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Generalized group soft topology

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Abstract. In the present paper, a notion of generalized group soft topology is introduced and some of the important properties of the resulting generalized group soft topological space are studied. Also some definitions and results on soft sets, soft mappings and soft topologies are established in this context.

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1. Introduction

The theory of soft sets which was introduced by Molodtsov [16] in 1999 has become an important tool for dealing with uncertainties inherent in the problems of physical science, biological science, engineering, economics, social science, medical science etc. Researchers are continuing their study to investigate the behaviour of different mathematical fields such as algebra, topology etc. and their hybrid structures in soft set theoretic form. In 2003, Maji et al. [12] worked on some mathematical aspects of soft sets, and, in 2007, Aktas and Cagman [1] introduced a basic version of soft group theory. Feng [6] in 2008, dealt with the concept of soft semirings; Shabir and Ali (2009) [23] studied soft semigroups and soft ideals; Kharal and Ahmed [11] as well as Majumdar and Samanta [14] defined soft mappings, Babitha and Sunil [3] worked on soft relations and functions etc. Recently, Shabir and Naz [24] came up with an idea of soft topological spaces. Later Zorlutuna et al. [26], Cagman et al. [4], Hussain, Ahmad [10] and Hazra et al. [8], Min [15], Varol et al. [25] and Peyghan et al. [22] studied on soft topological spaces. As a continuation of this, it is natural to investigate the behaviour of a combination of algebraic and topological structures in soft-set theoretic form. In view of this, and also considering the importance of topological group structure in developing Haar measure and Haar integral in [9], we have introduced in [20], a notion of group

soft topology where the continuity of the composition function $(x, y) \to xy^{-1}$ over the group is considered with respect to soft topology. In this paper we proceed further. In fact, in stead of taking ordinary group composition mapping, we have considered here soft composition mapping and studied the resulting soft topological group structure, which is defined here as generalized group soft topology. For this, we investigate various properties of soft mappings and soft topological spaces. In this connection, it is also worth noting that, in fuzzy setting, some significant works have been done on fuzzy topological group structure by Foster [7], Liang and Hai [5]. The present authors have already established, in soft setting, the concept of topological groups in [17, 18, 20, 21], however, approaching from different perspectives from that of the present paper. The organization of the paper is as follows:
Section 2 is the preliminary section where definitions and some properties of a soft set, soft mapping are given. In section 3, definitions of soft topological space and enriched soft topological spaces are given and studied their properties in the context of soft mappings. In section 4, we introduce generalized group soft topology and some properties of it are studied. The straightforward proofs of the theorems are omitted.

2. Preliminaries

Following Molodtsov [16] and Maji et al. [12] some definitions and preliminary results of soft sets are presented in this section. Unless otherwise stated, X will be assumed to be an initial universal set and E will be taken to be a set of parameters. Let $P(X)$ denote the power set of X and $S(X, E)$ denote the set of all soft sets over X under the parameter set E.

Definition 2.1 ([16, 12])**.** A pair (F, A) is called a *soft set* over X, where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

Definition 2.2 ([12])**.** Let (F, A) and (G, B) be two soft sets over X. Then their *union* is a soft set (H, C) over X where $C = A \cup B$ and for all $\alpha \in C$,

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A/B \\ G(\alpha) & \text{if } \alpha \in B/A \\ F(\alpha) \cup G(\alpha) & \text{if } \alpha \in A \cap B \end{cases}$$

This relationship is written as $(F, A) \,\widetilde{\cup}\, (G, B) = (H, C)$.

Definition 2.3 ([12])**.** Let (F, A) and (G, B) be two soft sets over X. Then their *intersection* is a soft set (H, C) over X where $C = A \cap B$ and for all $\alpha \in C$, $H(\alpha) = F(\alpha) \cap G(\alpha)$.
This relationship is written as $(F, A) \,\widetilde{\cap}\, (G, B) = (H, C)$.

Remark 2.4 ([13])**.** Let E be the set of parameters and $A \subseteq E$. Then for each soft set (F, A) over X we construct a soft set (H, E) over X, where for all $\alpha \in E$,

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A \\ \phi & \text{if } \alpha \in E/A \end{cases}$$

Thus the soft sets (F, A) and (H, E) are equivalent to each other and the usual set operations of the soft sets $(F_i, A_i),\ i \in \Delta$ is the same as the soft sets $(H_i, E),\ i \in \Delta$.

The following definitions and results are presented in this form.

Definition 2.5 ([17]). A pair (F, A), where F is a mapping from A to $P(X)$, is called a *soft set* over X.

Let (F_1, A) and (F_2, A) be two soft sets over a common universe X. Then (F_1, A) is said to be *soft subset* of (F_2, A) if $F_1(\alpha) \subseteq F_2(\alpha)$, for all $\alpha \in A$. This relation is denoted by $(F_1, A)\widetilde{\subseteq}(F_2, A)$. (F_1, A) is said to be *soft equal* to (F_2, A) if $F_1(\alpha) = F_2(\alpha)$, for all $\alpha \in A$. It is denoted by $(F_1, A) = (F_2, A)$.

The *complement* of a soft set (F, A) is defined as $(F, A)^c = (F^c, A)$, where $F^c(\alpha) = (F(\alpha))^c = X - F(\alpha)$, for all $\alpha \in A$.

A soft set (F, A) over X is said to be *a null soft set*$\Big($*an absolute soft set*$\Big)$ if $F(\alpha) = \phi$ $\Big(F(\alpha) = X\Big)$, for all $\alpha \in A$. This is denoted by $\widetilde{\Phi}\Big(\widetilde{A}\Big)$.

Definition 2.6 ([17]). Let $\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe X. Then their

(a) *Intersection*, denoted by $\widetilde{\cap}_{i\in I}$, is defined by $\widetilde{\cap}_{i\in I}(F_i, A) = (\widetilde{\cap}_{i\in I}F_i, A)$, where $(\widetilde{\cap}_{i\in I}F_i)(\alpha) = \cap_{i\in I}(F_i(\alpha))$, for all $\alpha \in A$.

(b) *Union*, denoted by $\widetilde{\cup}_{i\in I}$, is defined by $\widetilde{\cup}_{i\in I}(F_i, A) = (\widetilde{\cup}_{i\in I}F_i, A)$, where $(\widetilde{\cup}_{i\in I}F_i)(\alpha) = \cup_{i\in I}(F_i(\alpha))$, for all $\alpha \in A$.

Definition 2.7. A soft set $(F, A) \in S(X, A)$ is called a *pseudo constant* soft set if $F(\alpha) = \phi$ *or* X, for all $\alpha \in A$; $CS(X, A)$ denotes the set of all pseudo constant soft sets over X under the parameter set A.

Definition 2.8 ([19]). A soft set (E, A) over X is said to be a *soft element* if there exists $\alpha \in A$ such that $E(\alpha)$ is a singleton, say, $\{x\}$ and $E(\beta) = \phi$, for all $\beta(\neq \alpha) \in A$. Such a soft element is denoted by E^x_α. Let $\mathcal{E}$ be the set of all soft elements of the universal set X.

Also for a given $P \subseteq X$, a soft set (E, A) where $E(\alpha) = P$, for all $\alpha \in A$ is called a *constant soft set* and is denoted by E_P and in particular if $P = \{x\}$, then E_x denote the constant soft set where $E(\alpha) = \{x\}$, for all $\alpha \in A$.

Definition 2.9 ([19]). The soft element E^x_α is said to be in the soft set (G, A), denoted by $E^x_\alpha \,\widetilde{\in}\, (G, A)$ if $x \in G(\alpha)$.

Proposition 2.10 ([19]). *$E^x_\alpha \,\widetilde{\in}\, (F, A)$ iff $E^x_\alpha \,\widetilde{\notin}(F^c, A)$.*

Definition 2.11 ([11]). Let $S(X, A)$ and $S(Y, B)$ be the families of all soft sets over X and Y respectively. The mapping $f_\varphi : S(X, A) \to S(Y, B)$ is called a *soft mapping* from X to Y, where $f : X \to Y$ and $\varphi : A \to B$ are two mappings. Also

(i) the *image* of a soft set $(F, A) \in S(X, A)$ under the mapping f_φ is denoted by $f_\varphi[(F, A)] = (f_\varphi(F), B)$, and is defined by

$$[f_\varphi(F)](\beta) = \begin{cases} \bigcup_{\alpha\in\varphi^{-1}(\beta)} \left[f[F(\alpha)]\right] & \text{if } \varphi^{-1}(\beta) \neq \phi \\ \phi & \text{otherwise} \end{cases}$$

for all $\beta \in B$.

(ii) the *inverse image* of a soft set $(G, B) \in S(Y, B)$ under the mapping f_φ is denoted by $f_\varphi^{-1}[(G, B)] = (f_\varphi^{-1}(G), A)$, and is defined by $[f_\varphi^{-1}(G)](\alpha) = f^{-1}[G[\varphi(\alpha)]]$, for all $\alpha \in A$.

(iii) If f_φ is bijective, then the inverse soft mapping, denoted by $(f_\varphi)^{-1}$, is defined by $(f_\varphi)^{-1} = f_{\varphi^{-1}}^{-1}$.

Note: If $(f_\varphi)^{-1}$ exists, then $f_\varphi^{-1}[(G, B)] = f_{\varphi^{-1}}^{-1}[(G, B)]$.

(iv) The soft mapping f_φ is called *injective* (*surjective*) if f and φ are both injcetive (surjective).

(v) The soft mapping f_φ is said to be *constant*, if f is constant.

(vi) The soft mapping f_φ is *identity soft mapping*, if f and φ are both clasical identity mappings.

Definition 2.12 ([2])**.** Let $f_\varphi : S(X, A) \to S(Y, B)$ and $g_\psi : S(Y, B) \to S(Z, C)$ be two soft mappings, then the composition of f_φ and g_ψ is denoted by $g_\psi \circ f_\varphi$ and is defined by $g_\psi \circ f_\varphi = (g \circ f)_{\psi\circ\varphi}$.

