Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 5, (May 2015), pp. 753–769
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy inner product and fuzzy product space

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Received 19 August 2014; Accepted 30 October 2014

Abstract. Properties of operations (addition and scalar multiplication), orderings, and fuzzy set-valued convex mappings for general fuzzy sets are considered. In the present paper, a fuzzy inner product and a fuzzy product space for general fuzzy sets are proposed, and their properties with respect to operations, orderings, and fuzzy set-valued convex mappings are investigated.

2010 AMS Classification: 03E72

Keywords: Fuzzy inner product, Fuzzy product space, Fuzzy max order, Fuzzy set-valued convex mapping.

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1. Introduction

The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [19] as fuzzy set theory. Then, fuzzy set theory has been applied in various areas such as economics, management science, engineering, optimization theory, operations research, etc. (Gupta and Dangar [5], Kurano et al. [8], Mahapatra et al. [12], Saraj and Sadeghi [14], Wu [15, 16, 17], and Yoshida [18]). Fuzzy numbers and fuzzy vectors are often used in applications. A fuzzy number is a fuzzy set on $\mathbb{R}$ with some restrictions, and is interpreted as the fuzzy set of real numbers around some real number. A fuzzy vector is a fuzzy set on $\mathbb{R}^n$ with some restrictions, and is interpreted as the fuzzy set of vectors around some vector. The usual restrictions are support boundedness, closedness, convexity, and normality (Bortolan and Degani [1], Dubois et al. [3], Furukawa [4], Kurano et al. [9], Maeda [11], and Ramík and Řimánek [13]). Properties of operations (addition and scalar multiplication), orderings, and fuzzy set-valued convex mappings for general fuzzy sets rather than fuzzy numbers or fuzzy vectors are investigated by Kon [7].

In the present paper, a fuzzy inner product and a fuzzy product space for general fuzzy sets are proposed, and their properties with respect to operations, orderings, and fuzzy set-valued convex mappings are investigated.

The remainder of the present paper is organized as follows. In Section 2, some notations and auxiliary results are presented. In Section 3, we investigate fundamental properties of an inner product and product sets for crisp sets with respect to operations and orderings. In Section 4, operations and the (strict) fuzzy max order of fuzzy sets are defined, and their properties are presented. In Section 5, the fuzzy inner product of fuzzy sets is defined, and relationships between it and the (strict) fuzzy max order are investigated. In Section 6, we consider the fuzzy product space, and investigate properties of operations, the (strict) fuzzy max order, and the fuzzy inner product on the fuzzy product space. In Section 7, the definition of fuzzy set-valued convex mappings is presented, and its properties are investigated. Finally, conclusions are presented in Section 8.

2. Preliminaries

In this section, some notations and auxiliary results are presented.

For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}, [a, b[= \{x \in \mathbb{R} : a \leq x < b\},]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. We set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \leq \boldsymbol{0}\}$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$ be the canonical basis of $\mathbb{R}^n$. For $S \subset \mathbb{R}^n$, we denote the interior of S by $\mathrm{int}(S)$. Let $\mathcal{C}(\mathbb{R}^n)$, $\mathcal{BC}(\mathbb{R}^n)$, $\mathcal{K}(\mathbb{R}^n)$, and $\mathcal{BCK}(\mathbb{R}^n)$ be sets of all closed, compact, convex, and compact convex subsets of $\mathbb{R}^n$, respectively.

For the notational convenience, we identify a fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ with its membership function $\widetilde{s} : \mathbb{R}^n \to [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$.

Let $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. For each $\alpha \in]0, 1]$, the set $[\widetilde{s}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) \geq \alpha\}$ is called the α-level set of $\widetilde{s}$. The set $\mathrm{supp}(\widetilde{s}) = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$ is called the support of $\widetilde{s}$, and $\mathrm{hgt}(\widetilde{s}) = \sup_{\boldsymbol{x} \in \mathbb{R}^n} \widetilde{s}(\boldsymbol{x})$ is called the height of $\widetilde{s}$. The fuzzy set $\widetilde{s}$ is said to be normal if $\mathrm{hgt}(\widetilde{s}) = 1$. We set $I(\widetilde{s}) = \{\alpha \in]0, 1] : [\widetilde{s}]_\alpha \neq \varnothing\}$.

For a crisp set $S \subset \mathbb{R}^n$, a function $c_S : \mathbb{R}^n \to \{0, 1\}$ defined as $c_S(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in S$, and $c_S(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin S$ for each $\boldsymbol{x} \in \mathbb{R}^n$ is called the indicator function of S. For $\boldsymbol{0} \in \mathbb{R}^n$, we set $\widetilde{\boldsymbol{0}} = c_{\{\boldsymbol{0}\}} \in \mathcal{F}(\mathbb{R}^n)$.

A fuzzy set $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{s} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{s}]_\alpha}, \tag{2.1}$$

which is well-known as the decomposition theorem (Dubois et al. [2]).

