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] := \max([\varphi], [\psi])$;

(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;

(4) $[\varphi \dot{\wedge} \psi] := [\neg(\varphi \rightarrow \neg\psi)] := \max(0, [\varphi] + [\psi] - 1)$;

(5) $[\exists x \varphi(x)] := [\neg \forall x \neg \varphi(x)] := \sup_{x \in X} [\varphi(x)]$;

(6) If $\widetilde{A}$, $\widetilde{B} \in \Im(X)$, then
$[\widetilde{A} \subseteq \widetilde{B}] := \big[\forall x (x \in \widetilde{A} \rightarrow x \in \widetilde{B})\big] := \inf_{x \in X} \min\big(1, 1 - \widetilde{A}(x) + \widetilde{B}(x)\big)$;

where $\Im(X)$ is the family of all fuzzy sets in X.

Often we do not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying [16]-[19] did.

Second, we give some definitions and results in fuzzifying topology.

Definition 2.1 ([16])**.** Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfy the following condition:

(1) $\tau(X) = 1, \tau(\phi) = 1$;
(2) for any A, B, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geq \bigvee_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 2.2 ([16])**.** The family of all fuzzifying closed sets, denoted by $\mathcal{F} \in \Im(P(X))$, is defined as follows: $A \in \mathcal{F} := (X - A) \in \tau$, where $X - A$ is the complement of A.

Definition 2.3 ([16])**.** The fuzzifying neighbourhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows: $N_x(A) = \sup_{x \in B \subseteq A} \tau(B)$.

Definition 2.4 ([16], Lemma 5.2.)**.** The closure $\bar{A}$ of A is defined as $\bar{A}(x) = 1 - N_x(X - A)$. In Theorem 5.3 [16], Ying proved that the closure $^- : P(X) \rightarrow \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [16]) because its extension $^- : \Im(X) \rightarrow \Im(X)$, $\overline{\widetilde{A}} = \bigcup_{\alpha \in [0,1]} \alpha \overline{\widetilde{A}}_\alpha$, $\widetilde{A} \in \Im(X)$, where $\widetilde{A}_\alpha = \{x : \widetilde{A}(x) \geq \alpha\}$ is the α-cut of $\widetilde{A}$ and $\alpha\widetilde{A}(x) = \alpha \wedge \widetilde{A}(x)$ satisfied the following kuratowski closure axioms:

(1) $\models \bar{\phi} = \phi$;
(2) for any $\widetilde{A} \in \Im(X)$, $\models \widetilde{A} \subseteq \overline{\widetilde{A}}$;
(3) for any $\widetilde{A}$, $\widetilde{B}$, $\in \Im(X)$, $\models \overline{\widetilde{A} \cup \widetilde{B}} \equiv \overline{\widetilde{A}} \cup \overline{\widetilde{B}}$; (4) for any $\widetilde{A}, \widetilde{B} \in \Im(X)$, $\models \overline{(\overline{\widetilde{A}})} \subseteq \overline{\widetilde{A}}$.

Definition 2.5 ([17])**.** For any $A \subseteq X$, the fuzzy set of interior points of A is called the interior of A, and given as follows: $A^\circ(x) := N_x(A)$. From Lemma 3.1 [16] and the definitions of $N_x(A)$ and A° we have $\tau(A) = \inf_{x \in A} A^\circ(x)$.

Definition 2.6 ([14])**.** For any $\widetilde{A} \in \Im(X)$, $\models (\widetilde{A})^\circ \equiv X - (\overline{X - \widetilde{A}})$.

Lemma 2.7 ([14])**.** *If* $[\widetilde{A} \subseteq \widetilde{B}] = 1$, *then (1)*$\models \overline{\widetilde{A}} \subseteq \overline{\widetilde{B}}$ *(2)*$\models (\widetilde{A})^\circ \subseteq (\widetilde{B})^\circ$.

Definition 2.8 ([14])**.** Let (X, τ) be a fuzzifying topological space.

(1) The family of all fuzzifying e-open sets, denoted by $\tau_e \in \Im(P(X))$, is defined as follows:
$A \in \tau_e := \forall x(x \in A \to x \in A^{-\circ\delta} \cup A^{\circ-\delta})$, i.e.,
$$\tau_e(A) = \inf_{x\in A} \max\left(A^{-\circ\delta}(x), A^{\circ-\delta}(x)\right).$$

(2) The family of all fuzzifying e-closed sets, denoted by $\mathcal{F}_e \in \Im(P(X))$, is defined as follows:
$$A \in \mathcal{F}_e := (X - A) \in \tau_e.$$

(3) The fuzzifying e-neighborhood system of a point $x \in X$ is denoted by $N_x^e \in \Im(P(X))$, is defined as follows:
$$N_x^e(A) = \sup_{x\in B\subseteq A} \tau_e(B).$$

(4) The fuzzifying e-closure of a set $A \in P(X)$ is denoted by $Cl_e \in \Im(X)$, is defined as follows:
$$Cl_e(A)(x) = 1 - N_x^e(X - A).$$

(5) Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $C_e \in \Im(Y^X)$, called fuzzifying e-continuity, is given as follows
$$C_e(f) := \forall B(B \in \sigma \to f^{-1}(B) \in \tau_e).$$

Definition 2.9 ([14])**.** Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicate T_2^e (fuzzifying e-Hausdorff) $\in \Im(\Omega)$ is defined as follows:
$T_2^e(X,\tau) := \forall x\forall y\Big((x \in X \wedge y \in X \wedge x \neq y) \to \exists B\exists C\big(B \in N_x^e \wedge C \in N_x^e \wedge B \cap C = \phi\big)\Big)$.

Definition 2.10 ([19])**.** Let X be a set. If $\widetilde{A} \in \Im(X)$ such that the support $\sup p\widetilde{A} = \{x \in X : \widetilde{A}(x) > 0\}$ of A is finite, then $\widetilde{A}$ is said to be finite and we write $F(\widetilde{A})$. A unary fuzzy predicate $FF \in \Im(\Im(X))$, called fuzzy finiteness, is given as $FF(\widetilde{A}) := (\exists\widetilde{B})\big(F(\widetilde{B}) \wedge (\widetilde{A} \equiv \widetilde{B})\big) = 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_{[\alpha]})\}$, where $\widetilde{A}_\alpha = \{x \in X : \widetilde{A}(x) \geq \alpha\}$ and $\widetilde{A}_{[\alpha]} = \{x \in X : \widetilde{A}(x) > \alpha\}$.

Definition 2.11 ([19])**.** Let X be a set.

(1) A binary fuzzy predicate $K \in \Im\big(\Im(P(X)) \times P(X)\big)$, called fuzzifying covering, is given as follows:
$$K(\Re, A) := \forall x\big(x \in A \to \exists B(B \in \Re \wedge x \in B)\big).$$

(2) Let (X, τ) be a fuzzifying topological space. A binary fuzzy predicate $K_\circ \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying open covering, is given as follows:
$$K_\circ(\Re, A) := K(\Re, A) \dot\wedge (\Re \subseteq \tau).$$

Definition 2.12 ([19])**.** Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness, is given as follows:
$$(X, \tau) \in \Gamma := (\forall\Re)\Big(K_\circ(\Re, X) \to (\exists\wp)\big((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp)\big)\Big),$$
where $\wp \leq \Re$ means that for any $M \in P(X)$, $\wp(M) \leq \Re(M)$.

Definition 2.13 ([19]). Let X be a set. A unary fuzzy predicate $fI \in \Im(\Im(P(X)))$, called fuzzifying finite intersection property, is given as follows:

$$fI(\Re) := (\forall \mathcal{B})\Big((\mathcal{B} \leq \Re) \wedge FF(\mathcal{B}) \rightarrow (\exists x)(\forall B)\big((B \in \mathcal{B}) \rightarrow (x \in B)\big)\Big).$$

Theorem 2.14 ([14]). *Let $(X,\ \tau)$ be a fuzzifying topological space. Then, we have*

(1) $\models \tau \subseteq \tau_e$
(2) $\models F \subseteq \mathcal{F}_e$
(3) $\models \mathcal{F}_e(\bigcap\limits_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge\limits_{\lambda\in\Lambda} \mathcal{F}_e(A_\lambda)$.