Proposition 2.13 ([11])**.** *Let X and Y be two nonempty sets and $f_\varphi : S(X, A) \to S(Y, B)$ be a soft mapping. If (F, A), $(F_i, A) \in S(X, A)$ and (G, B), $(G_i, B) \in S(Y, B)$, $i \in \Delta$ then*

(i) $(F_1, A) \widetilde{\subseteq} (F_2, A) \Rightarrow f_\varphi[(F_1, A)] \widetilde{\subseteq} f_\varphi[(F_2, A)]$;

(ii) $(G_1, B) \widetilde{\subseteq} (G_2, B) \Rightarrow f_\varphi^{-1}[(G_1, B)] \widetilde{\subseteq} f_\varphi^{-1}[(G_2, B)]$;

(iii) $(F, A) \widetilde{\subseteq} f_\varphi^{-1}[f_\varphi(F, A)]$, *the equality holds if f_φ is injective.*

(iv) $f_\varphi[f_\varphi^{-1}(G, B)] \widetilde{\subseteq} (G, B)$, *the equality holds if f_φ is surjective.*

(v) $f_\varphi\big[\widetilde{\bigcup}_{i\in\Delta}(F_i, A)\big] = \widetilde{\bigcup}_{i\in\Delta}[f_\varphi(F_i, A)]$;

(vi) $f_\varphi\big[\widetilde{\bigcap}_{i\in\Delta}(F_i, A)\big] \widetilde{\subseteq} \widetilde{\bigcap}_{i\in\Delta}[f_\varphi(F_i, A)]$, *the equality holds if f_φ is injective.*

(vii) $f_\varphi^{-1}\big[\widetilde{\bigcup}_{i\in\Delta}(F_i, A)\big] = \widetilde{\bigcup}_{i\in\Delta}[f_\varphi^{-1}(F_i, A)]$;

(viii) $f_\varphi^{-1}\big[\widetilde{\bigcap}_{i\in\Delta}(F_i, A)\big] = \widetilde{\bigcap}_{i\in\Delta}[f_\varphi^{-1}(F_i, A)]$;

Proposition 2.14. *Let X and Y be two nonempty sets and $f_\varphi : S(X, A) \to S(Y, B)$ be a soft mapping. If $(F, A) \in S(X, A)$ and $(G, B) \in S(Y, B)$ then*

(i) $(G, B) \in CS(Y, B) \Rightarrow f_\varphi^{-1}[(G, B)] \in CS(X, A)$;

(ii) $(F, A) \in CS(X, A)$ *and f is surjective* $\Rightarrow f_\varphi[(F, A)] \in CS(Y, B)$.

Proof. (i) If $(G, B) \in CS(Y, B)$ and $G(\beta_j) = Y,\ G(\beta_k) = \phi,\ \beta_j,\ \beta_k \in B$, then for all $\alpha \in A$

$$[f_\varphi^{-1}(G)](\alpha) = \begin{cases} X & \text{if } \varphi(\alpha) = \beta_j \\ \phi & \text{if } \varphi(\alpha) = \beta_k \end{cases}$$

So, $(G, B) \in CS(Y, B) \Rightarrow f_\varphi^{-1}[(G, B)] \in CS(X, A)$.

(ii) If $(F, A) \in CS(X, A)$ and $F(\alpha_i) = X,\ F(\alpha_j) = \phi,\ \alpha_i,\ \alpha_j \in A$, then for all $\beta \in B$

$$[f_\varphi(F)](\beta) = \begin{cases} Y & \text{if } \varphi^{-1}(\beta) \neq \phi, \text{ there exists } \alpha_i \in \varphi^{-1}(\beta) \text{ and } f \text{ is surjective} \\ \phi & \text{if } \varphi^{-1}(\beta) \neq \phi \text{ and for all } \alpha_i \notin \varphi^{-1}(\beta) \\ \phi & \text{if } \varphi^{-1}(\beta) = \phi \end{cases}$$

So, $(F, A) \in CS(X, A)$ and f is surjective $\Rightarrow f_\varphi[(F, A)] \in CS(Y, B)$. $\square$

Proposition 2.15 ([2]). *Let $f_\varphi : S(X,A) \to S(Y,B)$ be a soft mapping. If $E^x_\alpha \in S(X,A)$, then $f_\varphi[E^x_\alpha] = E^{f(x)}_{\varphi(\alpha)} \in S(Y,B)$.*

Proposition 2.16. *Let $f_\varphi : S(X,A) \to S(Y,B)$ be a constant mapping and $\varphi : A \to B$ is surjective. Then for all $(F,A) \in S(X,A)$, $f_\varphi[(F,A)]$ is a constant soft set $E_{y_0} \in S(Y,B)$ if $f(x) = y_0$, for all $x \in X$.*

Proof. Let $\beta \in B$. Since φ is surjective, it follows that $\varphi^{-1}(\beta) \neq \phi$.
So, $[f_\varphi[(F,A)]](\beta) = \bigcup_{\alpha \in \varphi^{-1}(\beta)}[f[F(\alpha)]] = \{y_0\}$, for all $\beta \in B$.
Therefore, $f_\varphi[(F,A)] = E_{y_0} \in S(Y,B)$. □

Definition 2.17 ([3]). Let $(F,A) \in S(X,A)$ and $(G,B) \in S(Y,B)$. The *Cartesian product* of (F,A) and (G,B) is denoted by $(F,A)\widetilde{\times}(G,B) = (F\widetilde{\times}G, A \times B) \in S(X \times Y, A \times B)$ and defined as for all $(\alpha,\beta) \in A \times B$, $(F\widetilde{\times}G)(\alpha,\beta) = F(\alpha) \times G(\beta)$.

Proposition 2.18 ([2]). *Let X, Y be two nonempty sets and A, B be two sets of parameters. If $\pi^1 : X \times Y \to X$, $\pi^2 : X \times Y \to Y$, $\varphi_1 : A \times B \to A$ and $\varphi_2 : A \times B \to B$ are projection mappings, then,*
$[\pi^1_{\varphi_1}[(F,A) \widetilde{\times} (G,B)]](\alpha) = \bigcup_{\alpha=\varphi_1(\alpha,\beta)} \pi^1[F(\alpha) \times G(\beta)] = F(\alpha)$, for all $\alpha \in A$ and $[\pi^2_{\varphi_2}[(F,A) \widetilde{\times} (G,B)]](\beta) = \bigcup_{\beta=\varphi_2(\alpha,\beta)} \pi^2[F(\alpha) \times G(\beta)] = G(\beta)$, for all $\beta \in B$. Thus $\pi^1_{\varphi_1}$ and $\pi^2_{\varphi_2}$ are soft projection mappings, where $[\pi^1_{\varphi_1}][(F,A)\widetilde{\times}(G,B)] = (F,A)$ and $[\pi^2_{\varphi_2}][(F,A)\widetilde{\times}(G,B)] = (G,B)$.

3. Soft topological spaces

In this section we have introduced the notion of product of soft topological spaces and studied some continuity properties of mappings over product spaces; the soft topology that we have considered is in the sense of Shabir and Naz [24] and an enriched soft topological space in the sence of Aygun et.al and the authors [2, 20]. Unless otherwise stated, X is an initial universal set, A is the nonempty set of parameters and $S(X,A)$ denotes the collection of all soft sets over X under the parameter set A.

Definition 3.1 ([24]). Let τ be a collection of soft sets over X. Then τ is said to be a *soft topology* on X if

(i) $(\widetilde{\phi}, A)$, $(\widetilde{X}, A) \in \tau$ where $\widetilde{\phi}(\alpha) = \phi$ and $\widetilde{X}(\alpha) = X$, for all $\alpha \in A$.
(ii) the intersection of any two soft sets in τ belongs to τ.
(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X,A,τ) is called a *soft topological space* over X.

Definition 3.2 ([2, 20]). A subcollection τ of $S(X,A)$ is said to be an *enriched soft topology* on X if

(i) $(F,A) \in \tau$, for all $(F,A) \in CS(X,A)$;
(ii) the intersection of any two soft sets in τ belongs to τ;
(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X,A,τ) is called an *enriched soft topological space* over X.

Proposition 3.3 ([24]). *Let (X,A,τ) be a soft topological space over X. Then the collection $\tau^\alpha = \{F(\alpha) : (F,A) \in \tau\}$ for each $\alpha \in A$, defines a topology on X.*

Proposition 3.4 ([19])**.** *If (X, A, τ) is a soft topological space and if $\tau^* = \{(F, A) \in S(X, A) : \ F(\alpha) \in \tau^\alpha$, for all $\alpha \in A\}$, then τ^* is a soft topology on X such that $[\tau^*]^\alpha = \tau^\alpha$, for all $\alpha \in A$.*

Remark 3.5. If (X, A, τ) be a soft topological space which is enriched or not and if $\tau^* = \{(F, A) \in S(X, A) : \ F(\alpha) \in \tau^\alpha$, for all $\alpha \in A\}$ then τ^* is an enriched soft topology on X such that $[\tau^*]^\alpha = \tau^\alpha$, for all $\alpha \in A$.

Proposition 3.6 ([24])**.** *Let (X, A, τ_1) and (X, A, τ_2) be two soft topological spaces over X. Then $(X, A, \tau_1 \widetilde{\cap} \tau_2)$ where $\tau_1 \widetilde{\cap} \tau_2 = \{(F, A) : \ (F, A) \in \tau_1$ & $(F, A) \in \tau_2\}$ is a soft topological space over X. But the union of two soft topological spaces over X may not be a soft topological space over X.*

Definition 3.7 ([2])**.** Let (X, A, τ) and (Y, B, ν) be two soft topological spaces. A soft mapping $f_\varphi : (X, A, \tau) \to (Y, B, \nu)$ is said to be *soft continuous* if for each open soft set $(G, B) \in \nu$, the inverse image $(f_\varphi)^{-1}[(G, B)]$ is in τ.

