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On closed graph theorem and riesz theorem in intuitionistic 2-fuzzy 2-normed linear space

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Abstract. In this paper intuitionistic 2-fuzzy 2-Banach space is defined with respect to α-2-norm. A bounded intuitionistic 2-fuzzy 2-linear functional F on $A \times_{*,\diamond} B$ where A and B are subspaces of $(F(X), N, M)$ is also defined. 2-fuzzy 2-linear operator T and intuitionistic 2-fuzzy 2- compact space are introduced. Using these notions Closed Graph Theorem and Riesz Theorem are also established.

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1. Introduction

Eversince the introduction of the seminal concept of "Fuzzy Sets" by L.A.Zadeh [11] in 1964 which pervaded through the fields of scientific knowledge including Engineering and Computer Science, it revolutionized mathematical thinking by developing myriad theories in the realm of fuzzy mathematics.

The notion of fuzzy norm on a linear space was introduced in 1984 by Katsaras[6]. Almost similar work built around fuzzy normed linear space was carried out by Chang and Moderson [4] and Bag and Samanta [2]. Later, the concept of fuzzy linear space was introduced by Jialu Zhang [12] in a different way. Using the satisfactory theory of 2-norm on a linear space developed by Gahler [5], a tremendous extension to fuzzy n-normed linear space was proposed by A.L. Narayanan and S. Vijayabalaji [7]. Fuzzy normed linear space was projected to the next higher level of 2-fuzzy 2-normed linear space and established by R.M.Soma Sundaram and Thangaraj Beaula [10] in 2009. This concept is used to solve Mazur-Ulam problem done by C.Park and C.Alaca [8, 9]. As a further development in the same line, the concept of intuitionistic fuzzy set developed by K. T. Atanassov [1] in 1986 was used in establishing the new notion of intuitionistic 2-fuzzy 2-normed linear space by Thangaraj Beaula and Lilly Esthar Rani [3] in 2012.

In this paper we have defined intuitionistic 2-fuzzy 2-Banach space with respect to α-2-norm. The definition of intuitionistic 2-fuzzy 2-linear operator on $A \times_{*,\diamond} B$ is introduced where A and B are subspaces of an intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ and using these notions the famous Closed Graph Theorem is differently established. Riesz theorem is also established by using the concept of intuitionistic 2-fuzzy 2-compact subspace which we have introduced.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gahler [5], Bag and Samanta [2] and Jialu Zhang [12].

Definition 2.1 ([5])**.** Let X be a real linear space of dimension greater than one and let $||\cdot,\cdot||$ be a real valued function on $X \times X$ satisfying the following conditions:
1. $||x, y|| = 0$ if and only if x and y are linearly dependent,
2. $||x, y|| = ||y, x||$
3. $||\alpha x, y|| = |\alpha|||x, y||$
4. $||x, y + z|| \leq ||x, y|| + ||x, z||$.

$||\cdot,\cdot||$ is called a 2- norm on X and the pair $(X, ||\cdot,\cdot||)$ is called a 2-normed linear space.

Definition 2.2 ([2])**.** Let X be a linear space over K (the field of real or complex numbers). A fuzzy subset N of $X \times R$ (R, the set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in K$:
(N1) For all $t \in R$ with $t \leq 0, N(x, t) = 0$,
(N2) for all $t \in R$ with $t > 0, N(x, t) = 1$, if and only if $x = 0$,
(N3) for all $t \in R$ with $t > 0, N(cx, t) = N(x, t/|c|)$, if $c \neq 0$,
(N4) for all $s, t \in R, x, u \in X, N(x + u, s + t) \geq \min\{\ N(x, s), N(u, t)\}$,
(N5) $N(x, t)$ is a non decreasing function of R and $lim_{t\to\infty} N(x, t) = 1$

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.3 ([12])**.** Let X be any nonempty set and $F(X)$ be the set of all fuzzy sets on X.

For $U, V \in F(X)$ and $k \in K$ the field of real numbers define
$U + V = \{\ (x + y, \lambda \wedge u)|(x, \lambda) \in U, (y, \mu) \in V\}$,
$kU = \{\ (kx, \lambda)|(x, \lambda) \in U\}$

Definition 2.4 ([12])**.** A fuzzy vector space $\tilde{X} = X \times (0, 1]$ over the number field K where the addition and scalar multiplication operations on $\tilde{X}$ are defined by

$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu), k(x, \lambda) = (kx, \lambda)$ is a fuzzy normed space if to every $(x, \lambda) \in \tilde{X}$ there is associated a non-negative real number, $||(x, \lambda)||$, called the fuzzy norm of (x, λ), in such a way that
1. $||(x, \lambda)|| = 0 iff x = 0$ the zero element of $\tilde{X}, \lambda \in (0, 1]$
2. $||k(x, \lambda)|| = |k|||(x, \lambda)||$ for all $(x, \lambda) \in \tilde{X}$ and all $k \in K$
3. $||(x, \lambda) + (y, \mu)|| \leq ||(x, \lambda \wedge \mu)|| + ||(y, \lambda \wedge \mu)||$ for all $(x, \lambda)$and$(y, \mu) \in \tilde{X}$

4. $||(x, V\lambda t|| = t \wedge ||(x, \lambda t||$, for $\lambda, t \in (0, 1]$

Definition 2.5 ([10]). Let X be a non-empty and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{\ (x, \mu)/x \in X$ and $\mu \in (0, 1]\}$ f is a bounded function for$|f(x)| \leq 1$. Let K be the space of real numbers, then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$f + g = \{\ (x, \mu) + (y, \eta)\}\ = \{\ (x + y, \mu \wedge \eta), (x, \mu) \in f$ and $(y, \eta) \in g\}$

$kf = \{\ (kx, \mu)/(x, \mu) \in f\}$, where $k \in K$.