Theorem 2.15 ([14]). *The mapping $N^e : X \rightarrow \Im^N(P(X))$, $x \mapsto N^e_x$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:*

(1) $\models A \in N^e_x \rightarrow x \in A$;
(2) $\models A \subseteq B \rightarrow (A \in N^e_x \rightarrow B \in N^e_x)$;
(3) $\models A \in N^e_x \wedge B \in N^e_x \rightarrow A \cap B \in N^e_x$.

Conversely, if a mapping N^e_x satisfies (2) and (3), then N^e_x assigns a fuzzifying topology on X which is denoted by $\tau_e \in \Im(P(X))$ and defined as

$$A \in \tau_e := \forall x(x \in A \rightarrow A \in N^e_x).$$
$$\Big(i.e., \quad \tau_e(A) = \inf_{x\in A} N^e_x(A)\Big)$$

3. Fuzzifying e-irresolute functions

Definition 3.1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I_e \in \Im(Y^X)$, called fuzzifying e-irresolute, is given as follows: $I_e(f) := \forall B(B \in \sigma_e \rightarrow f^{-1}(B) \in \tau_e)$. Intuitively, the degree to which f is fuzzifying e-irresolute function is

$$[I_e(f)] = \inf_{B\subseteq Y} \min\Big(1, 1 - \sigma_e(B) + \tau_e\big(f^{-1}(B)\big)\Big)$$

Theorem 3.2. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. Then*

$$\models f \in I_e \rightarrow f \in C_e.$$

Proof. From Theorem 2.14 we have $\sigma(B) \leq \sigma_e(B)$ and the result holds. □

Definition 3.3. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. we define the unary fuzzy predicates $e_j \in \Im(Y^X)$ where $j = 1,\dots,5$ as follows:

(1) $f \in e_1 := \forall B(B \in \mathcal{F}^Y_e \rightarrow f^{-1}(B) \in \mathcal{F}^X_e)$, where $\mathcal{F}^X_e$ and $\mathcal{F}^Y_e$ are the fuzzifying e-closed subsets of X and Y, respectively;

(2) $f \in e_2 := \forall x \forall u(u \in N^{e^Y}_{f(x)} \rightarrow f^{-1}(u) \in N^{e^X}_x)$, where N^{e^X} and N^{e^Y} are the family of fuzzifying e-neighborhood systems of X and Y, respectively;

(3) $f \in e_3 := \forall x \forall u\Big(u \in N^{e^Y}_{f(x)} \rightarrow \exists v\big(f(v) \subseteq u \rightarrow v \in N^{e^X}_x\big)\Big)$;

(4) $f \in e_4 := \forall A\Big(f\big(Cl_e^X(A)\big) \subseteq Cl_e^Y\big(f(A)\big)\Big)$;

(5) $f \in e_5 := \forall B\Big(Cl_e^X\big(f^{-1}(B)\big) \subseteq f^{-1}\big(Cl_e^Y(B)\big)\Big)$.

Theorem 3.4. *$\models f \in I_e \leftrightarrow f \in e_j$, for $j = 1, \ldots, 5$.*

Proof. (a) We will prove that $\models f \in I_e \leftrightarrow f \in e_1$.
$[f \in e_1] = \inf_{B\in P(Y)} \min\Big(1, 1 - \mathcal{F}_e^Y(B) + \mathcal{F}_e^X(f^{-1}(B))\Big)$
$= \inf_{B\in P(Y)} \min\Big(1, 1 - \sigma_e(Y - B) + \tau_e\big(X - f^{-1}(B)\big)\Big)$
$= \inf_{B\in P(Y)} \min\Big(1, 1 - \sigma_e(Y - B) + \tau_e\big(f^{-1}(Y - B)\big)\Big)$
$= \inf_{u\in P(Y)} \min\Big(1, 1 - \sigma_e(u) + \tau_e\big(f^{-1}(u)\big)\Big)$
$= [f \in I_e]$.

(b) We will prove that $\models f \in I_e \leftrightarrow f \in e_2$. First, We prove that $[f \in e_2] \geq [f \in I_e]$. If $N_{f(x)}^{e^Y}(u) \leq N_x^{e^X}(f^{-1}(u))$, then $\min\Big(1, 1 - N_{f(x)}^{e^Y}(u) + N_x^{e^X}\big(f^{-1}(u)\big)\Big) = 1 \geq [f \in I_e]$. Suppose $N_{f(x)}^{e^Y}(u) > N_x^{e^X}(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$N_{f(x)}^{e^Y}(u) - N_x^{e^X}(f^{-1}(u)) = \sup_{f(x)\in A\subseteq u} \sigma_e(A) - \sup_{x\in B\subseteq f^{-1}(u)} \tau_e(B)$
$\leq \sup_{f(x)\in A\subseteq u} \sigma_e(A) - \sup_{f(x)\in A\subseteq u} \tau_e\big(f^{-1}(A)\big)$
$\leq \sup_{f(x)\in A\subseteq u} \Big(\sigma_e(A) - \tau_e\big(f^{-1}(A)\big)\Big)$.

So
$1 - N_{f(x)}^{e^Y}(u) + N_x^{e^X}\big(f^{-1}(u)\big) \geq \inf_{f(x)\in A\subseteq u} \Big(1 - \sigma_e(A) + \tau_e\big(f^{-1}(A)\big)\Big)$,

Therefore
$\min\Big(1, 1 - N_{f(x)}^{e^Y}(u) + N_x^{e^X}\big(f^{-1}(u)\big)\Big) \geq \inf_{f(x)\in A\subseteq u} \min\Big(1, 1 - \sigma_e(A) + \tau_e\big(f^{-1}(A)\big)\Big)$
$\geq \inf_{v\in P(Y))} \min\Big(1, 1 - \sigma_e(v) + \tau_e\big(f^{-1}(v)\big)\Big) = [f \in I_e]$.

Hence

$$\inf_{x\in X} \inf_{u\in P(Y)} \min\Big(1, 1 - N_{f(x)}^{e^Y}(u) + N_x^{e^X}\big(f^{-1}(u)\big)\Big) \geq [f \in I_e].$$

Second, we prove that $[f \in I_e] \geq [f \in e_2]$. From Theorem 2.15, we have

$$\begin{aligned}
[f \in I_e] &= \inf_{u\in P(Y)} \min\Big(1, 1 - \sigma_e(u) + \tau_e\big(f^{-1}(u)\big)\Big) \\
&\geq \inf_{u\in P(Y)} \min\Big(1, 1 - \inf_{f(x)\in u} N_{f(x)}^{e^Y}(u) + \inf_{x\in f^{-1}(u)} N_x^{e^X}\big(f^{-1}(u)\big)\Big) \\
&\geq \inf_{u\in P(Y)} \min\Big(1, 1 - \inf_{x\in f^{-1}(u)} N_{f(x)}^{e^Y}(u) + \inf_{x\in f^{-1}(u)} N_x^{e^X}\big(f^{-1}(u)\big)\Big) \\
&\geq \inf_{x\in X} \inf_{u\in P(Y)} \min\Big(1, 1 - N_{f(x)}^{e^X}(u) + N_x^{e^X}\big(f^{-1}(u)\big)\Big) = [f \in e_2].
\end{aligned}$$

(c) We prove that $[f \in e_2] = [f \in e_3]$. From Theorem 2.15 we have,

$$[f \in e_3] = \inf_{x\in X} \inf_{u\in P(Y)} \min\Big(1, 1 - N^{e^Y}_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} N^{e^X}_x(v)\Big)$$
$$\geq \inf_{x\in X} \inf_{u\in P(X)} \min\Big(1, 1 - N^{e^Y}_{f(x)}(u) + N^{e^X}_x\big(f^{-1}(u)\big)\Big) = [f \in e_2].$$