Definition 3.8 ([2])**.** Let (X, A, τ) and (Y, B, ν) be two soft topological spaces. A soft mapping $f_\varphi : (X, A, \tau) \to (Y, B, \nu)$ is said to be *soft open* if for each open soft set (F, A) in τ, the image $f_\varphi[(F, A)]$ is in ν.

Proposition 3.9. *Let (X, A, τ) and (Y, B, ν) be two soft topological spaces. A soft mapping $f_\varphi : (X, A, \tau) \to (Y, B, \nu)$ is soft continuous iff for all $x \in X$, for all $\alpha \in A$ and for all $(V, B) \in \nu$ with $E_{\varphi(\alpha)}^{f(x)} \widetilde{\in} (V, B)$, there exists $(U, A) \in \tau$ with $E_\alpha^x \widetilde{\in} (U, A)$ such that $f_\varphi[(U, A)] \widetilde{\subseteq} (V, B)$.*

Proof. Let $f_\varphi : (X, A, \tau) \to (Y, B, \nu)$ be soft continuous.
Let $x \in X$, $\alpha \in A$ and $(V, B) \in \nu$ such that $E_{\varphi(\alpha)}^{f(x)} \widetilde{\in} (V, B)$.
Since $E_{\varphi(\alpha)}^{f(x)} \widetilde{\in} (V, B)$, it follows that $\varphi(\alpha) = \beta \in B$ and $f(x) \in V(\beta)$.
Again since f_φ is soft continuous, $f_\varphi^{-1}[(V, B)] \in \tau$.
Let $(U, A) = f_\varphi^{-1}[(V, B)]$.
Also, $E_{\varphi(\alpha)}^{f(x)} \widetilde{\subseteq} (V, B) \Rightarrow E_\alpha^x \widetilde{\subseteq} f_\varphi^{-1}[f_\varphi(E_\alpha^x)] = f_\varphi^{-1}[E_{\varphi(\alpha)}^{f(x)}] \widetilde{\subseteq} f_\varphi^{-1}[(V, B)] = (U, A)$.
Then $x \in U(\alpha)$. Thus $E_\alpha^x \widetilde{\in} (U, A)$ and
$f_\varphi[(U, A)] = f_\varphi[f_\varphi^{-1}(V, B)] \widetilde{\subseteq} (V, B)$.
Therefore the given condition is satisfied.

Conversely let the given condition be satisfied.
Let $(V, B) \in \nu$ and $E_\alpha^x \widetilde{\in} f_\varphi^{-1}[(V, B)]$.
Then $x \in [f_\varphi^{-1}[(V, B)]](\alpha) = f^{-1}[V[\varphi(\alpha)]]$.
So $f(x) \in ff^{-1}[V(\varphi(\alpha))] \ \subseteq \ V[\varphi(\alpha)]$ and hence $E_{\varphi(\alpha)}^{f(x)} \widetilde{\in} (V, B)$. Then there exists $(U(x, \alpha), A) \in \tau$ such that $E_\alpha^x \widetilde{\in} (U(x, \alpha), A)$ and $f_\varphi[(U(x, \alpha), A)] \widetilde{\subseteq} (V, B)$ i.e. $(U(x, \alpha), A) \widetilde{\subseteq} f_\varphi^{-1}[(V, B)]$.
Therefore,

$$\begin{aligned} f_\varphi^{-1}[(V, B)] &= \widetilde{\cup}\{E_\alpha^x : \ E_\alpha^x \widetilde{\in} f_\varphi^{-1}[(V, B)]\} \\ &\widetilde{\subseteq} \widetilde{\cup}\{(U(x, \alpha), A) : \ E_\alpha^x \widetilde{\in} f_\varphi^{-1}[(V, B)]\} \\ &\widetilde{\subseteq} f_\varphi^{-1}[(V, B)]. \end{aligned}$$

Thus $f_{\varphi}^{-1}[(V,B)] = \widetilde{\bigcup}\{(U(x,\alpha),A) : \ E_{\alpha}^{x} \,\widetilde{\in}\, f_{\varphi}^{-1}[(V,B)]\} \in \tau$.
So $f_{\varphi} : (X,A,\tau) \to (Y,B,\nu)$ is soft continuous. □

Proposition 3.10 ([2]). *Let (X,A,τ) and (Y,B,ν) be two enriched soft topological spaces and $\varphi : A \to B$ be any mapping. Then the mapping $f_{\varphi} : (X,A,\tau) \to (Y,B,\nu)$ where $f(x) = y_0$ (a fixed element of Y), for all $x \in X$, is soft continuous.*

Remark 3.11. It can be easily shown that the above result is also true if (Y,B,ν) is not an enriched soft topological space.

Proposition 3.12. *Let (X,A,τ) be a soft topological space. Then the identity soft mapping $f_{\varphi} : (X,A,\tau) \to (X,A,\tau)$ is soft continuous.*

Proof. Let $(F,A) \in \tau$.
Then $f_{\varphi}^{-1}[(F,A)](\alpha) = f^{-1}[F(\varphi(\alpha))] = f^{-1}[F(\alpha)] = F(\alpha)$, for all $\alpha \in A$ and hence $f_{\varphi}^{-1}[(F,A)] = (F,A) \in \tau$.
Therefore, f_{φ} is soft continuous. □

Proposition 3.13 ([2]). *Let (X,A,τ), (Y,B,ν) and (Z,C,ω) be soft topological spaces. If $f_{\varphi} : (X,A,\tau) \to (Y,B,\nu)$ and $g_{\psi} : (Y,B,\nu) \to (Z,C,\omega)$ are soft continuous, then the mapping $g_{\psi} \circ f_{\varphi} : (X,A,\tau) \to (Z,C,\omega)$ is soft continuous.*

Proposition 3.14 ([20]). *Let (X,A,τ) and (Y,B,ν) be two soft topological spaces. Then $\mathcal{F} = \{(F,A) \,\widetilde{\times}\, (G,B) : \ (F,A) \in \tau, \ (G,B) \in \nu\}$ forms an open base of the soft topology on $X \times Y$.*

Definition 3.15 ([20]). The soft topology in $X \times Y$ induced by the open base $\mathcal{F}$ is said to be the *product soft topology* of the soft topologies τ and ν. It is denoted by $\tau \,\widetilde{\times}\, \nu$.

Proposition 3.16. *Let (X,A,τ) and (Y,B,ν) be two enriched soft topological spaces. Then $(X \times Y, \ A \times B, \ \tau \,\widetilde{\times}\, \nu)$ is an enriched soft topological space.*

Proof. Since (X,A,τ) and (Y,B,ν) be two soft topological spaces, it follows that $(X \times Y, \ A \times B, \ \tau \,\widetilde{\times}\, \nu)$ is a soft topological space. Now we are to prove that $(\tau \,\widetilde{\times}\, \nu)$ is an enriched soft topology.
For this, let $(H, A \times B) \in CS(X \times Y)$ and $(H, A \times B) = \widetilde{\bigcup}_{i,j}(H_{ij}, A \times B)$ where $H_{ij}(\alpha_k, \beta_l) = X \times Y$ if $i = k, \ j = l$ and $H_{ij}(\alpha_k, \beta_l) = \phi$, otherwise.
Let $(F_i, A) \in S(X,A)$ such that $F_i(\alpha_i) = X, \ F(\alpha_k) = \phi$ if $k \neq i$ and
$(G_j, B) \in S(Y,B)$ such that $G_j(\beta_j) = Y, \ G_j(\beta_k) = \phi$ if $k \neq j$.
Then $(F_i, A) \in CS(X,A), \ (G_j, B) \in CS(Y,B)$ such that
$(F_i, A) \,\widetilde{\times}\, (G_j, B) = (H_{ij}, A \times B)$.
Since (X,A,τ) and (Y,B,ν) be two enriched soft topological spaces, it follows that $(F_i, A) \in \tau, \ (G_j, B) \in \nu$ and hence $(H_{ij}, A \times B) \in (\tau \,\widetilde{\times}\, \nu)$.
Thus $(H, A \times B) \in (\tau \,\widetilde{\times}\, \nu)$, for all $(H, A \times B) \in CS(X \times Y, A \times B)$.
Therefore $(\tau \,\widetilde{\times}\, \nu)$ is an enriched soft topology on $X \times Y$ and hence
$(X \times Y, \ A \times B, \ \tau \,\widetilde{\times}\, \nu)$ is an enriched soft topological space. □

Proposition 3.17. *Let $(X \times Y, A \times B, \tau \widetilde{\times} \nu)$ be the product soft topological space of two soft topological spaces (X,A,τ) and (Y,B,ν). Then the soft projection mappings $\pi_{\varphi_1}^{1} : (X \times Y, A \times B, \tau \widetilde{\times} \nu) \to (X,A,\tau)$ and $\pi_{\varphi_2}^{2} : (X \times Y, A \times B, \tau \widetilde{\times} \nu) \to (Y,B,\nu)$*

are soft continuous and soft open. Also $\tau\widetilde{\times}\nu$ *is the smallest enriched soft topology in* $X \times Y$ *for which the soft projection mappings are soft continuous.*

Proof. Let $(F, A) \in \tau$.
Then $([\pi^1_{\varphi_1}]^{-1}[(F, A)])(\alpha, \beta) = [\pi^1]^{-1}(F[\varphi_1(\alpha, \beta)]) = [\pi^1]^{-1}[F(\alpha)] = F(\alpha) \times Y$.
Therefore, $[\pi^1_{\varphi_1}]^{-1}[(F, A)] = (F, A) \ \widetilde{\times} \ (\widetilde{Y}, B)$, is a basic open set in $\tau\widetilde{\times}\nu$.
So $\pi^1_{\varphi_1}$ is soft continuous.