The linear space $F(X)$ is said to be normed space if to every $f \in F(X)$, there is associated a non- negative real number $||f||$ called the norm of f in such a way that

1. $||f|| = 0$ if and only if $f = 0$.
   For $||f|| = 0 \Leftrightarrow \{\ ||(x, \mu)||/(x, \mu) \in f\}\ = 0 \Leftrightarrow x = 0, \mu \in (0, 1] \Leftrightarrow f = 0$
2. $||kf|| = |k|\ ||f||, k \in K$.
   For $||kf|| = \{\ ||k(x, \mu)||/(x, \mu) \in f, k \in K\}$
   $= \{\ |k|||x, \mu||)/(x, \mu) \in f\}\ = |k|||f||$.
3. $||f + g|| \leq ||f|| + ||g||$, for every $f, g \in F(X)$

For, $||f + g|| = \{\ ||(x, \mu) + (y, \eta)||/x, y \in X, \mu, \eta \in (0, 1]\}$
$= \{\ ||(x + y), (\mu \wedge \eta)||/x, y \in X, \mu, \eta \in (0, 1]\}$
$\leq \{\ ||x, \mu \wedge \eta|| + ||y, \mu \wedge \eta||/(x, \mu) \in f$ and $(y, \eta) \in g\}\ = ||f|| + ||g||$.

and $(F(X), ||\ ||)$ is a normed linear space.

Definition 2.6 ([10]). A 2-fuzzy set on X is a fuzzy set on $F(X)$

Definition 2.7 ([10]). Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$.($R$, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

(N1) for all $t \in R$ with $t \leq 0, N(f_1, f_2, t) = 0$,
(N2) for all $t \in R$ with $t > 0, N(f_1, f_2, t) = 1$, if and only if f_1 and f_2 are linearly dependent,
(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2,
(N4) for all $t \in R$, with $t \geq 0, N(f_1, cf_2, t) = N(f_1, f_2, \frac{t}{|c|})$ if $c \neq 0, c \in K$ (field),
(N5) for all $s, t \in R, N(f_1, f_2 + f_3, s + t) \geq \min\{\ N(f_1, f_2, s), N(f_1, f_3, t)\}$,
(N6) $N(f_1, f_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(N7) $\lim_{t\to\infty} N(f_1, f_2, t) = 1$.

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.8. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is commutative and associative,
2. $*$ is continuous,
3. $a * 1 = a$, for all $a \in [0, 1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.9. A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

1. $\diamond$ is commutative and associative,
2. $\diamond$ is continuous,
3. $a \diamond 0 = a$, for all $a \in [0, 1]$,

4. $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Note 2.10. (1) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$ there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamond r_2$.

(2) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \geq r_5$.

Definition 2.11. An intuitionistic fuzzy 2-normed linear space (i.f-2-NLS) is of the form $A = \{\ (F(X), N(f_1, f_2, t), M(f_1, f_2, t))/(f_1, f_2) \in F[(X)]^2\}$ where $F(X)$ is a linear space over a field K of real numbers, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm, N and M are fuzzy sets on $[F(X)]^2 \times (0, \infty)$ such that N denotes the degree of membership and M denotes the degree of non-membership of $(f_1, f_2, t) \in [F(X)]^2 \to (0, \infty)$ satisfying the following conditions:

(1) N (f_1, f_2, t) + M (f_1, f_2, t) $\leq$ 1

(2) N(f_1, f_2, t) $>$ 0

(3) N(f_1, f_2, t) = 1 if and only if f_1, f_2 are linearly dependent

(4) N(f_1, f_2, t) is invariant under any permutation of f_1, f_2

(5) N(f_1, f_2, t) : (0, ∞) $\to$ [0,1] is continuous in t.

(6) N(f_1, cf_2, t) = N (f_1, f_2, $\frac{t}{|c|}$), if c $\neq$ 0, c $\in$ K

(7) N (f_1, f_2, s) * N(f_1, f_3, t) $\leq$ N(f_1, f_2 + f_3, s + t) where * is a continuous t-norm

(8) M (f_1, f_2, t)$>$ 0

(9) M(f_1, f_2, t) = 0 if and only if f_1, f_2 are linearly dependent

(10) M (f_1, f_2, t) is invariant under any permutation of f_1, f_2

(11) M (f_1, cf_2, t) = M (f_1, f_2, $\frac{t}{|c|}$) if c $\neq$ 0, c $\in$ k

(12) M (f_1, f_2, s) $\Diamond$ M (f_1, f_3, t) $\geq$ M (f_1, f_2 + f_2, s + t) where $\Diamond$ is a continuous t-co-norm .

(13) M (f_1, f_2, t) : (0, ∞) $\to$ [0,1] is continuous in t.

Example 2.12. Let $F(X) = \{\ f/f : X \to [0, 1]\}$

Define $D : [F(X)]^2 \to R^+$ as $D(f, g) = \sqrt[3]{\sup[|f(x)| + |g(x)|]^3}$

Let $s * t = \min\{\ s, t\}$ and $s \diamond t = \max\{\ s, t\}$ for all $s, t \in [0, 1]$

Define $N(f, g, t) = \frac{t}{t+D(f,g)}$ and $M(x, t) = \frac{D(f,g)}{t+D(f,g)}$,

Then $A = (F(X), N, M)$ is an intuitionistic fuzzy 2-normed linear space.