(d) We prove that $[f \in e_4] = [f \in e_5]$. First, since for any fuzzy set $\widetilde{A}$ we have $[f^{-1}(f(\widetilde{A})) \supseteq \widetilde{A}] = 1$, then for any $B \in P(Y)$ we have $\Big[f^{-1}\Big(f\big(Cl^X_e(f^{-1}(B))\big)\Big) \supseteq Cl^X_e\big(f^{-1}(B)\big)\Big] = 1$. Also, since $[f(f^{-1}(B)) \subseteq B] = 1$, then we have that

$$\big[Cl^X_e\big(f(f^{-1}(B))\big) \subseteq Cl^Y_e(B)\big] = 1.$$

We have

$$\Big[Cl^X_e\big(f^{-1}(B)\big) \subseteq f^{-1}\big(Cl^Y_e(B)\big)\Big] \geq \Big[f^{-1}\Big(f\big(Cl^X_e(f^{-1}(B))\big)\Big) \subseteq f^{-1}\big(Cl^Y_e(B)\big)\Big].$$
$$\geq \Big[f^{-1}\Big(f\big(Cl^X_e(f^{-1}(B))\big)\Big) \subseteq f^{-1}\Big(Cl^Y_e\big(f(f^{-1}(B))\big)\Big)\Big].$$
$$\geq \Big[f\Big(Cl^Y_e\big(f^{-1}(B)\big)\Big) \subseteq Cl^Y_e\Big(f\big(f^{-1}(B)\big)\Big)\Big].$$

Therefore

$$[f \in e_5] = \inf_{B\in P(Y)} \Big[Cl^X_e(f^{-1}(B)) \subseteq f^{-1}\big(Cl^Y_e(B)\big)\Big]$$
$$\geq \inf_{B\in P(Y)} \Big[f\Big(Cl^X_e\big(f^{-1}(B)\big)\Big) \subseteq Cl^Y_e\Big(f(f^{-1}(B))\Big)\Big]$$
$$\geq \inf_{A\in P(X)} \Big[f\big(Cl^X_e(A)\big) \subseteq Cl^Y_e\big(f(A)\big)\Big] = [f \in e_4].$$

Second, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, we have

$$[f \in e_4] = \inf_{A\in P(X)} \big[f\big(Cl^X_e(A)\big) \subseteq Cl^Y_e\big(f(A)\big)\big]$$
$$\geq \inf_{A\in P(Y)} \Big[f\big(Cl^X_e(A)\big) \subseteq f\big(f^{-1}(Cl^Y_e(f(A)))\big)\Big]$$
$$\geq \inf_{A\in P(Y)} \Big[Cl^X_e(A) \subseteq f^{-1}\Big(Cl^Y_e\big(f(A)\big)\Big)\Big]$$
$$\geq \inf_{B\in P(Y), B=f(A)} \Big[Cl^X_e(f^{-1}(B)) \subseteq f^{-1}\big(Cl^Y_e(B)\big)\Big]$$
$$\geq \inf_{B\in P(Y)} \Big[Cl^X_e\big(f^{-1}(B)\big) \subseteq f^{-1}(Cl^Y_e(B))\Big] = [f \in e_5].$$

(e) We want to prove that $\models f \in e_2 \leftrightarrow f \in e_5$.

$$[f \in e_5] = \inf_{B\in P(Y)} \Big[Cl^X_e\big(f^{-1}(B)\big) \subseteq f^{-1}(Cl^Y_e(B))\Big]$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min\Big(1, 1 - \Big(1 - N^{e^X}_x(X - f^{-1}(B))\Big) + 1 - N^{e^Y}_{f(x)}(Y - B)\Big)$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min\Big(1, 1 - N^{e^Y}_{f(x)}(Y - B) + N^{e^X}_x\big(f^{-1}(Y - B)\big)\Big)$$
$$= \inf_{u\in P(Y)} \inf_{x\in X} \min\Big(1, 1 - N^{e^Y}_{f(x)}(u) + N^{e^X}_x\big(f^{-1}(u)\big)\Big) = [f \in e_2]. \qquad \square$$

4. Fuzzifying e-compact space

Definition 4.1. A fuzzifying topological space (X, τ) is said to be e-fuzzifying topological space if $\tau_e(A \cap B) \geq \tau_e(A) \wedge \tau_e(B)$.

Definition 4.2. A binary fuzzy predicate $K_e \in \Im\Big(\Im\big(P(X)\big) \times P(X)\Big)$, called fuzzifying e-open covering, is given as $K_e(\Re, A) := K(\Re, A) \dot{\wedge} (\Re \subseteq \tau_e)$.

Definition 4.3. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma_e \in \Im(\Omega)$, called fuzzifying e-compactness, is given as follows:
(1) $(X,\tau) \in \Gamma_e := (\forall \Re)\Big(K_e(\Re, X) \to (\exists \wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\Big)$;
(2) If $A \subseteq X$, then $\Gamma_e(A) := \Gamma_e(A, \tau/A)$.

Lemma 4.4. $\models K_\circ(\Re, A) \to K_e(\Re, A)$.

Proof. Since from Theorem 2.14 $\models \tau \subseteq \tau_e$, then we have $[\Re \subseteq \tau] \leq [\Re \subseteq \tau_e]$. So, $[K_\circ(\Re, A)] \leq [K_e(\Re, A)]$. □

Theorem 4.5. $\models (X,\tau) \in \Gamma_e \to (X,\tau) \in \Gamma$.

Proof. From Lemma 4.4 the proof is immediate. □

Theorem 4.6. *For any fuzzifying topolopgical spaces (X,τ) and $A \subseteq X$,*

$$\Gamma_e(A) \leftrightarrow (\forall \Re)\Big(K_e(\Re, A) \to (\exists \wp)\big((\wp \leq \Re) \wedge K\big(\wp, A) \dot{\wedge} FF(\wp)\big)\Big),$$

Where K_e is related to τ.

Proof. For any $\Re \in \Im\big(\Im(X)\big)$, we get $\bar{\Re} \in \Im\big(\Im(A)\big)$ defined as $\bar{\Re}(C) = \Re(B)$ with $C = A \cap B$, $B \subseteq X$. Then

$$K(\bar{\Re}, A) = \inf_{x\in A} \sup_{x \in C} \bar{\Re}(C) = \inf_{x \in A} \sup_{x \in C = A\cap B} \Re(B) = \inf_{x\in A} \sup_{x\in B} \Re(B) = K(\Re, A),$$

becaus $x \in A$ and $x \in B$ if and only if $x \in A \cap B$. Therefore

$$\begin{aligned}[\bar{\Re} \subseteq \tau_e/_A] &= \inf_{c \subseteq A} \min\big(1, 1 - \bar{\Re}(C) + \tau_e/_A(C)\big)\\ &= \inf_{c\subseteq A} \min\Big(1, 1 - \sup_{C = A\cap B, B\subseteq X} \Re(B) + \sup_{C = A\cap B, B \subseteq X} \tau_e(B)\Big)\\ &\geq \inf_{c\subseteq A, C = A\cap B, B \subseteq X} \min\big(1, 1 - \Re(B) + \tau_e(B)\big)\\ &\geq \inf_{B \subseteq X} \min\big(1, 1 - \Re(B) + \tau_e(B)\big) = [\Re \subseteq \tau_e].\end{aligned}$$

For any $\wp \leq \bar{\Re}$, we define $\wp' \in \Im\big(P(X)\big)$ as follows:

$$\wp'(B) = \begin{cases} \wp(B) & \text{if } B \subseteq A \\ 0 & \text{otherwise.}\end{cases}$$

Then $\wp' \leq \Re$, $FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$.
Furthermore, we have

$$\begin{aligned}\big[\Gamma_e(A) \dot{\wedge} K_e(\Re, A)\big] &\leq \big[\Gamma_e(A) \dot{\wedge} K'_e(\bar{\Re}, A)\big]\\ &\leq \Big[(\exists \wp)\big((\wp \leq \bar{\Re}) \wedge K(\wp, A) \dot{\wedge} FF(\wp)\big)\Big]\\ &\leq \Big[(\exists \wp')\big((\wp' \leq \Re) \wedge K(\wp', A) \dot{\wedge} FF(\wp')\big)\Big]\\ &\leq \Big[(\exists \mathcal{B})\big((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \dot{\wedge} FF(\mathcal{B})\big)\Big].\end{aligned}$$