Again let $(H, A \times B) \in \tau\widetilde{\times}\nu$. Then there exists a subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that $(H, A \times B)$ is the union of the members of $\mathcal{F}'$. So,

$$\begin{aligned} [\pi^1_{\varphi_1}][(H, A \times B)] &= \widetilde{\cup}\{\pi^1_{\varphi_1}[(U, A)\widetilde{\times}(V, B)] : \ (U, A)\widetilde{\times}(V, B) \in \mathcal{F}'\} \\ &= \widetilde{\cup}\{(U, A) : \ (U, A)\widetilde{\times}(V, B) \in \mathcal{F}'\} \in \tau\big(\text{since}(U, A) \in \tau\big). \end{aligned}$$

Therefore $\pi^1_{\varphi_1}$ is soft open.

Similarly it can be shown that $\pi^2_{\varphi_2}$ is also soft continuous and soft open.

Next let ω be any enriched soft topology on $X \times Y$ such that the mappings $\pi^1_{\varphi_1} : (X \times Y, A \times B, \omega) \to (X, A, \tau)$ and $\pi^2_{\varphi_2} : (X \times Y, A \times B, \omega) \to (Y, B, \nu)$ are soft continuous.
Let $[(U, A) \ \widetilde{\times} \ (V, B)]$ be any basic open soft set in $\tau \ \widetilde{\times} \ \nu$.

$$\begin{aligned} \text{Now}, [(U, A) \ \widetilde{\times} \ (V, B)] &= [(U, A) \ \widetilde{\cap} \ (\widetilde{X}, A)] \ \widetilde{\times} \ [(\widetilde{Y}, B) \ \widetilde{\cap} \ (V, B)] \\ &= [(U, A) \ \widetilde{\times} \ (\widetilde{Y}, B)] \ \widetilde{\cap} \ [(\widetilde{X}, A) \ \widetilde{\times} \ (V, B)] \\ &= [\pi^1_{\varphi_1}]^{-1}[(U, A)] \ \widetilde{\cap} \ [\pi^2_{\varphi_2}]^{-1}[(V, B)] \in \omega \\ &\quad \big(\text{since}[\pi^1_{\varphi_1}]^{-1}[(U, A)], \ [\pi^2_{\varphi_2}]^{-1}[(V, B)] \in \omega\big). \end{aligned}$$

Thus $\tau \ \widetilde{\times} \ \nu$ is a subset of ω.

Therefore $\tau \ \widetilde{\times} \ \nu$ is the smallest enriched soft topology in $X \times Y$ for which the soft projection mappings are soft continuous. □

Proposition 3.18. *Let* $(X \times Y, A \times B, \tau\widetilde{\times}\nu)$ *be the product space of two soft topological spaces* (X, A, τ) *and* (Y, B, ν) *and* $\pi^1_{\varphi_1} : (X \times Y, A \times B, \tau\widetilde{\times}\nu) \to (X, A, \tau)$ *and* $\pi^2_{\varphi_2} : (X \times Y, A \times B, \tau\widetilde{\times}\nu) \to (Y, B, \nu)$ *be the soft projection mappings. If* (Z, C, ω) *is any soft topological space, then the mapping* $f_\varphi : (Z, C, \omega) \to (X \times Y, A \times B, \tau\widetilde{\times}\nu)$ *is soft continuous iff the soft mappings* $\pi^1_{\varphi_1} \circ f_\varphi : (Z, C, \omega) \to (X, A, \tau)$ *and* $\pi^2_{\varphi_2} \circ f_\varphi : (Z, C, \omega) \to (Y, B, \nu)$ *are soft continuous.*

Proof. First let $f_\varphi : (Z, C, \omega) \to (X \times Y, A \times B, \tau\widetilde{\times}\nu)$ be soft continuous.
Also the soft mappings $\pi^1_{\varphi_1} : (X \times Y, A \times B, \tau\widetilde{\times}\nu) \to (X, A, \tau)$ and $\pi^2_{\varphi_2} : (X \times Y, A \times B, \tau\widetilde{\times}\nu) \to (Y, B, \nu)$ are soft continuous.
Then by Proposition 3.13, we have the composition soft mappings $\pi^1_{\varphi_1} \circ f_\varphi : (Z, C, \omega) \to (X, A, \tau)$, $\pi^2_{\varphi_2} \circ f_\varphi : (Z, C, \omega) \to (Y, B, \nu)$ are soft continuous.

Next let the soft mappings $\pi^1_{\varphi_1} \circ f_\varphi : (Z, C, \omega) \to (X, A, \tau)$ and $\pi^2_{\varphi_2} \circ f_\varphi : (Z, C, \omega) \to (Y, B, \nu)$ be soft continuous.
Let $(F, A \times B) \in (\tau\widetilde{\times}\nu)$ and $\mathcal{F} = \{(U, A) \ \widetilde{\times} \ (V, B) : \ (U, A) \in \tau, \ (V, B) \in \nu\}$.
Then there exists a subfamily $\mathcal{F}' = \{(U_k, A) \ \widetilde{\times} \ (V_k, B) : \ k \in \Delta\}$ of $\mathcal{F}$ such that $(F, A \times B) = \widetilde{\bigcup}_{k\in\Delta}[(U_k, A) \ \widetilde{\times} \ (V_k, B)]$.

Thus,

$$\begin{aligned} [f_\varphi]^{-1}[(F, A\times B)] &= (f_\varphi)^{-1}[\tilde{\cup}_{k\in\Delta}[(U_k, A)\ \tilde{\times}\ (V_k, B)]] \\ &= (f_\varphi)^{-1}[\tilde{\cup}_{K\in\Delta}[(\pi^1_{\varphi_1})^{-1}(U_k, A)\ \tilde{\cap}\ (\pi^2_{\varphi_2})^{-1}(V_k, B)]] \\ &= \tilde{\cup}_{k\in\Delta}[(\pi^1_{\varphi_1}\circ f_\varphi)^{-1}(U_k, A)\ \tilde{\cap}\ (\pi^2_{\varphi_2}\circ f_\varphi)^{-1}(V_k, B)] \in \omega. \end{aligned}$$

Therefore f_φ is soft continuous. □

Proposition 3.19. *Let $(X\times Y, A\times B, \tau\tilde{\times}\nu)$ be the product space of two enriched soft topological spaces (X, A, τ) and (Y, B, ν) and $\varphi : B \to A\times B$ be a mapping such that $\varphi(\beta) = (\alpha, \beta)$ for some $\alpha \in A$. Then for $a \in X$, the soft mapping $f_\varphi : (Y, B, \nu) \to (X\times Y, A\times B, \tau\tilde{\times}\nu)$, where $f(y) = (a, y)$, is soft continuous.*

Proof. Let $\pi^1_{\varphi_1} : (X\times Y, A\times B, \tau\tilde{\times}\nu) \to (X, A, \tau)$ and $\pi^2_{\varphi_2} : (X\times Y, A\times B, \tau\tilde{\times}\nu) \to (Y, B, \nu)$ be the soft projection mappings.
Now $\pi^1_{\varphi_1}\circ f_\varphi : (Y, B, \nu) \to (X, A, \tau)$ is such that $[\pi^1\circ f](y) = a$, for all $y \in Y$ and $\pi^2_{\varphi_2}\circ f_\varphi : (Y, B, \nu) \to (Y, B, \nu)$ is such that $[\pi^2\circ f](y) = y$, for all $y \in Y$.
So, by Propositions 3.10 and 3.12, the soft mappings $\pi^1_{\varphi_1}\circ f_\varphi$ and $\pi^2_{\varphi_2}\circ f_\varphi$ are soft continuous and hence by Proposition 3.18, f_φ is soft continuous. □

Proposition 3.20. *Let $(X\times Y, A\times B, \tau\tilde{\times}\nu)$ be the product space of two enriched soft topological spaces (X, A, τ) and (Y, B, ν) and $\varphi : A \to A\times B$ be a mapping such that $\varphi(\alpha) = (\alpha, \beta)$ for some $\beta \in B$. Then for $a \in Y$, the soft mapping $f_\varphi : (X, A, \tau) \to (X\times Y, A\times B, \tau\tilde{\times}\nu)$, where $f(x) = (x, a)$, is soft continuous.*

Proof. Proof is similar to that of Proposition 3.19. □

Proposition 3.21. *Let (X, A, τ) and (Y, B, ν) be two soft topological spaces. Then for each $(\alpha, \beta) \in (A\times B)$, $(\tau\tilde{\times}\nu)^{(\alpha,\beta)} = \tau^\alpha \times \nu^\beta$.*

Proof. Let $(\alpha, \beta) \in (A\times B)$ and $U \in (\tau\tilde{\times}\nu)^{(\alpha,\beta)}$.
Then there exists $(F, A\times B) \in (\tau\tilde{\times}\nu)$ such that $U = F(\alpha, \beta)$.
Since $(F, A\times B) \in (\tau\tilde{\times}\nu)$, it follows that there exist $(U_i, A) \in \tau$, $(V_i, B) \in \nu$, $i \in \Delta$ such that $(F, A\times B) = \tilde{\bigcup}_{i\in\Delta}[(U_i, A)\ \tilde{\times}\ (V_i, B)]$.
So, $U = F(\alpha, \beta) = \bigcup_{i\in\Delta}[U_i(\alpha)\times V_i(\beta)] \in (\tau^\alpha\times\nu^\beta)$.
Therefore, $(\tau\tilde{\times}\nu)^{(\alpha,\beta)} \subseteq (\tau^\alpha\times\nu^\beta)$.
Next, let $U \in (\tau^\alpha\times\nu^\beta)$.
Then we have $U_i \in \tau^\alpha$, $V_i \in \nu^\beta$, $i \in \Delta$ such that $U = \bigcup_{i\in\Delta}[U_i\times V_i]$ and hence there exist $(F_i, A) \in \tau$, $(G_i, B) \in \nu$ such that $F_i(\alpha) = U_i$, $G_i(\beta) = V_i$, $i \in \Delta$.
Thus $\tilde{\bigcup}_{i\in\Delta}[(F_i, A)\ \tilde{\times}\ (G_i, B)] \in (\tau\ \tilde{\times}\ \nu)$ and $U = \bigcup_{i\in\Delta}[F_i(\alpha)\times G_i(\beta)] \in (\tau\ \tilde{\times}\ \nu)^{(\alpha,\beta)}$.
So, $\tau^\alpha\times\nu^\beta \subseteq (\tau\ \tilde{\times}\ \nu)^{(\alpha,\beta)}$.
Therefore, for each $(\alpha, \beta) \in (A\times B)$, $(\tau\tilde{\times}\nu)^{(\alpha,\beta)} = \tau^\alpha\times\nu^\beta$. □