Note 2.13. N and M are membership and nonmembership functions satisfying the condition $0 < N(f, g, t) + M(f, g, t) < 1$ and so if $N(f, g, t) \geq \alpha$ then

$M(f, g, t) \leq 1 - \alpha$, using this concept the following theorem is proved.

3. Closed graph theorem on intuitionistic 2-fuzzy 2-normed linear space

Theorem 3.1. *Let $(F(X), N, M)$ be an intuitionistic 2-fuzzy 2-normed linear space. Assume that $N(f_1, f_2, t) > 0$ and $M(f_1, f_2, t) < 1$ for all $t > 0 - - - - (1.1)$ implies f_1 and f_2 are linearly dependent.*

Define $||f_1, f_2||_\alpha = inf\{\ t : N(f_1, f_2, t) \geq \alpha, \alpha \in (0, 1)\}$ and in terms of the non membership function it can be defined as

$||f_1, f_2||_\alpha = \sup\{\ t : M(f_1, f_2, t) \geq 1 - \alpha, \alpha \in (0, 1)\}$.

Then $\{\ ||\cdot, \cdot||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2-norms on $F(X)$. These 2-norms are called α-2-norms on $F(X)$ corresponding to the fuzzy 2-norm.

Proof. Let $f_1, f_2||\alpha = 0$. This implies

(i) $\inf\{\ t : N(f_1, f_2, t) \geq \alpha\}\ = 0$ and $\sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha\}\ = 1$

(ii) For all $t \in R, t > 0$

$N(f_1, f_2, t) \geq \alpha > 0$

$M(f_1, f_2, t) \leq 1 - \alpha \leq 1, \alpha \in (0, 1)$

(iii) f_1, f_2 are linearly dependent from (1.1)

Conversely assume that f_1, f_2 are linearly dependent. This implies

(i) $N(f_1, f_2, t) = 1$ and $M(f_1, f_2, t) = 0$ for all $t > 0$

(ii) For all $\alpha \in (0, 1)$ $\inf\{\ t : N(f_1, f_2, t) \geq \alpha\}\ = 0$ and $\sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha, \alpha \in (0, 1)\}\ = 1$

(iii) $||f_1, f_2||_\alpha = 0$

2. As $N(f_1, f_2, t)$ and $M(f_1, f_2, t)$ are invariant under any permutation it follows that $||f_1, f_2||_\alpha$ is invariant under any permutation

3. If $c \neq 0$ then

$$\begin{aligned} ||f_1, cf_2||_\alpha &= \inf\{\ s : N(f_1, cf_2, s) \geq \alpha\} \\ &= \inf\{\ s : N(f_1, f_2, \tfrac{s}{|c|}) \geq \alpha\} \end{aligned}$$

Similarly

$$\begin{aligned} ||f_1, cf_2||_\alpha &= \sup\{\ s : M(f_1, cf_2, s) \leq 1 - \alpha\} \\ &= \sup\{\ s : M(f_1, f_2, \tfrac{s}{|c|}) \leq 1 - \alpha\} \end{aligned}$$

Let $t = \frac{s}{|c|}$

$$\begin{aligned} ||f_1, cf_2||_\alpha &= \inf\{\ |c|t : N(f_1, f_2, t) \geq \alpha\}\ = |c| \inf\{\ t : N(f_1, f_2, t) \geq \alpha\} \\ &= |c|\ ||f_1, f_2||_\alpha \\ ||f_1, cf_2||_\alpha &= \sup\{\ |c|t : M(f_1, f_2, t) \geq 1 - \alpha\} \\ &= |c| \sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha\}\ = ||f_1, f_2||_\alpha \end{aligned}$$

If $c = 0$ then $||f_1, cf_2||_\alpha = ||f_1, 0||$

$$\begin{aligned} &= 0 \\ &= 0.||f_1, f_2|| \\ &= |c|\ ||f_1, f_2||_\alpha \text{ for every } c \in R \end{aligned}$$

4. $||f_1, f_2||_\alpha + ||f_1, f_3||_\alpha$

$$\begin{aligned} &= \inf\{\ t : N(f_1, f_2, t) \geq \alpha\}\ + \inf\{\ s : N(f_1, f_3, s) \geq \alpha\} \\ &\geq \inf\{\ t + s : N(f_1, f_2, t) \geq \alpha, N(f_1, f_3, s) \geq \alpha\} \\ &\geq \inf\{\ t + s : N(f_1, f_2 + f_3, t + s) \geq \alpha\} \\ &\geq \inf\{\ r : N(f_1, f_2 + f_3, r) \geq \alpha\} \end{aligned}$$

$= ||f_1, f_2 + f_3||_\alpha$

$||f_1, f_2||_\alpha + ||f_1, f_3||_\alpha$

$= \sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha\}\ + \sup\{\ s : M(f_1, f_3, s) \leq 1 - \alpha\}$

$\geq \sup\{\ t + s : M(f_1, f_2, t) \leq 1 - \alpha, M(f_1, f_3, s) le 1 - \alpha\}$