Then $\Gamma_e(A) \leq \big[K_e(\Re, A)\big] \to \Big[(\exists \mathcal{B})\big((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \dot{\wedge} FF(\mathcal{B})\big)\Big]$, where $K'_e(\bar{\Re}A) = \big[K(\bar{\Re}, A) \dot{\wedge} (\bar{\Re} \subseteq \tau_e/_A)\big]$. Therefore

$$\Gamma_e(A) \leq \inf_{\Re\in\Im(P(X))}\Big[K_e(\Re,A)\to(\exists\mathcal{B})\Big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\Big)\Big]$$

$$= \Big[(\forall\Re)\Big(K_e(\Re,A)\to(\exists\mathcal{B})\big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\big)\Big)\Big].$$

Conversely, for any $\Re\in\Im\big(P(A)\big)$, if $\big[\Re\subseteq\tau_e/_A\big]=\inf_{B\subseteq A}\min\big(1,1-\Re(B)+\tau_e/_A(B)\big)=\lambda$, then for any $n\in N$ and $B\subseteq A$, $\sup_{B=A\cap C,C\subseteq X}\tau_e(C)=\tau_e/_A(B)>\lambda+\Re(B)-1-\frac{1}{n}$, and there exists $C_B\subseteq X$ such that $C_B\cap A=B$ and $\tau_e(C_B)>\lambda+\Re(B)-1-\frac{1}{n}$. Now, we define $\bar{\Re}\in\Im\big(P(X)\big)$ as $\bar{\Re}(C)=\max_{B\subseteq A}(0,\lambda+\Re(B)-1-\frac{1}{n})$.
Then $[\bar{\Re}\subseteq\tau_e]=1$ and

$$K(\bar{\Re},A)=\inf_{x\in A}\sup_{x\in C\subseteq X}\bar{\Re}(C)=\inf_{x\in A}\sup_{x\in B}\bar{\Re}(C_B)\geq\inf_{x\in A}\sup_{x\in B}\big(\lambda+\Re(B)-1-\tfrac{1}{n}\big)$$
$$=\inf_{x\in A}\sup_{x\in B}\Re(B)+\lambda-1-\tfrac{1}{n}=K(\Re,A)+\lambda-1-\tfrac{1}{n},$$
$$K_e(\bar{\Re},A)=\big[K(\bar{\Re},A)\dot{\wedge}(\bar{\Re}\subseteq\tau_e)\big]=\big[K(\bar{\Re},A)\big]\geq\max\big(0,K(\Re,A)+\lambda-1-\tfrac{1}{n}\big)$$
$$\geq\max\big(0,K(\Re,A)+\lambda-1\big)-\tfrac{1}{n}=K'_e(\Re,A)-\tfrac{1}{n}.$$

For any $\wp\leq\bar{\Re}$, we set $\wp'\in\Im\big(P(A)\big)$ as $\wp'(B)=\wp(C_B),B\subseteq A$. Then $\wp'\leq\Re$, $FF(\wp')=FF(\wp)$ and $K(\wp',A)=K(\wp,A)$. Therefore

$$\Big[(\forall\Re)\Big(K_e(\Re,A)\to(\exists\wp)\big((\wp\leq\Re)\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big)\Big]\dot{\wedge}[K'_e(\Re,A)]-\tfrac{1}{n}$$
$$\leq\Big[(\forall\Re)\Big(K_e(\Re,A)\to(\exists\wp)\big((\wp\leq\Re)\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big)\Big]\dot{\wedge}\Big([K'_e(\Re,A)-\tfrac{1}{n}]\Big)$$
$$\leq\Big[K_e(\bar{\Re},A)\to(\exists\wp)\Big((\wp\leq\bar{\Re})\wedge K(\wp,A)\dot{\wedge} FF(\wp)\Big)\Big]\dot{\wedge}\Big[K_e(\bar{\Re},A)\Big]$$
$$\leq\Big[(\exists\wp)\big((\wp\leq\bar{\Re})\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big]$$
$$\leq\Big[(\exists\wp')\big((\wp'\leq\Re)\wedge K(\wp',A)\dot{\wedge} FF(\wp')\big)\Big]$$
$$\leq\Big[(\exists\mathcal{B})\big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\big)\Big].$$

Let $n\to\infty$. We obtain

$$\Big[(\forall\Re)\Big(K_e(\Re,A)\to(\exists\wp)\big((\wp\leq\Re)\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big)\Big]\dot{\wedge}\big[K'_e(\Re,A)\big]\leq\Big[(\exists\mathcal{B})\big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\big)\Big].$$

Then

$$\Big[(\forall\Re)\Big(K_e(\Re,A)\to(\exists\wp)\big((\wp\leq\Re)\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big)\Big]$$
$$\leq\Big[K'_e(\Re,A)\to(\exists\mathcal{B})\big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\big)\Big]$$
$$\leq\inf_{\Re\in\Im(P(X))}\Big[K'_e(\Re,A)\to(\exists\mathcal{B})\big((\mathcal{B}\leq\Re)\wedge K(\mathcal{B},A)\dot{\wedge} FF(\mathcal{B})\big)\Big]$$
$$=\Gamma_e(A).$$
□

Theorem 4.7. *Let (X,τ) be a fuzzifying topological space.*

$$\pi_1:=(\forall\Re)\Big(\big(\Re\in\Im\big(P(X)\big)\big)\wedge(\Re\subseteq\mathcal{F}_e)\dot{\wedge} fI(\Re)\to(\exists x)(\forall A)(A\in\Re\to x\in A)\Big);$$

$\pi_2 := (\forall \Re)(\exists B)\Big(\big((\Re \subseteq \mathcal{F}_e) \wedge (B \in \tau_e)\big) \dot{\wedge}$
$(\forall \wp)\big((\wp \leq \Re) \dot{\wedge} FF(\wp) \rightarrow \neg(\cap\wp \subseteq B)\big) \rightarrow \neg(\cap\Re \subseteq B)\Big).$

Then $\models \Gamma_e(X,\tau) \leftrightarrow \pi_i$, $i = 1, 2$.

Proof. (a) We prove $\Gamma_e(X,\tau) = [\pi_1]$. For any $\Re \in \Im(P(X))$, we set $\mathcal{R}^c(X - A) = \Re(A)$. Then

$[\Re \subseteq \tau_e] = \inf_{A \in P(X)} \min\big(1, 1 - \Re(A) + \tau_e(A)\big)$
$= \inf_{X-A \in P(X)} \min\big(1, 1 - \Re^c(X - A) + \mathcal{F}_e(X - A)\big) = [\Re^c \subseteq \mathcal{F}_e],$
$FF(\Re) = 1 - \inf\{\alpha \in [0,1] : F(\mathcal{R}_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\mathcal{R}^c_\alpha)\} = FF(\Re^c)$
and

$$\mathcal{B} \leq \Re^c \Leftrightarrow \mathcal{B}(M) \Leftrightarrow \mathcal{B}^c(X - M) \leq \Re(X - M) \Leftrightarrow \mathcal{B}^c \leq \Re.$$