Proposition 3.22. *Let (X, A, τ) and (Y, B, ν) be two soft topological spaces which are enriched or not and define $T^* = \{(F, A\times B) \in S(X\times Y, A\times B)$ such that $F(\alpha, \beta) \in (\tau^\alpha\times\nu^\beta)$, for all $(\alpha, \beta) \in (A\times B)\}$. Then T^* is an enriched soft topology over $X\times Y$ and $T^* = \tau^*\ \tilde{\times}\ \nu^*$, where τ^*, ν^* are as in Proposition 3.4.*

Proof. Since ϕ and $X \times Y \in (\tau^\alpha \times \nu^\beta)$, for all $(\alpha, \beta) \in (A \times B)$, we have for all $(F, A \times B) \in CS(X \times Y)$, $(F, A \times B) \in T^*$.
Again, let $(F_1, A \times B)$, $(F_2, A \times B) \in T^*$. Then $F_1(\alpha, \beta)$, $F_2(\alpha, \beta) \in (\tau^\alpha \times \nu^\beta)$.
So, $(F_1 \tilde{\cap} F_2)(\alpha, \beta) = F_1(\alpha, \beta) \cap F_2(\alpha, \beta) \in (\tau^\alpha \times \nu^\beta)$, for all $(\alpha, \beta) \in (A \times B)$.
Thus $(F_1, A \times B) \tilde{\cap} (F_2, A \times B) \in T^*$.
Next, let $(F_i, A \times B) \in T^*$, $i \in \Delta$. Then $F_i(\alpha, \beta) \in (\tau^\alpha \times \nu^\beta)$, $i \in \Delta$.
So, $(\bigcup_{i\in\Delta} F_i)(\alpha, \beta) = \bigcup_{i\in\Delta}[F_i(\alpha, \beta)] \in (\tau^\alpha \times \nu^\beta)$, for all $(\alpha, \beta) \in (A \times B)$.
Thus $\tilde{\bigcup}_{i\in\Delta}(F_i, A \times B) \in T^*$.
Therefore, T^* is an enriched soft topology over $X \times Y$.

Now let $(F, A \times B) \in T^*$ and $(\alpha, \beta) \in (A \times B)$. Then $F(\alpha, \beta) \in (\tau^\alpha \times \nu^\beta)$ and hence there exist $U_i \in \tau^\alpha$, $V_i \in \nu^\beta$, $i \in \Delta$ such that $F(\alpha, \beta) = \bigcup_{i\in\Delta}(U_i \times V_i)$.
For each pair $U_i \in \tau^\alpha$, $V_i \in \nu^\beta$, take soft sets (F_{U_i}, A) and (F_{V_i}, B) such that $F_{U_i}(\alpha) = U_i$, $F_{U_i}(\alpha') = \phi$, for all α' $(\neq\ \alpha) \in A$ and $F_{V_i}(\beta) = V_i$, $F_{V_i}(\beta') = \phi$, for all β' $(\neq\ \beta) \in B$.
So, $(F_{U_i}, A) \in \tau^*$, $(F_{V_i}, B) \in \nu^*$ and hence $(F_{U_i} \tilde{\times} F_{V_i}, A \times B) \in (\tau^* \tilde{\times} \nu^*)$.
Also $[F_{U_i} \tilde{\times} F_{V_i}](\alpha, \beta) = F_{U_i}(\alpha) \times F_{V_i}(\beta) = U_i \times V_i$ and
$[F_{U_i} \tilde{\times} F_{V_i}](\gamma, \delta) = F_{U_i}(\gamma) \times F_{V_i}(\delta) = \phi$, for all (γ, δ) $[\neq\ (\alpha, \beta)] \in (A \times B)$.
Let $(G_{(\alpha,\beta)}, A \times B) = \tilde{\bigcup}_{i\in\Delta}(F_{U_i} \tilde{\times} F_{V_i}, A \times B)$. Then $(G_{(\alpha,\beta)}, A \times B) \in (\tau^* \tilde{\times} \nu^*)$ and $G_{(\alpha,\beta)}(\alpha, \beta) = \bigcup_{i\in\Delta}(U_i \times V_i) = F(\alpha, \beta)$, $G_{(\alpha,\beta)}(\gamma, \delta) = \phi$, for all (γ, δ) $[\neq\ (\alpha, \beta)] \in (A \times B)$.

Again, let $(G, A \times B) = \tilde{\bigcup}_{(\alpha,\beta)\in(A\times B)}(G_{(\alpha,\beta)}, A \times B)$.
Then $(G, A \times B) \in (\tau^* \tilde{\times} \nu^*)$ and $G(\alpha, \beta) = F(\alpha, \beta)$, for all $(\alpha, \beta) \in (A \times B)$.
Thus, $(F, A \times B) = (G, A \times B) \in (\tau^* \tilde{\times} \nu^*)$.
Therefore, $T^* \subseteq (\tau^* \tilde{\times} \nu^*)$.

Also, let $(F, A \times B) \in (\tau^* \tilde{\times} \nu^*)$. Then there exist $(U_i, A) \in \tau^*$, $(V_i, B) \in \nu^*$, $i \in \Delta$ such that $(F, A \times B) = \tilde{\bigcup}_{i\in\Delta}[(U_i, A) \tilde{\times} (V_i, B)]$.
Also, $F(\alpha, \beta) = \bigcup_{i\in\Delta}[U_i(\alpha) \times V_i(\beta)] \in (\tau^\alpha \times \nu^\beta)$, for all $(\alpha, \beta) \in (A \times B)$ and hence $(F, A \times B) \in T^*$.
Thus, $(\tau^* \tilde{\times} \nu^*) \subseteq T^*$.
Therefore, $T^* = \tau^* \tilde{\times} \nu^*$. □

Proposition 3.23. *Let $f_\varphi : (X, A, \tau) \to (X, A, \tau)$ and $g_\psi : (Y, B, \nu) \to (Y, B, \nu)$ be two soft continuous mappings. Then the soft mapping $h_\xi : (X \times Y, A \times B, \tau \tilde{\times} \nu) \to (X \times Y, A \times B, \tau \tilde{\times} \nu)$, where $h(x, y) = (f(x), g(y))$ and $\xi(\alpha, \beta) = (\varphi(\alpha), \psi(\beta))$ is soft continuous.*

Proof. Let $(F, A \times B) \in (\tau \tilde{\times} \nu)$. Then there exist $(G_i, A) \in \tau$ and $(H_i, B) \in \nu$, $i \in \Delta$ such that $(F, A \times B) = \tilde{\bigcup}_{i\in\Delta}[(G_i, A) \tilde{\times} (H_i, B)]$.
Since f_φ and g_ψ are soft continuous, it follows that $f_\varphi^{-1}[(G_i, A)] \in \tau$ and $g_\psi^{-1}[(H_i, B)] \in \nu$.

Now for all $(\alpha,\beta) \in (A \times B)$, we have

$$\begin{aligned}
\big[h_\xi^{-1}[(G_i,A)\widetilde{\times}(H_i,B)]\big](\alpha,\beta) &= \big[h^{-1}[(G_i\widetilde{\times}H_i)]\big]\xi(\alpha,\beta)\\
&= \big[h^{-1}[(G_i\widetilde{\times}H_i)]\big]\big(\varphi(\alpha),\psi(\beta)\big)\\
&= h^{-1}\big[G_i\big(\varphi(\alpha)\big) \times H_i\big(\psi(\beta)\big)\big]\\
&= f^{-1}[G_i(\varphi(\alpha))] \times g^{-1}[H_i(\psi(\beta))]\\
&= \big[f_\varphi^{-1}(G_i)\widetilde{\times}g_\psi^{-1}(H_i)\big](\alpha,\beta).
\end{aligned}$$

Therefore, $h_\xi^{-1}[(G_i,A)\widetilde{\times}(H_i,B)] = \big(f_\varphi^{-1}(G_i),A\big)\widetilde{\times}\big(g_\psi^{-1}(H_i),B\big) \in (\tau\widetilde{\times}\nu)$ and hence

$$\begin{aligned}
h_\xi^{-1}[(F,A\times B)] &= h_\xi^{-1}\big[\widetilde{\cup}_{i\in\Delta}[(G_i,A)\ \widetilde{\times}\ (H_i,A)]\big]\\
&= \widetilde{\cup}_{i\in\Delta}h_\xi^{-1}[(G_i,A)\ \widetilde{\times}\ (H_i,A)] \in (\tau\ \widetilde{\times}\ \nu)
\end{aligned}$$

Therefore, the soft mapping $h_\xi : (X\times Y, A\times B, \tau\widetilde{\times}\nu) \to (X\times Y, A\times B, \tau\widetilde{\times}\nu)$ defined by $h(x,y) = (f(x), g(y))$ where $\xi(\alpha,\beta) = (\varphi(\alpha),\psi(\beta))$ is soft continuous. □

Definition 3.24. Let τ and ν be two soft topologies on X, Y under the parameter sets A, B respectively and $f : X \to Y$, $\varphi : A \to B$ be two mappings. The *image* of τ and the *preimage* of ν under the soft mapping f_φ respectively are defined by

(i) $f_\varphi(\tau) = \{(G,B) \in S(Y,B) :\ f_\varphi^{-1}[(G,B)] \in \tau\}$ and
(ii) $f_\varphi^{-1}(\nu) = \{f_\varphi^{-1}[(G,B)] :\ (G,B) \in \nu\}$.