$\geq \sup\{\ t + s : M(f_1, f_2 + f_3, t + s) \geq 1 - \alpha\}$

$\geq \sup\{\ r : M(f_1, f_2 + f_3, r) \geq 1 - \alpha\}$

$= ||f_1, f_2 + f_3||_\alpha$

Therefore $||f_1, f_2 + f_3||_\alpha \leq ||f_1, f_2||_\alpha + ||f_1, f_3||_\alpha$

Thus $\{\ ||\cdot,\cdot||_\alpha : \alpha \in (0, 1)\}$ is an α-2-norm on $F(X)$

5. Let $0 < \alpha_1 < \alpha_2$

Then

$||f_1, f_2||_{\alpha_1} = \inf\{\ t : N(f_1, f_2, t) \geq \alpha_1\}$

$||f_1, f_2||_{\alpha_2} = \inf\{\ t : N(f1, f2, t) \geq \alpha_1\}$

$||f_1, f_2||_{\alpha_1} = \sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha_1\}$

$||f_1, f_2||_{\alpha_2} = \sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha_2\}$

As $\alpha_1 < \alpha_2, \{\ t : N(f_1, f_2, t) \geq \alpha_2\}\ \subset \{\ t : N(f_1, f_2, t) \geq \alpha_1\}$

implies $\inf\{\ t : N(f_1, f_2, t) \leq \alpha_2\}\ \geq \inf\{\ t : N(f_1, f_2, t) \geq \alpha_1\}$

and $\{\ t : M(f_1, f_2, t) \leq 1 - \alpha_2\}\ \subset \{\ t : M(f_1, f_2, t) \leq 1 - \alpha_1\}$

implies $\sup\{\ t : N(f_1, f_2, t) \leq 1 - \alpha_2\}\ \geq \sup\{\ t : N(f_1, f_2, t) \leq 1 - \alpha_1\}$

Therefore $||f_1, f_2||_{\alpha_2} \geq ||f_1, f_2||_{\alpha_1}$

Hence $\{\ ||\cdot,\cdot||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α-2-norms on $F(X)$ corresponding to the fuzzy 2-norm on $F(X)$. □

Definition 3.2. The intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ in which every Cauchy sequence converges is said to be a complete intuitionistic 2-fuzzy 2-normed linear space . The intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ is an intuitionistic 2-fuzzy 2-Banach spaces with respect to α-2-norm if it is a complete intuitionistic 2-fuzzy 2-normed linear space with respect to α-2-norm.

Note 3.3. Let $\{\ f_n\}$ be a Cauchy sequence in $(F(X), N, M)$ with respect to α-2-norm if there exist g and h in $F(X)$ which are linearly independent such that

$\lim ||f_n - f_m, g||_\alpha = 0$ and $\lim ||f_n - f_m, h||_\alpha = 0$

That is $\inf\{\ t : N(f_n - fm, g, t) \geq \alpha\}\ = 0$ and $\sup\{\ t : M(f_n - f_m, g, t) \leq 1 - \alpha\}\ = 0$, $\inf\{\ t : N(f_n - f_m, h, t) \geq \alpha\}\ = 0$ and $\sup\{\ t : M(f_n - f_m, h, t) \leq 1 - \alpha\}$ =0 yields a sequence of real numbers that automatically converges. Therefore $(F(X), N, M)$ under α-2-norm is an intuitionistic 2-fuzzy 2-Banach space.

Definition 3.4. An intuitionistic 2-fuzzy 2-linear functional F is a real valued function on $A_{*,\diamond}B$ where A and B are subspaces of $(F(X), N, M)$ such that

(1) $F(f + h, g + h') = F(f, g) + F(f, h') + F(h, g) + F(h, h')$

(2) $F(\alpha f, \beta g) = \alpha\beta F(f, g)$ where $\alpha, \beta \in [0, 1]$

F is said to be bounded with respect to α-2-norm if there exists a constant $k \in [0, 1]$ such that $|F(f, g)| \leq k||(f, g)||$ for every $(f, g) \in A_{*,\diamond}B$

If F is bounded then define

$||F|| = glb\{\ k : ||F(f, g)|| \leq k||(f, g)||_\alpha$ for every $(f, g) \in A_{*,\diamond}B\}$

Example 3.5. Let $(F(X), N, M)$ be an intuitionistic fuzzy 2-normed linear space and A, B be subspaces of $F(X)$. Define a real valued function $F : A \times B \to R$ as

$F(f,g) = \eta(f,g,t)$ where η is the 2-fuzzy inner product defined in [**?**]. Then by properties of η, F satisfies the requirements and so F is 2- fuzzy 2-linear functional.

Definition 3.6. An intuitionistic 2-fuzzy 2-linear operator T is a function from $A \times_{*,\diamond} B$ to $C \times_{*,\diamond} D$ where A, B are subspaces of intuitionistic 2-fuzzy 2-normed linear space $(F(X), N_1, M_1)$ and C, D are subspaces of intuitionistic 2-fuzzy 2-normed linear space $(F(Y), N_2, M_2)$ such that
$T(f+h, g+h') = T(f,g) + T(f,h') + T(h,g) + T(h,h')$
and $T(\alpha f, \beta g) = \alpha\beta FT(f,g)$ where $\alpha, \beta \in [0,1]$