Therefore
$\Gamma_e(X,\tau) = \Big[(\forall\Re)\Big(K_e(\Re, A) \rightarrow (\exists\wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re \subseteq \tau_e) \dot{\wedge} K(\Re, X) \rightarrow (\exists\wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re \subseteq \tau_e) \rightarrow \big(K(\Re, X)(\exists\wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re^c \subseteq \mathcal{F}_e) \rightarrow \big((\forall x)(\exists A)(A \in \Re \wedge x \in A) \rightarrow$
$(\exists\wp)\big((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)\big)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re^c \subseteq \mathcal{F}_e) \rightarrow \big((\forall x)(\exists A)(A \in \Re \wedge x \in A) \rightarrow$
$(\exists\mathcal{B}^c)\big((\mathcal{B}^c \leq \Re) \wedge K(\mathcal{B}^c, A) \dot{\wedge} FF(\mathcal{B}^c)\big)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re^c \subseteq \mathcal{F}_e) \rightarrow \big((\forall x)(\exists A)(A \in \Re \wedge x \in A) \rightarrow$
$(\exists\mathcal{B})\big((\mathcal{B} \leq \Re^c) \wedge FF(\mathcal{B}) \dot{\wedge} K(\mathcal{B}^c, X)\big)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re^c \subseteq \mathcal{F}_e) \rightarrow \Big((\forall x)(\exists A)(A \in \Re \wedge x \in A) \rightarrow$
$(\exists\mathcal{B})\big((\mathcal{B} \leq \Re^c) \wedge FF(\mathcal{B}) \dot{\wedge} (\forall x)(\exists B)(B \in \mathcal{B}^c \wedge x \in B)\big)\Big)\Big)\Big]$
$= \Big[(\forall\Re)\Big(\Re^c \subseteq \mathcal{F}_e \rightarrow \big(\neg\big((\exists\mathcal{B})(\mathcal{B} \leq \Re^c \wedge FF(\mathcal{B}) \dot{\wedge}$
$(\forall x)(\exists B)(B \in \mathcal{B}^c \wedge x \in B)\big) \rightarrow \neg\big((\forall x)(\exists A)(A \in \Re \wedge x \in A)\big)\big)\big)\Big)\Big]$
$= \Big[(\forall\Re)\Big((\Re^c \subseteq \mathcal{F}_e) \rightarrow \Big(fI(\Re^c) \rightarrow \neg\big((\forall x)(\exists B)(A \in \Re \wedge x \in A)\big)\Big)\Big)\Big]$
$= \Big[(\forall\Re)\big((\Re^c \subseteq \mathcal{F}_e) \dot{\wedge} fI(\Re^c) \rightarrow (\exists x)(\forall A)(A \in \Re^c \rightarrow x \in A)\big)\Big] = [\pi_1].$

(b) We prove $[\pi_1] = [\pi_2]$. Let $X - B \in P(X)$. For any $\Re \in \Im(P(X))$.
$\big[(\Re \subseteq \mathcal{F}_e) \wedge (B \in \tau_e)\big] = \big[(\Re \subseteq \mathcal{F}_e) \wedge (X - B \in \mathcal{F}_e)\big]$
$= \inf_{A \in P(X)} \min\big(1, 1 - \Re(A) + \mathcal{F}_e(A)\big) \wedge \mathcal{F}_e(X - B)$
$= \inf_{A \in P(X)} \min\big(1, 1 - \Re(A) + \mathcal{F}_e(A)\big) \wedge \inf_{A \in P(X)} \min\big(1, 1 - [A \in$
$\{X - B\}] + \mathcal{F}_e(A)\big)$

$$= \inf_{A \in P(X)} \min\Big(1, 1 - \Big[\big(\Re \cup \{X - B\}\big)(A)\Big] + \mathcal{F}_e(A)\Big)$$
$$= \Big[\big(\Re \cup \{X - B\}\big) \subseteq \mathcal{F}_e\Big].$$

Therefore, for any $\mathcal{B} \in \Im\big(P(X)\big)$, let $\wp = \mathcal{B} \backslash \{X - B\} \in \Im\big(P(X)\big)$.

$$\wp(A) = \begin{cases} \mathcal{B}(A), & \text{if } A \neq X - B \\ 0, & A = X - B \end{cases}$$

Then $\wp \leq \mathcal{B}$, $\wp \cup \{X - B\} \geq \mathcal{B}$, $\big[FF(\wp)\big] = \big[FF(\mathcal{B})\big]$,

$$[\wp \leq \Re] = \big[\mathcal{B} \leq (\Re \cup \{X - B\})\big]$$

and

$$\Big[(\forall \wp)\Big((\wp \leq \Re) \dot{\wedge} FF(\wp) \to (\exists x)(\forall A)\big(A \in (\wp \cup \{X - B\}) \to (x \in A)\big)\Big)\Big]$$
$$= \inf_{\wp \leq \Re} \min\Big(1, 1 - \big[FF(\wp)\big] + \sup_{x \in X} \inf_{A \in P(X)} \big((\wp \cup \{X - B\})(A) \to A(x)\big)\Big)$$
$$\leq \inf_{\mathcal{B} \leq (\Re \cup \{X - B\})} \min\Big(1, 1 - \big[FF(\mathcal{B})\big] + \sup_{x \in X} \inf_{A \in P(X)} \big(\mathcal{B}(A) \to A(x)\big)\Big)$$
$$= fI\big(\Re \cup \{X - B\}\big).$$

Furthermore, we have

$$\pi_1 \dot{\wedge} \Big[\big((\Re \subseteq \mathcal{F}_e) \wedge (B \in \tau_e)\big) \dot{\wedge} (\forall \wp)\big((\wp \leq \Re) \dot{\wedge} FF(\wp) \to \neg(\cap \wp \subseteq B)\big)\Big]$$
$$= \pi_1 \dot{\wedge} \Big[\big(\Re \cup \{X - B\} \subseteq \mathcal{F}_e\big) \dot{\wedge} (\forall \wp)$$
$$\Big[(\wp \leq \Re) \dot{\wedge} FF(\wp) \to (\exists x)(\forall A)\big(A \in (\wp \cup \{X - B\}) \to x \in A\big)\Big]$$
$$= \pi_1 \dot{\wedge} \Big[\big(\Re \cup \{X - B\} \subseteq \mathcal{F}_e\big) \dot{\wedge} fI(\Re \cup \{X - B\})\Big]$$
$$\leq \Big[(\exists x)(\forall A)\big(A \in (\Re \cup \{X - B\}) \to x \in A\big)\Big] = \big[\neg(\cap \Re \subseteq B)\big].$$

Therefore

$$\pi_1 \leq \inf_{\Re \in \Im(P(X))} \sup_{B \subseteq X} \big((\Re \subseteq \mathcal{F}_e \wedge B \in \tau_e\big) \dot{\wedge} (\forall \wp)\big((\wp \leq \Re) \dot{\wedge} FF(\wp) \to$$
$$\neg(\cap \wp \subseteq B)\big) \to \neg(\cap \Re \subseteq B)\big) = \pi_2.$$

Conversely,

$$\pi_2 \dot{\wedge} \Big[(\Re \subseteq \mathcal{F}_e) \dot{\wedge} fI(\Re)\Big] = \pi_2 \dot{\wedge} \Big[\big((\Re \backslash \{B\}) \cup \{B\}\big) \subseteq \mathcal{F}_e\Big] \dot{\wedge} \Big[fI\big((\Re \backslash \{B\}) \cup \{B\}\big)\Big]$$
$$= \pi_2 \dot{\wedge} \Big[(\Re' \subseteq \mathcal{F}_e) \wedge (X - B \in \tau_e) \dot{\wedge} (\forall \wp)\big((\wp \leq \Re') \dot{\wedge} FF(\wp) \to$$
$$(\exists x)(\forall A)\big(A \in (\wp \cup \{B\}) \to x \in A\big)\big)\Big]$$
$$= \pi_2 \dot{\wedge} \Big[(\Re' \subseteq \mathcal{F}_e) \wedge (X - B \in \tau_e) \dot{\wedge} (\forall \wp)\big((\wp \leq \Re') \dot{\wedge} FF(\wp) \to$$
$$\neg(\cap \wp \subseteq X - B)\big)\Big]$$
$$\leq \big[\neg(\cap \Re' \subseteq X - B)\big] = \Big[(\exists x)(\forall A)\big(A \in (\Re' \cup \{B\}) \to (x \in A)\big)\Big]$$
$$= \Big[(\exists x)(\forall A)\big(A \in \Re \to (x \in A)\big)\Big].$$

Therefore

$$\pi_2 \leq \inf_{\Re \in \Im(P(X))} \Big[(\Re \subseteq \mathcal{F}_e) \dot{\wedge} fI(\Re) \to (\exists x)(\forall A)\big(A \in \Re \to (x \in A)\big)\Big] = \pi_1. \qquad \square$$

5. Some properties of fuzzifying e-compactness

Theorem 5.1. *For any fuzzifying topological space (X, τ) and $A \subseteq X$,*

$$\models \Gamma_e(X,\tau) \dot{\wedge} A \in \mathcal{F}_e \to \Gamma_e(A).$$

Proof. For any $\Re \in \Im(P(A))$, we define $\bar{\Re} \in \Im(P(X))$ as follows:

$$\bar{\Re}(B) = \begin{cases} \Re(B) & \text{if } B \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $FF(\bar{\Re}) = 1 - \inf\{\alpha \in [0,1] : F(\bar{\Re}_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\Re_\alpha)\} = FF(\Re)$ and

$$\begin{aligned}\sup_{x\in X}\inf_{x\notin B\subseteq X}(1-\bar{\Re}(B)) &= \sup_{x\in X}\Big(\big(\inf_{x\notin B\subseteq A}(1-\bar{\Re}(B))\big)\wedge\big(\inf_{x\notin B\subsetneq A}(1-\bar{\Re}(B))\big)\Big)\\ &= \sup_{x\in X}\Big(\inf_{x\notin B\subseteq A}\big(1-\bar{\Re}(B)\big)\Big)\wedge\sup_{x\in X}\Big(\inf_{x\notin B\subsetneq A}\big(1-\bar{\Re}(B)\big)\Big)\\ &= \sup_{x\in X}\Big(\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big)\Big)\\ &= \sup_{x\in A}\Big(\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big)\Big)\vee\sup_{x\notin A}\Big(\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big)\Big)\end{aligned}$$

If $x \notin A$, then for any $x' \in A$ we have

$\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big) = \inf_{B\subseteq A}\big(1-\Re(B)\big) \leq \inf_{x'\notin B\subseteq A}\big(1-\Re(B)\big)$.