Proposition 3.25. *Let τ and ν be two soft topologies on X, Y under the parameter sets A, B respectively and $f : X \to Y$, $\varphi : A \to B$ be two mappings. Then*

(i) *$f_\varphi^{-1}(\nu)$ is a soft topology on X under the parameter set A and*
(ii) *$f_\varphi(\tau)$ is a soft topology on Y under the parameter set B.*

Proof. (i) We observe that $(\widetilde{\phi},A) = f_\varphi^{-1}[(\widetilde{\phi},B)]$ and $[f_\varphi^{-1}[(\widetilde{Y},B)]](\alpha) =$
$f^{-1}[\widetilde{Y}(\varphi(\alpha))] = f^{-1}(Y) = X$, for all $\alpha \in A$ and hence $(\widetilde{X},A) = f_\varphi^{-1}[(\widetilde{Y},B)]$.
Therefore, $(\widetilde{\phi},A),\ (\widetilde{X},A) \in f_\varphi^{-1}(\nu)$.

Next let $(F_1,A),\ (F_2,A) \in f_\varphi^{-1}(\nu)$. Then there exist $(G_1,B),\ (G_2,B) \in \nu$ such that
$(F_1,A) = f_\varphi^{-1}[(G_1,B)]$ and $(F_2,A) = f_\varphi^{-1}[(G_2,B)]$.
Thus $(G_1,B)\ \widetilde{\cap}\ (G_2,B) \in \nu$ and hence

$$\begin{aligned}
f_\varphi^{-1}[(G_1,B)\ \widetilde{\cap}\ (G_2,B)] &= f_\varphi^{-1}[(G_1,B)]\ \widetilde{\cap}\ f_\varphi^{-1}[(G_2,B)]\\
&= (F_1,A)\ \widetilde{\cap}\ (F_2,A) \in f_\varphi^{-1}(\nu).
\end{aligned}$$

Again let $(F_i,A) \in f_\varphi^{-1}(\nu),\ i \in \Delta$. Then there exist $(G_i,B) \in \nu,\ i \in \Delta$ such that $(F_i,A) = f_\varphi^{-1}[(G_i,B)],\ i \in \Delta$. Thus $\widetilde{\bigcup}_{i\in\Delta}(G_i,B) \in \nu$ and hence
$\widetilde{\bigcup}_{i\in\Delta}f_\varphi^{-1}[(G_i,B)] = f_\varphi^{-1}[\widetilde{\bigcup}_{i\in\Delta}[(G_i,B)]] = \widetilde{\bigcup}_{i\in\Delta}[(F_i,A)] \in f_\varphi^{-1}(\nu)$.
Therefore, $f_\varphi^{-1}(\nu)$ is a soft topology on X under the parameter set A.

(ii) Proof is similar to part (i). □

Definition 3.26. A soft mapping $f_\varphi : (X,A,\tau) \to (Y,B,\nu)$ is said to be *soft homeomorphism* if f_φ is bijective and $f_\varphi,\ (f_\varphi)^{-1}$ are soft continuous.

Definition 3.27. A soft topology τ is called *indiscrete (discrete)* soft topology if τ^{α} is a indiscrete (discrete) topology, for all $\alpha \in A$. The discrete and indiscrete soft topologies are called the trivial soft topologies.

4. Group soft topology

Equipped with the results developed in sections 2 and 3, we are now in a position to investigate our main interest, i.e. we introduce the definition of a generalized group soft topology and to study some of its properties. Throughout this section $(G, \circ)$ is assumed to be a group. By abuse of notation, we use G instead of $(G, \circ)$ and xy instead of $x \circ y$.

Definition 4.1. Let G be a group and τ be a soft topology on G with respect to a parameter set A. Then τ is said to be a *generalized group soft topology* on G if there exist mappings $\varphi : A \times A \to A$ and $\psi : A \to A$ such that the soft mappings

(i) $f_{\varphi} : (G \times G, A \times A, \tau \mathbin{\widetilde{\times}} \tau) \to (G, A, \tau)$, where $f(x, y) = xy$ and
(ii) $g_{\psi} : (G, A, \tau) \to (G, A, \tau)$, where $g(x) = x^{-1}$
are soft continuous.

Example 4.2. (i) Let G be a group, $\tau = \{(\widetilde{\phi}, A),\ (\widetilde{G}, A)\},\ \nu = S(G, A)$ are soft topologies on G and $\varphi : A \times A \to A,\ \psi : A \to A$ be any two mappings. Then $\tau \mathbin{\widetilde{\times}} \tau = \{(\widetilde{\phi}, A \times A),\ (\widetilde{G \times G}, A \times A)\}$ and $\nu \mathbin{\widetilde{\times}} \nu = S(G \times G, A \times A)$. It can be easily shown that τ and ν are generalized group soft topologies. Thus, the trivial soft topologies on a group G are generalized group soft topology.

(ii) Let G be a group, $A = \{\alpha, \beta\}$ and
$\tau = \big\{\{\phi/\alpha,\ \phi/\beta\},\ \{\phi/\alpha,\ G/\beta\},\ \{G/\alpha,\ \phi/\beta\},\ \{G/\alpha,\ G/\beta\}\big\}$. Then

$$
\begin{aligned}
\tau \mathbin{\widetilde{\times}} \tau \ = \ & \big\{\{\phi/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ \phi/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ \phi/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ \phi/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ \phi/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ \phi/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ \phi/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ \phi/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ \phi/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ \phi/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ \phi/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{\phi/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\},\\
& \{(G\times G)/(\alpha,\alpha),\ (G\times G)/(\alpha,\beta),\ (G\times G)/(\beta,\alpha),\ (G\times G)/(\beta,\beta)\}\big\}
\end{aligned}
$$

Since $f(x,y) = xy$ and $g(x) = x^{-1}$, it follows that $f^{-1}(\phi) = \phi$, $f^{-1}(G) = G \times G$, $g^{-1}(\phi) = \phi$ and $g^{-1}(G) = G$. Now if $(F,A) \in \tau$, then

$$f_\varphi^{-1}[(F,A)](\alpha,\alpha) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G \times G & \text{if } F(\alpha) = G \end{cases} & \text{if } \varphi(\alpha,\alpha) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G \times G & \text{if } F(\beta) = G \end{cases} & \text{if } \varphi(\alpha,\alpha) = \beta \end{cases},$$

$$f_\varphi^{-1}[(F,A)](\alpha,\beta) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G \times G & \text{if } F(\alpha) = G \end{cases} & \text{if } \varphi(\alpha,\beta) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G \times G & \text{if } F(\beta) = G \end{cases} & \text{if } \varphi(\alpha,\beta) = \beta \end{cases},$$

$$f_\varphi^{-1}[(F,A)](\beta,\alpha) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G \times G & \text{if } F(\alpha) = G \end{cases} & \text{if } \varphi(\beta,\alpha) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G \times G & \text{if } F(\beta) = G \end{cases} & \text{if } \varphi(\beta,\alpha) = \beta \end{cases},$$

$$f_\varphi^{-1}[(F,A)](\beta,\beta) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G \times G & \text{if } F(\alpha) = G \end{cases} & \text{if } \varphi(\beta,\beta) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G \times G & \text{if } F(\beta) = G \end{cases} & \text{if } \varphi(\beta,\beta) = \beta \end{cases},$$

$$g_\psi^{-1}[(F,A)](\alpha) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G & \text{if } F(\alpha) = G \end{cases} & \text{if } \psi(\alpha) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G & \text{if } F(\beta) = G \end{cases} & \text{if } \psi(\alpha) = \beta \end{cases},$$

$$g_\psi^{-1}[(F,A)](\beta) = \begin{cases} \begin{cases} \phi & \text{if } F(\alpha) = \phi \\ G & \text{if } F(\alpha) = G \end{cases} & \text{if } \psi(\beta) = \alpha \\ \\ \begin{cases} \phi & \text{if } F(\beta) = \phi \\ G & \text{if } F(\beta) = G \end{cases} & \text{if } \psi(\beta) = \beta \end{cases}$$

Thus, clearly $f_\varphi^{-1}[(F,A)] \in \tau \widetilde{\times} \tau$ and $g_\psi^{-1}[(F,A)] \in \tau$, for all $(F,A) \in \tau$. Therefore, G is a generalized group soft topology.