4. Closed graph theorem

Theorem 4.1. *Let T be an intuitionistic 2-Fuzzy 2-linear mapping from an intuitionistic 2-fuzzy 2-Banach space $(F(X), N_1, M_1 \times F(X), N_1, M_1)$ to intuitionistic 2-fuzzy 2-Banach space $(F(Y), N_2, M_2 \times F(X), N_2, M_2)$. Suppose for every sequence $\{ f_n, f_{n'}\}$ in$(F(X) \times F(X))$ such that $(fn, fn') \to (f, f')$ and $T(f_n, f_{n'}) \to (g, g')$ for some $f, f' \in F(X), g, g' \in F(Y)$ it follows that $T(f, f') = (g, g')$. Then T is continuous.*

Proof. The fuzzy 2- norm on $(F(X), N_1, M_1) \times_{*,\diamond} (F(Y), N_2, M_2)$ is given by
$N((f_1, f_2), (g_1, g_2), t) = N_1(f_1, f_2, t_1) * N_2(g_1, g_2, t_2)$where $t_1 + t_2 = t$
$M((f_1, f_2), (g_1, g_2), t) = M_1(f_1, f_2, t_1) \diamond M_2(g_1, g_2, t_2)$ where $t_1 + t_2 = t$
where $*$ is the usual continuous t-norm and $\diamond$ is the continuous t-conorm.

With these norms $(F(X), N_1, M_1) \times_{*,\diamond} (F(Y), N_2, M_2)$ is a complete intuitionistic 2-fuzzy 2-normed linear space. For each $(f_1, f_2), (f_1', f_2') \in F(X) \times F(X)$ and $(g_1, g_2), (g_1', g_2') \in F(Y) \times F(Y)$ and $t, s > 0$ it follows that

$$\begin{aligned}
&N((f_1, f_2), (g_1, g_2), t) * N((f_1', f_2'), (g_1', g_2'), s) \text{ where } t = t_1 + t_2, s = s_1 + s_2\\
&\quad = [N_1((f_1, f_2, t_1) * N_2((g_1, g_2, t_2)] * [N_1((f_1', f_2', s_1) * N_2((g_1', g_2', s_2)]\\
&\quad = [N_1((f_1, f_2, t_1) * N_1((f_1', f_2', s_1)] * [N_2((g_1, g_2, t_2) * N_2((g_1', g_2', s_2)]\\
&\quad \le N_1(f_1 + f_1', f_2 + f_2', s_1 + t_1)) * N_2((g_1 + g_1', g_2 + g_2', s_2 + t_2)\\
&\quad = N(f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s_1 + s_2 + t_1 + t_2)\\
&\quad = N(f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s + t)
\end{aligned}$$

$$\begin{aligned}
&M((f_1, f_2), (g_1, g_2), t) \diamond M((f_1', f_2'), (g_1', g_2'), s)\\
&\quad = [M_1((f_1, f_2, t_1) \diamond M_2((g_1, g_2, t_2)] * [M_1((f_1', f_2', s_1) \diamond M_2((g_1', g_2', s_2)]\\
&\quad = [M_1((f_1, f_2, t_1) \diamond M_1((f_1', f_2', s_1)] \diamond [M_2((g_1, g_2, t_2) \diamond M_2((g_1', g_2', s_2)]\\
&\quad \le M_1(f_1 + f_1', f_2 + f_2', s_1 + t_1)) \diamond M_2((g_1 + g_1', g_2 + g_2', s_2 + t_2)\\
&\quad = M(f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s_1 + s_2 + t_1 + t_2)\\
&\quad = M(f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s + t)
\end{aligned}$$

Now if $\{ (f_n, f_n'), (g_n, g_n')\}$ is a Cauchy sequence in $(F(X) \times F(X) \times F(Y) \times F(Y), N, M)$ then there exists $n_0 \in N$ such that
$N((f_n, f_n'), (g_n, g_n') - (f_m, f_m'), (g_m, g_m'), t)) > 1 - \varepsilon$ and
$M((f_n, f_n'), (g_n, g_n') - (f_m, f_m'), (g_m, g_m'), t)) < \varepsilon$ for every $\varepsilon > 0$ and $t > 0$

$$\begin{aligned}
&N_1(f_n - f_m, f_n' - f_m, t) * N_2(g_n - g_m, g_n' - g_m', t)\\
&\quad = N((f_n - f_m, f_n' - f_m', t), (g_n - g_m, g_n' - g_m', t))\\
&\quad = N((f_n, f_n'), (g_n, g_n')), ((f_m, f_m'), (g_m, g_m')), t) > 1 - \varepsilon
\end{aligned}$$

$M_1(f_n - f_m, f'_n - f_m, t) \diamond M_2(g_n - g_m, g'_n - g'_m, t)$
$= M((f_n - f_m, f'_n - f'_m, t), (g_n - g_m, g'_n - g'_m, t))$
$= M((f_n, f'_n), (g_n, g'_n)), ((f_m, f'_m), (g_m, g'_m)), t) < \varepsilon$

Therefore $\{\ (f_n, f'_n)\}$ and $\{\ (g_n, g'_n)\}$ are Cauchy sequences and so there exist $f, f' \in F(X) \times F(X)$ and $g, g' \in F(Y) \times F(Y)$ such that
$(f_n, f'_n) \to (f, f')$ and $(g_n, g'_n) \to (g, g')$. Consequently
$\{\ (f_n, f'_n), (g_n, g'_n)\}$ converges to $((f, f'), (g, g'))$

Hence $(F(X) \times F(X) \times F(Y) \times F(Y), N, M)$ is a complete intuitionistic 2-fuzzy 2-normed linear space .