Therefore, $\sup_{x\in X}\inf_{x\notin B\subseteq A}\big(1-\bar{\Re}(B)\big) = \sup_{x\in A}\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big)$,

$$\begin{aligned}\big[fI(\bar{\Re})\big] &= \Big[(\forall\bar{\mathcal{B}})\Big((\bar{\mathcal{B}}\leq\bar{\Re})\wedge FF(\bar{\mathcal{B}})\to(\exists x)(\forall B)\big((B\in\bar{\Re})\to(x\in B)\big)\Big)\Big]\\ &= \inf_{\bar{\mathcal{B}}\leq\bar{\Re}}\min\Big(1,1-FF(\bar{\mathcal{B}})+\sup_{x\in X}\inf_{x\notin B\subseteq X}\big(1-\bar{\Re}(B)\big)\Big)\\ &= \inf_{\mathcal{B}\leq\Re}\min\Big(1,1-FF(\mathcal{B})+\sup_{x\in X}\inf_{x\notin B\subseteq A}\big(1-\Re(B)\big)\Big) = [fI(\Re)].\end{aligned}$$

We want to prove that $\mathcal{F}_e(A) \dot{\wedge} \big[\Re \subseteq F_e/_A\big] \leq \big[\bar{\Re} \subseteq \mathcal{F}_e\big]$. In fact, from Theorem 2.14 (3)

we have

$$\begin{aligned}\mathcal{F}_e(A)\dot{\wedge}[\Re\subseteq F_e/_A] &= \max\Big(0,\mathcal{F}_e(A)+\inf_{B\subseteq A}\min\big(1,1-\Re(B)+F_e/_A(B)\big)-1\Big)\\ &\leq \inf_{B\subseteq A}\big(1-\Re(B)\big)+\big(\mathcal{F}_e(A)+F_e/_A(B)-1\big)\\ &\leq \inf_{B\subseteq A}\big(1-\Re(B)\big)+\big(\mathcal{F}_e(A)\wedge F_e/_A(B)\big)\\ &= \inf_{B\subseteq A}\big(1-\Re(B)\big)+\big(\mathcal{F}_e(A)\wedge\sup_{B'\cap A=B,B'\subseteq X}\mathcal{F}_e(B')\big)\\ &= \inf_{B\subseteq A}\big(1-\Re(B)\big)+\sup_{B'\cap A=B,B'\subseteq X}\big(\mathcal{F}_e(A)\wedge\mathcal{F}_e(B')\big)\\ &= \inf_{B\subseteq A}\big(1-\Re(B)\big)+\sup_{B'\cap A=B,B'\subseteq X}\big(\mathcal{F}_e(A\cap B')\big)\\ &= \inf_{B\subseteq A}\big(1-\Re(B)\big)+\mathcal{F}_e(B)\\ &= \inf_{B\subseteq A}\min\big(1,1-\Re(B)+\mathcal{F}_e(B)\big)\\ &= \inf_{B\subseteq A}\min\big(1,1-\bar{\Re}(B)+\mathcal{F}_e(B)\big) = [\bar{\Re}\subseteq\mathcal{F}_e].\end{aligned}$$

Furthermore, from Theorem 4.7 we have

$$\Gamma_e(X,\tau) \dot\wedge \mathcal{F}_e(A) \dot\wedge [\Re \subseteq F_e/_A] \dot\wedge fI(\Re) \leq \Gamma_e(X,\tau) \dot\wedge [\bar\Re \subseteq F_e] \dot\wedge fI(\bar\Re)$$
$$\leq \sup_{x\in A} \inf_{x\notin B\subseteq A} \big(1-\bar\Re(B)\big) = \sup_{x\in A} \inf_{x\notin B\subseteq A} \big(1-\Re(B)\big).$$

Then

$$\Gamma_e(X,\tau) \dot\wedge \mathcal{F}_e(A) \leq [\Re \subseteq F_e/_A] \dot\wedge fI(\Re) \to \sup_{x\in X} \inf_{x\notin B\subseteq A} \big(1-\Re(B)\big)$$
$$\leq \inf_{\Re\in\Im(P(X))} \Big([\Re \subseteq F_e/_A] \dot\wedge fI(\Re) \to \sup_{x\in X} \inf_{x\notin B\subseteq A} \big(1-\Re(B)\big)\Big) = \Gamma_e(A). \qquad \square$$

Theorem 5.2. *Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f \in Y^X$ is surjection. Then $\models \Gamma_e(X,\tau) \dot\wedge C_e(f) \to \Gamma(f(X))$.*

Proof. For any $\mathcal{B} \in \Im(P(Y))$, we define $\Re \in \Im(P(X))$ as follows:

$$\Re(A) = f^{-1}(\mathcal{B})(A) = \mathcal{B}(f(A)).$$

Then

$$K(\Re, X) = \inf_{x\in X} \sup_{x\in A} \Re(A) = \inf_{x\in X} \sup_{x\in A} \mathcal{B}(f(A))$$
$$= \inf_{x\in X} \sup_{f(x)\in B} \mathcal{B}(B) = \inf_{y\in f(x)} \sup_{y\in B} \mathcal{B}(B) = K(\mathcal{B}, f(X)),$$

$$[\mathcal{B} \subseteq \sigma] \dot\wedge [C_e(f)] = \inf_{B\subseteq Y} \min\big(1, 1-\mathcal{B}(B)+\sigma(B)\big) \dot\wedge \inf_{B\subseteq Y} \min\Big(1, 1-\sigma(B)+ \tau_e\big(f^{-1}(B)\big)\Big)$$
$$= \max\Big(0, \inf_{B\subseteq Y} \min\big(1, 1-\mathcal{B}(B)+\sigma(B)\big) + \inf_{B\subseteq Y} \min\Big(1, 1-\sigma(B)+ \tau_e\big(f^{-1}(B)\big)\Big) - 1\Big)$$
$$\leq \inf_{B\subseteq Y} \max\Big(0, \min\big(1, 1-\mathcal{B}(B)+\sigma(B)\big) + \min\Big(1, 1-\sigma(B)+\tau_e\big(f^{-1}(B)\big)\Big) - 1\Big)$$
$$\leq \inf_{B\subseteq Y} \min\Big(1, 1-\mathcal{B}(B)+\tau_e\big(f^{-1}(B)\big)\Big) = \inf_{A\subseteq X} \inf_{f^{-1}(B)=A} \min\Big(1, 1-\mathcal{B}(B)+ \tau_e\big(f^{-1}(B)\big)\Big)$$
$$= \inf_{A\subseteq X} \inf_{f^{-1}(B)=A} \min\big(1, 1-\mathcal{B}(B)+\tau_e(A)\big) = \inf_{A\subseteq X} \min\Big(1, 1- \sup_{f^{-1}(B)=A} \mathcal{B}(B)+\tau_e(A)\Big)$$
$$= \inf_{A\subseteq X} \min\big(1, 1-\Re(A)+\tau_e(A)\big) = [\Re \subseteq \tau_e].$$