Proposition 4.3. *Let τ be a generalized group soft topology on G and f_φ, g_ψ be the corresponding soft continuous mappings. If further $\varphi(\alpha,\alpha) = \alpha$ and $\psi(\alpha) = \alpha$, for all $\alpha \in A$, then τ^α is a group topology on G, for all $\alpha \in A$.*

Proof. Let $U \in \tau^\alpha$. Then there exists $(F,A) \in \tau$ such that $F(\alpha) = U$.
Since τ is a generalized group soft topology and the mappings
$f_\varphi : (G \times G, A \times A, \tau \widetilde{\times} \tau) \to (G, A, \tau)$, where $f(x,y) = xy$,
$g_\psi : (G, A, \tau) \to (G, A, \tau)$, where $g(x) = x^{-1}$ are soft continuous, it follows that

$f_{\varphi}^{-1}[(F,A)] \in (\tau \mathbin{\widetilde{\times}} \tau)$ and $g_{\psi}^{-1}[(F,A)] \in \tau$.
Also since $\varphi(\alpha,\alpha) = \alpha$ and $\psi(\alpha) = \alpha$, for all $\alpha \in A$, we have,

$$\begin{aligned}\big[f_{\varphi}^{-1}[(F,A)]\big](\alpha,\alpha) &= f^{-1}\big[F[\varphi(\alpha,\alpha)]\big] = f^{-1}[F(\alpha)] \\ &= f^{-1}[U] \in (\tau \mathbin{\widetilde{\times}} \tau)^{(\alpha,\alpha)} = (\tau^{\alpha} \times \tau^{\alpha})\end{aligned}$$

and $\big[g_{\psi}^{-1}[(F,A)]\big](\alpha) = g^{-1}\big[F[\psi(\alpha)]\big] = g^{-1}[F(\alpha)] = g^{-1}[U] \in \tau^{\alpha}$, for all $\alpha \in A$.
Thus the mappings $f : (G \times G, \tau^{\alpha} \times \tau^{\alpha}) \to (G, \tau^{\alpha})$, where $f(x,y) = xy$ and $g : (G, \tau^{\alpha}) \to (G, \tau^{\alpha})$, where $g(x) = x^{-1}$ are continuous for all $\alpha \in A$. Therefore τ^{α} is a group topology on G, for all $\alpha \in A$. $\square$

Definition 4.4. Let (F,A), (H,B) be two soft sets over a group G. Then

(i) $(F,A) \mathbin{\widetilde{\circ}} (H,B) = (F \mathbin{\widetilde{\circ}} H, A \times B)$ where $(F \mathbin{\widetilde{\circ}} H)(\alpha,\beta) = F(\alpha) \circ H(\beta)$, for all $(\alpha,\beta) \in (A \times B)$.
(ii) $(F,A)^{-1} = (F^{-1}, A)$ where $F^{-1}(\alpha) = [F(\alpha)]^{-1}$, for all $\alpha \in A$.

Proposition 4.5. *Let (G,A,τ) be a soft topological space over a group G. Then τ is a generalized group soft topology iff the following conditions are satisfied:*

(i) *there exists a mapping $\varphi : A \times A \to A$ and for all x, $y \in G$, for all $(\alpha,\beta) \in A \times A$ and for all $(W,A) \in \tau$ with $E^{xy}_{\varphi(\alpha,\beta)} \mathbin{\widetilde{\in}} (W,A)$, there exist (U,A), $(V,A) \in \tau$ such that $E^{x}_{\alpha} \mathbin{\widetilde{\in}} (U,A)$, $E^{y}_{\beta} \mathbin{\widetilde{\in}} (V,A)$ with $[U(\alpha) \circ V(\beta)] \subseteq W(\gamma)$, for all α, β, $\gamma \in A$ such that $\varphi(\alpha,\beta) = \gamma$.*
(ii) *there exists a mapping $\psi : A \to A$ and for all $x \in G$, for all $\alpha \in A$ and for all $(W,A) \in \tau$ with $E^{x^{-1}}_{\psi(\alpha)} \mathbin{\widetilde{\in}} (V,A)$, there exists $(U,A) \in \tau$ such that $E^{x}_{\alpha} \mathbin{\widetilde{\in}} (U,A)$ with $U^{-1}(\alpha) \subseteq V(\beta)$, for all α, $\beta \in A$ such that $\psi(\alpha) = \beta$.*

Proof. Let (G,A,τ) be a generalized group soft topological space.
Then there exists a mapping $\varphi : A \times A \to A$ defined by $\varphi(\alpha,\beta) = \gamma$ (say) such that $f_{\varphi} : (G \times G, A \times A, \tau \mathbin{\widetilde{\times}} \tau) \to (G, A, \tau)$, where $f(x,y) = xy$, is soft continuous.
Let $x, y \in G$, $(\alpha,\beta) \in A \times A$ and $(W,A) \in \tau$ such that $E^{f(x,y)}_{\varphi(\alpha,\beta)} = E^{xy}_{\gamma} \mathbin{\widetilde{\in}} (W,A)$. Then by Proposition 3.9, there exists $(P, A \times A) \in (\tau \mathbin{\widetilde{\times}} \tau)$ such that $E^{(x,y)}_{(\alpha,\beta)} \mathbin{\widetilde{\in}} (P, A \times A)$ and $f_{\varphi}[(P, A \times A)] \mathbin{\widetilde{\subseteq}} (W,A)$.
Since $(P, A \times A) \in (\tau \mathbin{\widetilde{\times}} \tau)$, there exist (U_i,A), $(V_i,A) \in \tau$, $i \in \Delta$ such that $(P, A \times A) = \bigcup_{i\in\Delta}[(U_i,A) \mathbin{\widetilde{\times}} (V_i,A)]$ and also there exists $i \in \Delta$ such that $E^{x}_{\alpha} \mathbin{\widetilde{\in}} (U_i,A)$ and $E^{y}_{\beta} \mathbin{\widetilde{\in}} (V_i,A)$.
Therefore, $f_{\varphi}[(P, A \times A)] = f_{\varphi}[\bigcup_{i\in\Delta}[(U_i,A) \mathbin{\widetilde{\times}} (V_i,A)]] \mathbin{\widetilde{\subseteq}} (W,A)$ and hence

$$\begin{aligned}\big[f_{\varphi}[\cup_{i\in\Delta}[(U_i,A) \mathbin{\widetilde{\times}} (V_i,A)]]\big](\gamma) &= \cup_{\varphi(\alpha,\beta)=\gamma} \cup_{i\in\Delta} f[U_i(\alpha) \times V_i(\beta)] \\ &= \cup_{\varphi(\alpha,\beta)=\gamma} \cup_{i\in\Delta} [U_i(\alpha) \circ V_i(\beta)] \subseteq W(\gamma).\end{aligned}$$

Therefore, $[U_i(\alpha) \circ V_i(\beta)] \subseteq W(\gamma)$, for all α, β, $\gamma \in A$ such that $\varphi(\alpha,\beta) = \gamma$.

Condition (ii) can be proved similarly.

Conversely let the given conditions be satisfied.
Let $(W,A) \in \tau$ and $E^{(x,y)}_{(\alpha,\beta)} \mathbin{\widetilde{\in}} f_{\varphi}^{-1}[(W,A)]$. Then $f_{\varphi}[E^{(x,y)}_{(\alpha,\beta)}] = E^{f(x,y)}_{\varphi(\alpha,\beta)} = E^{xy}_{\gamma} \mathbin{\widetilde{\in}} (W,A)$ and hence by the given condition (i), there exist (U_x,A), $(V_y,A) \in \tau$ such that

$E^x_\alpha \,\widetilde{\in}\, (U_x, A)$,
$E^y_\beta \,\widetilde{\in}\, (V_y, A)$ and $[U_x(\alpha) \circ V_y(\beta)] \subseteq W(\gamma)$, for all $\alpha,\ \beta,\ \gamma$ with $\varphi(\alpha,\beta) = \gamma$.

$$\begin{aligned}\text{Thus, } \big[f_\varphi[(U_x, A) \,\widetilde{\times}\, (V_y, A)]\big](\gamma) &= \cup_{\varphi(\alpha,\beta)=\gamma} f[U_x(\alpha) \times V_y(\beta)] \\ &= \cup_{\varphi(\alpha,\beta)=\gamma} [U_x(\alpha) \circ V_y(\beta)] \subseteq W(\gamma),\end{aligned}$$

for all $\gamma \in A$. Therefore, $f_\varphi[(U_x, A) \,\widetilde{\times}\, (V_y, A)] \,\widetilde{\subseteq}\, (W, A)$ and hence

$$[(U_x, A) \,\widetilde{\times}\, (V_y, A)] \,\widetilde{\subseteq}\, f_\varphi^{-1}[(W, A)].$$

Again

$$\begin{aligned} f_\varphi^{-1}[(W, A)] &= \widetilde{\cup}\{E^{(x,y)}_{(\alpha,\beta)} : E^{(x,y)}_{(\alpha,\beta)} \widetilde{\in}\, f_\varphi^{-1}[(W, A)]\} \\ &\widetilde{\subseteq}\ \widetilde{\cup}\{(U_x, A) \,\widetilde{\times}\, (V_y, A) : E^{(x,y)}_{(\alpha,\beta)} \widetilde{\in}\, f_\varphi^{-1}[(W, A)]\} \,\widetilde{\subseteq}\, f_\varphi^{-1}[(W, A)].\end{aligned}$$

Thus $f_\varphi^{-1}[(W, A)] = \widetilde{\bigcup}\{(U_x, A) \,\widetilde{\times}\, (V_y, A) : E^{(x,y)}_{(\alpha,\beta)} \widetilde{\in}\, f_\varphi^{-1}[(W, A)]\} \in (\tau \,\widetilde{\times}\, \tau)$ and hence, the soft mapping $f_\varphi : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$, where $f(x, y) = xy$ is soft continuous.

Continuity of the soft mapping $g_\psi : (G, A, \tau) \to (G, A, \tau)$, where $g(x) = x^{-1}$, can be proved similarly.