Here let $G = \{\ ((f_n, f_n), T(f_n, f_n))$ for every $(f_n, f'_n) \in F(X) \times F(X)\}$ be the graph of the intuitionistic 2-fuzzy 2-normed linear space. Suppose $(f_n, f'_n) \to (f, f')$ and $T(f_n, f'_n) \to (g, g')$

Then from the previous argument $((f_n, f_n), (T(f_n), T(f'_n))$ converge to
$((f, f'), (g, g')$ which belongs to G .

We abserve that $T(f, f') = (g, g')$ and hence T is continuous. □

5. Riesz theorem

Definition 5.1. A subspace $(F(Y), N, M)$ of an intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ is called intuitionistic 2-fuzzy 2-compact space if for every sequence $\{\ f_n\}$ in $F(Y)$ there exists a subsequence $\{\ f_n\}$ which converges to an element $f \in F(Y)$

In other words given $t > 0, 0 < \varepsilon < 1$ there exists a positive integer n_0 such that $N(f_{n_k} - f, g, t) > 1 - \varepsilon$ and $M(f_{n_k} - f, g, t) < \varepsilon$ for all $n, n_k \geq n_0$

Lemma 5.2. *Let $(F(X), N, M)$ be an intuitionistic 2-fuzzy 2-normed linear space and $(F(Y), N, M)$ a fuzzy 2-compact space of $(F(X), N, M)$. For $f_1, f_2 \in F(X)$*
$\min[\inf\{\ t : N(f_1 - f, f_2 - g, t)\}\ \geq \alpha] = 0$ *where $g \in F(Y)$ and*
$\max[\sup\{\ t : M(f_1 - f, f_2 - g, t)\}\ \leq 1 - \alpha] = 1$ *where $g \in F(Y)$*
then there exists a function $g_0 \in F(Y)$ such that $\inf\{\ t : N(f_1 - f, f_2 - g, t)\}\ = 0$ $\sup\{\ t : M(f_1 - f, f_2 - g, t)\}\ = 1$

Proof. For each integer k, there exists a function $g_k \in F(Y)$ such that
$\inf\{\ t : N(f_1 - g_k, f_2 - g_k, t) \geq \alpha\}\ < \frac{1}{k}$
$\sup\{\ t : N(f_1 - g_k, f_2 - g_k, t) \leq 1 - \alpha\}\ > 1 - \frac{1}{k} \quad - - - - - (1)$

Since $\{\ g_k\}$ is a sequence in a fuzzy 2-compact space $F(Y)$, we can consider a convergent sequence $\{\ g_k\}$ in $F(Y)$, without loss of generality. Let $g_k \to g_0$ as $k \to 0$ for some $g_0 \in F(Y)$

For every $\varepsilon > 0$ with $0 < \varepsilon < 1$ there exists a positive integer K with $\frac{1}{K} < \frac{\varepsilon}{3}$
such that $k < K$ implies $\inf\{\ t : N(g_k - g_0, h, t) \geq \alpha, \alpha \in (0,1)\}\ < \frac{\varepsilon}{3}$
$\sup\{\ t : M(g_k - g_0, h, t) \leq 1 - \alpha, \alpha \in (0,1)\}\ < 1 - \frac{\varepsilon}{3}$ for all $h \in F(X)$ $- - - - (2)$

Consider
$\inf\{\ t + t' : N(f_1 - g_0, f_2 - g_0, t + t') \geq \alpha\}$
$\leq \inf\{\ t + t' : N(f_1 - g_k + g_k - g_0, f_2 - g_0, t + t') \geq \alpha\}$
$\leq \inf\{\ t : N(f_1 - g_k, f_2 - g_0, t) \geq \alpha\}\ + \inf\{\ t' : N(g_k - g_0, f_2 - g_0, t') \geq \alpha\}$
$= \inf\{\ t + s : N(f_1 - g_k, f_2 - g_k + g_k - g_0, t + s) \geq \alpha\}$
$+ \inf\{\ t' : N(g_k g_0, f_2 - g_0, t') \geq \alpha\}$
$\leq \inf\{\ t : N(f_1 - g_k, f_2 - g_k, t) \geq \alpha\}$

$+inf\{\ s : N(f_1 - g_k, g_k - g_0, s) \geq \alpha\}$
$+\inf\{\ t' : N(g_k - g_0, f_2 - g_0, t') \geq \alpha\}$
$< \frac{1}{k} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$
$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}$
$= \varepsilon$

Since ε is arbitrary

$\min[\inf_{g\in F(X)}\{\ t : N(f_1 - f, f_2 - g, t)\}\ ] = 0$

Further more,

$\sup\{\ t + t' : M(f_1 - g_0, f_2 - g_0, t + t') \leq 1 - \alpha\}$
$= \sup\{\ t + t' : M(f_1 - g_k + g_k - g_0, f_2 - g_0, t + t') \leq 1 - \alpha\}$
$\geq \sup\{\ t : M(f_1 - g_k, f_2 - g_0, t) \leq 1 - \alpha\} + \sup\{\ t' : M(g_k - g_0, f_2 - g_0, t') \leq 1 - \alpha\}$
$= \sup\{\ t + s : M(f_1 - g_k, f_2 - g_k + g_k - g_0, t + s) \leq 1 - \alpha\}$
$+ \sup\{\ t' : M(g_k - g_0, f_2 - g_0, t') \leq 1 - \alpha\}$
$\geq \sup\{\ t : M(f_1 - g_k, f_2 - g_k, t) \leq 1 - \alpha\} + \sup\{\ s : M(f_1 - g_k, g_k - g_0, s) \leq 1 - \alpha\}$
$+ \sup\{\ t' : M(g_k - g_0, f_2 - g_0, t') \leq 1 - \alpha\}$
$> 2(1 - \frac{\varepsilon}{3}) + (1 - \frac{1}{k})$
$> 2(1 - \frac{\varepsilon}{3}) + (1 - \frac{\varepsilon}{3})$ (k is chosen arbitrarily)
$= 1 - \frac{\varepsilon}{3}$