For any $\wp \leq \Re$, we set $\bar\wp \in \Im(P(Y))$ defined as follows:

$$\bar\wp(f(A)) = f(\wp)(f(A)) = \wp(A),\ A \subseteq X.$$

Then $\bar\wp(f(A)) = f(\wp)(f(A)) \leq f(\Re)(f(A)) = f\big(f^{-1}(\mathcal{B})(f(A))\big) \leq \mathcal{B}(f(A))$,

$$FF(\wp) = 1 - \inf\Big\{\alpha \in [0,1] : F(\wp_{[\alpha]})\Big\} = 1 - \inf\Big\{\alpha \in [0,1] : F\big(f(\wp_{[\alpha]})\big)\Big\}$$
$$FF\big(f(\wp)\big) \leq FF(\bar\wp)$$

and

$$K(\bar\wp, f(X)) = \inf_{y\in f(X)} \sup_{y\in B} \bar\wp(B) = \inf_{y\in f(X)} \sup_{y\in B=f(A)} \wp(A) \geq \inf_{y\in f(X)} \sup_{f^{-1}(y)\in A} \wp(A)$$
$$= \inf_{x\in X} \sup_{x\in A} \wp(A) = K(\wp, X).$$

Futhermore

$[\Gamma_e(X,\tau)] \dot{\wedge} [C_e(f)] \dot{\wedge} \Big[K'_\circ\big(\mathcal{B}, f(X)\big)\Big]$

$$= [\Gamma_e(X,\tau)] \dot{\wedge} [C_e(f)] \dot{\wedge} \Big[K\big(\mathcal{B}, f(X)\big)\Big] \dot{\wedge} [\mathcal{B} \subseteq \sigma]$$
$$\leq [\Gamma_e(X,\tau)] \dot{\wedge} [\Re \subseteq \tau_e] \dot{\wedge} \big[K(\Re, X)\big] = [\Gamma_e(X,\tau)] \dot{\wedge} \big[K_e(\Re, X)\big]$$
$$\leq \Big[(\exists \wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\Big]$$
$$\leq \Big[(\exists \wp')\big((\wp' \leq \Re) \wedge K(\wp', f(X)) \dot{\wedge} FF(\wp')\big)\Big]$$

where K' is related to σ. Therefore, from Theorem 4.6 we obtain

$[\Gamma_e(X,\tau)] \dot{\wedge} [C_e(f)]$

$$\leq K'_\circ(\mathcal{B}, f(X)) \to (\exists \wp')\Big((\wp' \leq \Re) \wedge K(\wp', f(X)) \dot{\wedge} FF(\wp')\Big)$$
$$\leq \inf_{\mathcal{B} \in \Im(P(X))} \Big(K'_\circ\big(\mathcal{B}, f(X)\big) \to (\exists \wp')\big((\wp' \leq \Re) \wedge K\big(\wp', f(X)\big) \dot{\wedge} FF(\wp')\big)\Big)$$
$$= [\Gamma(f(X))].$$

□

Theorem 5.3. *Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f \in Y^X$ is surjection. Then $\models \Gamma_e(X,\tau) \dot{\wedge} I_e(f) \to \Gamma(f(X))$.*

Proof. From the proof of Theorem 5.2 we have for any $\mathcal{B} \in \Im(P(Y))$ we define $\Re \in \Im(P(X))$ as

$$\Re(A) = f^{-1}(\mathcal{B})(A) = \mathcal{B}(f(A)).$$

Then $K(\Re, X) = K(\mathcal{B}, f(X))$ and $[\mathcal{B} \subseteq \sigma_e] \dot{\wedge} [I_e(f)] \leq [\Re \subseteq \tau_e]$. For any $\wp \leq \Re$, we get $\bar{\wp} \in \Im(P(Y))$ defined as $\bar{\wp}(f(A)) = f(\wp)(f(A)) = \wp(A)$, $A \subseteq X$ and we have $FF(\wp) \leq FF(\bar{\wp})$, $K(\bar{\wp}, f(X)) \geq K(\wp, X)$. Therefore

$[\Gamma_e(X,\tau)] \dot{\wedge} [I_e(f)] \dot{\wedge} \Big[K'_e\big(\mathcal{B}, f(X)\big)\Big]$

$$= [\Gamma_e(X,\tau)] \dot{\wedge} [I_e(f)] \dot{\wedge} \Big[K\big(\mathcal{B}, f(X)\big)\Big] \dot{\wedge} [\mathcal{B} \subseteq \sigma_e]$$
$$\leq [\Gamma_e(X,\tau)] \dot{\wedge} [\Re \subseteq \tau_e] \dot{\wedge} \big[K(\Re, X)\big] = [\Gamma_e(X,\tau)] \dot{\wedge} \big[K(\Re, X)\big]$$
$$\leq \Big[(\exists \wp)\big((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)\big)\Big]$$
$$\leq \Big[(\exists \wp)\big((\wp \leq \Re) \wedge K(\bar{\wp}, f(X)) \dot{\wedge} FF(\bar{\wp})\big)\Big]$$
$$\leq \Big[(\exists \wp')\big((\wp' \leq \mathcal{B}) \wedge K(\wp', f(X)) \dot{\wedge} FF(\wp')\big)\Big],$$

Where K'_e is related to σ. Therefore, from Theorem 4.6 we obtain

$[\Gamma_e(X,\tau)] \dot{\wedge} [I_e(f)]$

$$\leq K'_e(\mathcal{B}, f(X)) \to (\exists \wp')\big((\wp' \leq \mathcal{B}) \wedge K(\wp', f(X)) \dot{\wedge} FF(\wp')\big)$$
$$\leq \inf_{\mathcal{B} \in \Im(P(X))} \Big(K'_e\big(\mathcal{B}, f(X)\big) \to (\exists \wp')\big((\wp' \leq \mathcal{B}) \wedge K\big(\wp', f(X)\big) \dot{\wedge} FF(\wp')\big)\Big)$$
$$= \Big[\Gamma_e\big(f(X)\big)\Big].$$

□

Theorem 5.4. *Let (X,τ) be any fuzzifying e-topological space and $A, B \subseteq X$. Then (1) $T_2^e(X,\tau) \dot{\wedge} \big(\Gamma_e(A) \wedge \Gamma_e(B)\big) \wedge A \cap B = \phi \models^{ws} T_2^e(X,\tau) \to$*

$(\exists U)(\exists V)\big((U \in \tau_e) \wedge (V \in \tau_e) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi)\big)$;
(2) $T_2^e(X,\tau) \dot{\wedge} \Gamma_e(A) \models^{ws} T_2^e(X,\tau) \rightarrow A \in \mathcal{F}_e$.

Proof. (1) Assume $A \cap B = \phi$ and $T_2^e(X,\tau) = t$. Let $x \in A$. Then for any $y \in B$ and $\lambda < t$, we have from Theorem 2.15 that

$$\begin{aligned}&\sup\Big\{\tau_e(P) \wedge \tau_e(Q) : x \in P, y \in Q, P \cap Q = \phi\Big\}\\ &= \sup\Big\{\tau_e(P) \wedge \tau_e(Q) : x \in P \subseteq U, y \in Q \subseteq V, U \cap V = \phi\Big\}\\ &= \sup_{U\cap V=\phi}\Big\{\sup_{x\in P\subseteq U}\tau_e(P) \wedge \sup_{y\in Q\subseteq V}\tau_e(Q)\Big\} = \sup_{U\cap V=\phi}\Big\{N_x^e(U) \wedge N_y^e(V)\Big\}\\ &\geq \inf_{x\neq y}\sup_{U\cap V=\phi}\Big\{N_x^e(U) \wedge N_y^e(V)\Big\} = T_2^e(X,\tau) = t > \lambda, \text{ i.e.,}\end{aligned}$$

there exist P_y, Q_y such that $x \in P_y$, $y \in Q_y$, $P_y \cap Q_y = \phi$ and $\tau_P(P_y) > \lambda$, $\tau_P(Q_y) > \lambda$. Set $\mathcal{B}(Q_y) = \tau_P(Q_y)$ for $y \in B$. Since $[\mathcal{B} \subseteq \tau_e] = 1$, we have
$[K_e(\mathcal{B}, B)] = [K(\mathcal{B}, B)] = \inf_{y\in B}\sup_{y\in C}\mathcal{B}(C) \geq \inf_{y\in B}\mathcal{B}(Q_y) = \inf_{y\in B}\tau_P(Q_y) \geq \lambda$.