Therefore (G, A, τ) is a generalized group soft topology on G. □

Proposition 4.6. *Let τ be a generalized group soft topology on a group G and $f_\varphi,\ g_\psi$ be the corresponding soft mappings. Then the soft mapping $h_\xi : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau)$, where $h(x, y) = (x, y^{-1})$ and $\xi(\alpha, \beta) = (\alpha, \psi(\beta))$, is soft continuous.*

Proof. By Proposition 3.17, we have the mappings $\pi^1_{\varphi_1} : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$ and $\pi^2_{\varphi_2} : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$, where $\pi^1(x, y) = x,\ \pi^2(x, y) = y$, for all $x,\ y \in G$ and $\varphi_1(\alpha, \beta) = \alpha,\ \varphi_2(\alpha, \beta) = \beta$ are soft continuous.
Now $\pi^1_{\varphi_1} h_\xi = (\pi^1 h)_{\varphi_1 \xi} : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$, where $[\pi^1 h](x, y) = \pi^1(x, y^{-1}) = x = \pi^1(x, y)$ and $[\varphi_1 \xi](\alpha, \beta) = \varphi_1(\alpha, \psi(\beta)) = \alpha = \varphi_1(\alpha, \beta)$.
So $\pi^1_{\varphi_1} h_\xi = \pi^1_{\varphi_1}$ is soft continuous.
Again since $g_\psi : (G, A, \tau) \to (G, A, \tau)$, where $g(y) = y^{-1}$, is soft continuous, by Proposition 3.13, we get $g_\psi \pi^2_{\varphi_2} = (g\pi^2)_{\psi\varphi_2} : (G \times G, A \times A, \tau \,\widetilde{\times} \tau) \to (G, A, \tau)$, where $g\pi^2(x, y) = y^{-1}$, is soft continuous.
Thus $\pi^2_{\varphi_2} h_\xi = [\pi^2 h]_{\varphi_2 \xi} : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$ defined by $\pi^2 h(x, y) = \pi^2(x, y^{-1}) = y^{-1} = g\pi^2(x, y)$ and $\varphi_2 \xi(\alpha, \beta) = \varphi_2(\alpha, \psi(\beta)) = \psi(\beta) = \psi\varphi_2(\alpha, \beta)$.
Thus $\pi^2_{\varphi_2} h_\xi = (g\pi^2)_{\psi\varphi_2} = g_\psi \pi^2_{\varphi_2}$ is soft continuous.
Therefore from Proposition 3.18, we get that the soft mapping
$h_\xi : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau)$ defined by $h(x, y) = (x, y^{-1})$ where $\xi(\alpha, \beta) = (\alpha, \psi(\beta))$, is soft continuous. □

Proposition 4.7. *A soft topology τ on a group G is a generalized group soft topology iff there exists a mapping $\chi : A \times A \to A$ such that the soft mapping $p^r_\chi \big(p^l_\chi\big) : (G \times G, A \times A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$, where $p^r(x, y) = xy^{-1}$ $\big(p^l(x, y) = x^{-1}y\big)$, for all $x,\ y \in G$, is soft continuous.*

Proof. Let τ be a generalized group soft topology.
Then there exists $\varphi : A \times A \to A$ and $\psi : A \to A$ such that $f_\varphi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G, A, \tau)$ and $g_\psi : (G, A, \tau) \to (G, A, \tau)$, where $f(x, y) = xy$, for all $x,\ y \in G$ and $g(x) = x^{-1}$,
for all $x \in G$ are soft continuous.
So by Proposition 4.6, $h_\xi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G \times G, A \times A, \tau \tilde{\times} \tau)$, where $h(x, y) = (x, y^{-1})$, for all $x,\ y \in G$ and $\xi(\alpha, \beta) = (\alpha, \psi(\beta))$, is soft continuous.
Let $\chi = \varphi\xi$.
Therefore $p^r_\chi = f_\varphi h_\xi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G, A, \tau)$ is such that
$p^r(x, y) = f(h(x, y)) = f(x, y^{-1}) = xy^{-1}$, for all $x,\ y \in G$ and $\chi(\alpha, \beta) = \varphi(\xi(\alpha, \beta)) = \varphi(\alpha, \psi(\beta))$, is soft continuous.

Conversely let $p^r_\chi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G, A, \tau)$, where $p^r(x, y) = xy^{-1}$,
for all $x,\ y \in G$ and $\chi : A \times A \to A$, be soft continuous.
Let e be the identity element of G, then from Proposition 3.19,
$j_\varpi : (G, A, \tau) \to (G \times G, A \times A, \tau \tilde{\times} \tau)$, where $j(x) = (e, x)$ and $\varpi : A \to A \times A$ is any mapping, is soft continuous.
Thus $g_\psi = p^r_\chi j_\varpi : (G, A, \tau) \to (G, A, \tau)$ where $g(x) = p^r j(x) = p^r(e, x) = x^{-1}$,
for all $x \in G$, is soft continuous.
Also since the identity soft mapping $i_\eta : (G, A, \tau) \to (G, A, \tau)$ where $i(x) = x$ and $\eta : A \to A$ is an identity mapping, is soft continuous, from Proposition 3.23, we have that the soft mapping $h_\xi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G \times G, A \times A, \tau \tilde{\times} \tau)$, where $h(x, y) = (x, y^{-1})$, for all $x,\ y \in G$ and $\xi(\alpha, \beta) = (\eta(\alpha), \psi(\beta))$, is soft continuous.
So $f_\varphi = p^r_\chi h_\xi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G, A, \tau)$, where
$[p^r h](x, y) = p^r(x, y^{-1}) = x(y^{-1})^{-1} = xy$, for all $x,\ y \in G$, is soft continuous.
Therefore τ is a generalized group soft topology.

Similarly we can prove that τ is a generalized group soft topology iff there exists a mapping $\chi : A \times A \to A$ such that the soft mapping $p^l_\chi : (G \times G, A \times A, \tau \tilde{\times} \tau) \to (G, A, \tau)$, where $p^l(x, y) = x^{-1}y$, for all $x,\ y \in G$, is soft continuous. □

Definition 4.8. A mapping $\varphi : A \times A \to A$ is said to be

(i) $left\ one-one$ if $a \neq b \Rightarrow \varphi(a, \alpha) \neq \varphi(b, \beta)$ for all $a,\ b,\ \alpha,\ \beta \in A$;
(ii) $right\ one-one$ if $a \neq b \Rightarrow \varphi(\alpha, a) \neq \varphi(\beta, b)$ for all $a,\ b,\ \alpha,\ \beta \in A$;
(iii) $left\ onto$ if $\varphi(A \times \{\alpha\}) = A$, for all $\alpha \in A$;
(iv) $right\ onto$ if $\varphi(\{\alpha\} \times A) = A$, for all $\alpha \in A$;
(v) $left\ bijective$ if φ is left one-one and left onto;
(vi) $right\ bijcetive$ if φ is right one-one and right onto.

Proposition 4.9. *Let (G, A, τ) be a generalized group soft topological space with the associated product soft mapping f_φ and inverse soft mapping g_ψ.*

(i) *If $\varphi : A \times A \to A$ is left bijective, then there exists a bijective mapping $\eta : A \to A$ such that the soft mapping $R^a_\eta : (G, A, \tau) \to (G, A, \tau)$, where $R^a(x) = xa$, is soft homeomorphism;*
(ii) *If $\varphi : A \times A \to A$ is right bijective, then there exists a bijective mapping $\zeta : A \to A$ such that the soft mapping $L^a_\eta : (G, A, \tau) \to (G, A, \tau)$, where $L^a(x) = ax$, is soft homeomorphism;*

(iii) *If $\varphi : A \times A \to A$ is both left and right bijective, then there exists a bijective mapping $\kappa : A \to A$ such that the soft mapping $C_\kappa : (G, A, \tau) \to (G, A, \tau)$, where $C(x) = axa^{-1}$, is soft homeomorphism.*

Proof. (i) It is clear that R^a is a bijective mapping.
Since τ is a generalized group soft topology with the associated product soft mapping f_φ, then $f_\varphi : (G\times G, A\times A, \tau \mathbin{\widetilde{\times}} \tau) \to (G, A, \tau)$, where $f(x, y) = xy$, is soft continuous.
Also by Proposition 3.20, the mapping $j_\varpi : (G, A, \tau) \to (G\times G, A\times A, \tau \mathbin{\widetilde{\times}} \tau)$, where $j(x) = (x, a)$ and $\varpi(\alpha) = (\alpha, \beta)$, is soft continuous.
So, by Proposition 3.13, $R^a_\eta = f_\varphi j_\varpi : (G, A, \tau) \to (G, A, \tau)$, where $R^a(x) = f[j(x)] = f[(x, a)] = xa$, is soft continuous.
Also since $\varphi : A \times A \to A$ is left bijective, it follows that $\eta = \varphi\varpi$ is bijective.
Therefore, R^a_η is bijective and soft continuous mapping.
Again $[R^a_\eta]^{-1} = R^{[a^{-1}]}_{\eta^{-1}}$ implies that $[R^a_\eta]^{-1}$ is bijective and soft continuous.
Therefore $R^a_\eta : (G, A, \tau) \to (G, A, \tau)$ is a soft homeomorphism.

(ii) The proof is similar to that of (i).

(iii) From part (i) and (ii) we have the soft mappings L^a_ζ and $R^{[a^{-1}]}_\eta$ are soft continuous.
Therefore by Proposition 3.13, the soft mapping $C_\kappa = L^a_\zeta R^{[a^{-1}]}_\eta : (G, A, \tau) \to (G, A, \tau)$, where $C(x) = L^a[R^{a^{-1}}(x)] = L^a[xa^{-1}] = axa^{-1}$, for all $x \in G$, is soft continuous and $\kappa : A \to A$ is bijective.
Therefore, C_κ is bijective and soft continuous mapping.
Similarly the soft mapping $[C_\kappa]^{-1}$ is bijective and soft continuous.
Therefore $C_\kappa : (G, A, \tau) \to (G, A, \tau)$ is a soft homeomorphism.

□

5. Conclusion

In this paper, we have generalized the group soft topology by taking soft composition mapping instead of ordinary group composition mapping and parallel product is replaced by usual product. We have called the resulting structure as generalized group soft topological space and studied some basic properties of this space. Topological groups have many applications in the field of abstract integration theory such as the development of the theory of Haar measure and Haar integral and also in the manifold theory through the development of Lie groups. Here we have studied the topological group theory in soft set setting which is best suited for modeling physical systems involving uncertainty. In this sense, this study has a great significance. There is an ample scope of further research in developing the theory of Haar measure, Haar integral and the theory of Lie group etc in this setting.

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