Since ε is arbitrary

$\max[\sup_{g\in F(Y)}\{\ t : M(f_1 - f, f_2 - g, t)\}\ ] = 1$ □

Theorem 5.3. ***RIESZ THEOREM*** *Let Y and Z be subspaces of intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ and Y be a fuzzy 2-compact proper subspace of Z with dimension greater than one. For each*
$\theta \in (0, 1)$ *there exists an element $(f_1, f_2) \in Z \times Z$ such that*
$\inf\{\ t : N(f_1, f_2, t) \geq \alpha\}\ = 1$ *and* $\sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha\}\ = 0$,
That is, $\inf\{\ t : N(f_1 - f, f_2 - g, t)\}\ \geq \alpha\}\ \geq \theta$
$\sup\{\ t : M(f_1 - f, f_2 - g, t)\}\ \leq 1 - \alpha\}\ \leq \theta$ *for all* $g \in Y$

Proof. Let $f_1, f_2 \in Z$ be linearly independent.

Let $\min[\inf\{\ t : N(f_1 - g_0, f_2 - g_0, t) \geq \alpha\}\ ] = a$ and
$\max[\sup\{\ t : M(f_1 - g_0, f_2 - g_0, t) \leq 1 - \alpha\}\ ] = b$

Assume $a = 0$ and $b = 1$ then by theorem 5.2 there exists $g_0 \in Y$, such that
$\inf\{\ t : N(f_1 - g_0, f_2 - g_0, t) \geq \alpha\}\ = 0$ and
$\sup\{\ t : M(f_1 - g_0, f_2 - g_0, t) \leq 1 - \alpha\}\ = 1$

If $g_0 = 0$ then $\inf\{\ t : N(f_1, f_2, t) \geq \alpha\}\ = 0$
$\sup\{\ t : M(f_1, f_2, t) \leq 1 - \alpha\}\ = 1$

This implies f_1 and f_2 are linearly dependent which is a contradiction to the fact that f_1, f_2, g_0 are linearly independent if g_0 is nonzero. From definition and from (1) it follows that $f_1 - g_0, f_2 - g_0$ are linearly dependent. So there exist scalars α_1, α_2 not all zero such that

Thus $\alpha_1(f_1 - g_0) + \alpha_2(f_2 - g_0) = 0$

Thus $\alpha_1 f_1 + \alpha_2 f_2 + (-1)(\alpha_1 + \alpha_2)g_0 = 0$

Then f_1, f_2, g_0 are linearly dependent which is a contradiction.

Hence $a > 0$ and $b < 1$. For each $\theta \in (0, 1)$ there exits an element $g_0 \in Y$ such that

$a \leq \inf\{\ t : N(f_1 - g_0, f_2 - g_0, t) \geq \alpha\}\ \leq \frac{a}{\theta}$ and

$b \geq \sup\{\ t : M(f_1 - g_0, f_2 - g_0, t) \leq 1 - \alpha\}\ \geq \frac{b}{\theta}$
Define
$d(f_1, f_2) = \{\ t : \inf\{\ t : N(f_1 - g_0, f_2 - g_0, t) \geq \alpha\}$ and
$d'(f_1, f_2) = \{\ t : \sup\{\ t : M(f_1 - g_0, f_2 - g_0, t) \leq 1 - \alpha\}$
For each $j = 1, 2$ define
$h_j = \frac{(f_1 - g_0)}{(d(f_1, f_2))^{\frac{1}{2}}}$
Consider
$\inf\{\ t : N(h_1, h_2, t) \geq \alpha\}$
$= \inf\{\ (t : N(\frac{f_1 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}}, \frac{f_2 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}}, t) \geq \alpha)\}$
$= \inf\{\ t : N(f_1 - g_0, f_2 - g_0, td(f_1, f_2) \geq \alpha\}$
Equivalently,
$\sup\{\ t : M(h_1, h_2, t) \leq 1 - \alpha\}$
$= \sup\{\ (t : M(\frac{f_1 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}}, \frac{f_2 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}}, t) \leq 1 - \alpha)\}$
$= \sup\{\ t : M(f_1 - g_0, f_2 - g_0, td(f_1, f_2) \leq 1 - \alpha\}$
$d(h_1, h_2) = \{\ t : N(f_1 - g_0, f_2 - g_0, td(f_1, f_2) \geq \alpha\}$ and
$d'(h_1, h_2) = \{\ t : \sup\{\ t : M(f_1 - g_0, f_2 - g_0, td(f - 1, f_2) \leq 1 - \alpha\}$
$\inf\{\ t : N(h_1 - g, h_2 - g, t) \geq \alpha\}$
$= \inf\{\ (t : N(\frac{f_1 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}} - g, \frac{f_2 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}} - g, t) \geq \alpha)\}$
$= \inf\{\ t : N(f_1 - g_0 - g(d(f_1, f_2))^{\frac{1}{2}}, f_2 - g_0 - g(d(f_1, f_2))^{\frac{1}{2}}, d(f_1, f_2)t) \geq \alpha\}$
$= \frac{a}{a/\theta} = \theta$
Similarly,
$\sup\{\ t : M(h_1 - g, h_2 - g, t) \leq 1 - \alpha\}$
$= \sup\{\ (t : M(\frac{f_1 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}} - g, \frac{f_2 - g_0}{(d(f_1, f_2))^{\frac{1}{2}}} - g, t) \leq 1 - \alpha)\}$
$= \sup\{\ t : N(f_1 - g_0 - g(d(f_1, f_2))^{\frac{1}{2}}, f_2 - g_0 - g(d(f_1, f_2))^{\frac{1}{2}}, d(f_1, f_2)t) \leq \alpha\}$
$= \frac{b}{B/\theta} = \theta$ for all $g \in Y$. □