On other hand, since $T_2^e(X,\tau) \dot{\wedge} \big(\Gamma_e(A) \wedge \Gamma_e(B)\big) > 0$, then $1-t < \Gamma_e(A) \wedge \Gamma_e(B) \leq \Gamma_e(A)$.

Therefore, for any $\lambda \in \big(1-\Gamma_e(A), t\big)$, it holds that

$$\begin{aligned}&1-\lambda < \Gamma_e(A) \leq 1 - [K_e(\mathcal{B}, B)] + \sup_{\wp\leq B}\Big\{[K(\wp, B)] \dot{\wedge} FF(\wp)\Big\}\\ &\leq 1-\lambda + \sup_{\wp\leq\mathcal{B}}\Big\{[K(\wp, B)] \dot{\wedge} FF(\wp)\Big\},\end{aligned}$$

i.e., $\sup_{\wp\leq\mathcal{B}}\Big\{K(\wp, B) \dot{\wedge} FF(\wp)\Big\} > 0$ and there exists $\wp \leq \mathcal{B}$ such that $K(\wp, B) + FF(\wp) - 1 > 0$, i.e., $1 - FF(\wp) < K(\wp, B)$. Then, $\inf\Big\{\theta : F(\wp_\theta)\Big\} < K(\wp, B)$. Now, there exist θ_1 such that $\theta_1 < K(\wp, B)$ and $F(\wp_{\theta_1})$. Since $\wp \leq \mathcal{B}$, we may write $\wp_{\theta_1} = \{Q_{y_1}, \ldots, Q_{y_n}\}$. We put $U_x = \{P_{y_1} \cap \cdots \cap P_{y_n}\}$, $V_x = \{Q_{y_1} \cap \cdots \cap Q_{y_n}\}$ and have $V_x \supseteq B$, $U_x \cap V_x = \phi$, $\tau_e(U_x) \geq \tau_e(P_{y_1}) \wedge \cdots \wedge \tau_e(P_{y_n}) > \lambda$ because (X,τ) is fuzzifying e-topological space. Also, $\tau_e(V_x) \geq \tau_e(Q_{y_1}) \wedge \cdots \wedge \tau_e(Q_{y_n}) > \lambda$. In fact, $\inf_{y\in B}\sup_{y\in D}\wp(D) = K(\wp, B) > \theta_1$, and for any $y \in B$, there exists D such that $y \in D$ and $\wp(D) > \theta_1$, $D \in \wp_{\theta_1}$. Similarly, if $\lambda \in \Big(1 - \big[\Gamma_e(A) \wedge \Gamma_e(B)\big], t\Big)$, then we can find $x_1, \ldots, x_m \in A$ with $U_\circ = U_{x_1} \cup \cdots \cup U_{x_m} \supseteq A$. By putting $V_\circ = V_{x_1} \cap \cdots \cap V_{x_m}$ we obtain $V_\circ \supseteq B$, $U_\circ \cap V_\circ = \phi$ and

$$\begin{aligned}&(\exists U)(\exists V)\Big((U \in \tau_e) \wedge (V \in \tau_e) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi)\Big)\\ &\geq \tau_e(U_\circ) \wedge \tau_e(V_\circ) \geq \min_{i=1,\ldots,n}\tau_e(U_{x_i}) \wedge \min_{i=1,\ldots,n}\tau_e(V_{x_i}) \geq \lambda.\end{aligned}$$

Finally, we let $\lambda \rightarrow t$ and complete the proof.

(2) Assume $\models^{ws} T_2^e(X,\tau) \dot{\wedge} \Gamma_e(A)$. For any $x \in X - A$we have from (1)

$$\sup_{x\in U\subseteq X-A}\tau_e(U) \geq \sup\Big\{\tau_e(U) \wedge \tau_e(V) : x \in U, A \subseteq V, U \cap V = \phi\Big\} \geq [T_2^e(X,\tau)].$$

From Theorem 2.15, we obtain,

$$\mathcal{F}_e(A) = \inf_{x\in X-A}N_x^e(X-A) = \inf_{x\in X-A}\sup_{x\in U\subseteq X-A}\tau_e(U) \geq [T_2^e(X,\tau)].$$

□

Definition 5.5. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_e \in \Im(Y^X)$, called fuzzifying e-closedness, is given as follows:

$$Q_e(f) := \forall B\big(B \in \mathcal{F}_e^X \to f^{-1}(B) \in \mathcal{F}_e^Y\big),$$

where $\mathcal{F}_e^X$ and $\mathcal{F}_e^Y$ are the fuzzy families of τ, σ-e-closed in X and Y respectively

Theorem 5.6. *Let (X,τ) a fuzzifying topological space, (Y,σ) be an e-fuzzifying topological space and $f \in Y^X$. Then $\models \Gamma_e(X,\tau) \dot\wedge T_2^e(Y,\sigma) \dot\wedge I_e(f) \to Q_e(f)$.*

Proof. For any $A \subseteq X$, we have the following:

(i) From Theorm 5.1 we have $\big[\Gamma_e(X,\tau) \dot\wedge \mathcal{F}_e^X(A)\big] \le \Gamma_e(A)$;

(ii)
$$\begin{aligned} I_e(f/_A) &= \inf_{U\in P(Y)} \min\Big(1, 1-\sigma_e(U) + \tau_e/_A\big((f/_A)^{-1}(U)\big)\Big)\\ &= \inf_{U\in P(Y)} \min\Big(1, 1-\sigma_e(U) + \tau_e/_A\big(A \cap f^{-1}(U)\big)\Big)\\ &= \inf_{U\in P(Y)} \min\Big(1, 1-\sigma_e(U) + \sup_{A\cap f^{-1}(U)=B\cap A} \tau_e(B)\Big)\\ &\ge \inf_{U\in P(Y)} \min\Big(1, 1-\sigma_e(U) + \tau_e\big(f^{-1}(U)\big)\Big) = I_e(f). \end{aligned}$$

(iii) From Theorem 5.3, we have $\Big[\Gamma_e(A) \wedge I_e(f/_A)\Big] \le \Gamma_e\big(f(A)\big)$.

(iv) From Theorem 5.4(2) we have $T_2^e(X,\tau) \dot\wedge \Gamma_e(A) \models^{ws} T_2^e(Y,\sigma) \to f(A) \in \mathcal{F}_e^Y$, which implies $\models T_2^e(Y,\sigma) \dot\wedge \Gamma_e\big(f(A)\big) \to f(A) \in \mathcal{F}_e^Y$. By combining (i)-(iv) we have

$$\begin{aligned} \Big[\Gamma_e(X,\tau) \dot\wedge T_2^e(Y,\sigma) \dot\wedge I_e(f)\Big] &\le \Big[\big(\mathcal{F}_e^X(A) \to \Gamma_e(A)\big) \dot\wedge I_e(f/_A) \dot\wedge T_2^e(Y,\sigma)\Big]\\ &\le \Big[\big(\mathcal{F}_e^X(A) \to \big(\Gamma_e(A)\big) \dot\wedge I_e(f/_A)\big) \dot\wedge T_2^e(Y,\sigma)\Big]\\ &\le \Big[\mathcal{F}_e^X(A) \to \big(\Gamma_e(A)\big) \dot\wedge T_2^e(Y,\sigma)\Big]\\ &\le \Big[\mathcal{F}_e^X(A) \to \mathcal{F}_e^Y\big(f(A)\big)\Big]. \end{aligned}$$

Therefore

$$\begin{aligned} \Big[\Gamma_e(A) \dot\wedge T_2^e(X,\tau) \dot\wedge I_e(f)\Big] &\le \Big[\mathcal{F}_e^X(A) \to \mathcal{F}_e^Y\big(f(A)\big)\Big]\\ &\le \inf_{A\subseteq X} \Big(\Big[\mathcal{F}_e^X(A) \to \mathcal{F}_e^Y\big(f(A)\big)\Big]\Big) = Q_e(f). \end{aligned}$$ □

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