Definition 5.4. A subset Y of an intuitionistic 2-fuzzy 2-normed linear space $(F(X), N, M)$ is said to be a fuzzy partially 2-closed subset if for every pair of linearly independent elements $f_1, f_2 \in F(X)$ there exits a sequence $\{\ g_k\}$ in Y such that
$\inf\{\ t : N(f_1 - g_k, f_2 - g_k, t) \geq 1 - \alpha\}$ tends to $'1'$ and
$\sup\{\ t : M(f_1 - g_k, f_2 - g_k, t) \leq \alpha\}$ tends to $'0'$ as $k \to \infty$ then $f_j \in Y$ for some j.

Example 5.5. Let B be the set consisting of all convergent real sequences in X. $\{\ g_n\}$ is convergent in $(F(X), ||\ ||)$ where $g_n(x) = x_n, (x_n) \in B$
define $||g|| = sup_{x \in X}[g(x)]$
$\{\ g_n\}$ is convergent in $(F(X), N, M)$. Let Y be subset of $(F(X), N, M)$
Then Y is partially closed
For,
$N(f - g_n, g - g_n, t) > 1 - \alpha$ and $M(f - g_n, g - g_n, t) < \alpha$ for $\alpha \in [0, 1]$.

Theorem 5.6. *Let Y, Z be subspaces of $F(X)$ then intuitionistic 2-fuzzy 2-normed linear space and Y be fuzzy partially closed subspace of Z. Assume dim $Z \geq 2$. For each $\theta \in$ (0,1) there exits an element $(h_1, h_2) \in Z^2$ such that*

$\inf\{\ t : N(h_1, h_2, t) \geq \alpha\}\ = 1$ *and*
$\sup\{\ t : M(h_1, h_2, t) \leq 1 - \alpha\}\ = 0$ *with*
$\inf\{\ t : N(h_1 - g, h_2 - g, t) \geq 1 - \alpha\}\ \geq \theta$ *and*
$\sup\{\ t : M(h_1 - g, h_2 - g, t) \leq \alpha\}\ \leq \theta$ *for all* $g \in Y$

Proof. Let $f_1, f_2 \in Z - Y$ be linearly independent and
let $a = \min\{\ \inf\{\ t : N(f_1 - g, f_2 - g, t) \geq \alpha\}\ \}$ and
$b = \max\{\ \sup\{\ t : M(f_1 - g, f_2 - g, t) \leq 1 - \alpha\}\ \}$
Assume $a = 0$ and $b = 1$
Then there exists a sequence $\{\ g_k\}\ \in Y$
such that $\inf\{\ t : N(f_1 - g_k, f_2 - g_k, t) \geq 1 - \alpha\}\ \to 0$ as $k \to \infty$ and
$\sup\{\ t : M(f_1 - g_k, f_2 - g_k, t) \leq 1 - \alpha\}\ \to 0$ as $k \to \infty$ which is a contradiction since Y is fuzzy partially 2-closed and $f_j \in Y$ for some j.

Hence $a > 0$ and $b < 0$. Repeating the same argument as in the previous theorem the result follows. □

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986) 87–96.
[2] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear space, J. Fuzzy Math. 11(3) (2003) 687–705.
[3] T. Beaula and L. E. Rani, Some aspects of intuitionistic 2-fuzzy 2-normed linear spaces, J. Fuzzy Math. 20(2) (2012) 371–378.
[4] S. C. Chang and J. N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. 86(5) (1994) 429–436.
[5] S. Gahler, Lineare 2-normierte, Math. Nachr. 28 (1964) 1–43.
[6] A. K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems 12 (1994) 143–154.
[7] A. L. Narayanan and S. Vijayabalaji, Fuzzy n-normed linear space, Int. J. Math. Math. Sci. 2005, no. 24, 3963–3977.
[8] C. Park and C. Alaca, An introduction to 2-fuzzy n-normed linear spaces and a new perspective to the Mazur-Ulam problem, J. Inequal. Appl. 2012, 2012:14, 17 pp.
[9] C. Park and C. Alaca, Mazur-Ulam problem under weaker conditions in the frame work of 2-fuzzy 2-normed linear spaces, J. Inequal. Appl. 2013(1) 1–9.
[10] R. M. Somasundaram and T. Beaula, Some aspects of 2-fuzzy 2-normed linear space, Bull. Malays. Math. Sci. Soc. 32(2) (2009) 211–221.
[11] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1964) 338–353.
[12] J. Zhang, The continuity and boundedness of fuzzy linear spaces, J. Fuzzy Math. 13 (2005) 519–536.

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