Let $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. The fuzzy set $\widetilde{s}$ is said to be closed if $\widetilde{s}$ is an upper semicontinuous function, and $\widetilde{s}$ is closed if and only if $[\widetilde{s}]_\alpha \in \mathcal{C}(\mathbb{R}^n)$ for any $\alpha \in]0, 1]$. The fuzzy set $\widetilde{s}$ is said to be convex if $\widetilde{s}(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \geq \min\{\widetilde{s}(\boldsymbol{x}), \widetilde{s}(\boldsymbol{y})\}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$. That is, $\widetilde{s}$ is said to be convex if $\widetilde{s}$ is a quasiconcave function, and $\widetilde{s}$ is convex if and only if $[\widetilde{s}]_\alpha \in \mathcal{K}(\mathbb{R}^n)$ for any $\alpha \in]0, 1]$. The fuzzy set $\widetilde{s}$ is said to be compact if $[\widetilde{s}]_\alpha \in \mathcal{BC}(\mathbb{R}^n)$ for any $\alpha \in]0, 1]$. If $\widetilde{s}$ is support bounded and closed, then $\widetilde{s}$ is compact. Note that $\widetilde{s}$ is not always support bounded even if $\widetilde{s}$ is compact. Let $\mathcal{FC}(\mathbb{R}^n)$, $\mathcal{FBC}(\mathbb{R}^n)$, $\mathcal{FK}(\mathbb{R}^n)$, and $\mathcal{FBCK}(\mathbb{R}^n)$ be sets of all closed, compact, convex, and compact convex fuzzy sets on $\mathbb{R}^n$, respectively.

We set

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in]0,1]} : \; & S_\alpha \subset \mathbb{R}^n, \alpha \in]0, 1], \\ & \text{and } S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in]0, 1] \text{ with } \beta < \gamma\}, \end{aligned} \tag{2.2}$$

and define $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M(\{S_\alpha\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha} \tag{2.3}$$

for each $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. For $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it follows that $M(\{S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\}$, where $\sup\varnothing = 0$. The decomposition theorem (2.1) can be represented as $\widetilde{s} = M(\{[\widetilde{s}]_\alpha\}_{\alpha\in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. When $\widetilde{s} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{s}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha\in]0,1]}$, and $\{S_\alpha\}_{\alpha\in]0,1]}$ is called the generator of $\widetilde{s}$.

The following proposition shows a relationship between the inclusion relation of two generators of two fuzzy sets and the inclusion relation of the two fuzzy sets.

Proposition 2.1. (Kon [6]) *Let* $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. *If* $S_\alpha \subset T_\alpha$ *for any* $\alpha \in]0,1]$, *then* $M(\{S_\alpha\}_{\alpha\in]0,1]}) \leq M(\{T_\alpha\}_{\alpha\in]0,1]})$.

The following proposition shows a relationship between a generator of a fuzzy set and level sets of the fuzzy set.

Proposition 2.2. (Kon [6]) *Let* $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, *and let* $\widetilde{s} = M(\{S_\alpha\}_{\alpha\in]0,1]})$. *Then,* $[\widetilde{s}]_\alpha = \bigcap_{\beta\in]0,\alpha[} S_\beta$ *for* $\alpha \in]0,1]$.

3. Fundamental properties of crisp sets

In this section, we investigate fundamental properties of an inner product and product sets for crisp sets with respect to operations and orderings.

For $A, B \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define $A + B$, $\lambda A \subset \mathbb{R}^n$ as $A + B = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\}$ and $\lambda A = \{\lambda\boldsymbol{x} : \boldsymbol{x} \in A\}$.

We define orders on $2^{\mathbb{R}^n}$.

Definition 3.1. (Kurano et al. [9], Kuroiwa et al. [10], Maeda [11]) Let $A, B \subset \mathbb{R}^n$.

(i) We write $A \leq B$ or $B \geq A$ if $B \subset A + \mathbb{R}^n_+$ and $A \subset B + \mathbb{R}^n_-$.
(ii) We write $A < B$ or $B > A$ if $B \subset A + \mathrm{int}(\mathbb{R}^n_+)$ and $A \subset B + \mathrm{int}(\mathbb{R}^n_-)$.

Let $A, B \subset \mathbb{R}^n$. $B \subset A + \mathbb{R}^n_+$ if and only if for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} \leq \boldsymbol{y}$. $A \subset B + \mathbb{R}^n_-$ if and only if for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} \leq \boldsymbol{y}$. $B \subset A + \mathrm{int}(\mathbb{R}^n_+)$ if and only if for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} < \boldsymbol{y}$. $A \subset B + \mathrm{int}(\mathbb{R}^n_-)$ if and only if for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} < \boldsymbol{y}$. It can be shown easily that the order $\leq$ in Definition 3.1 is a pseudo-order on $2^{\mathbb{R}^n}$.

We define the inner product of crisp sets.

Definition 3.2. (i) For $A, B \subset \mathbb{R}^n$,

$$\langle A, B\rangle = \{x \in \mathbb{R} : x = \langle \boldsymbol{y}, \boldsymbol{z}\rangle, \boldsymbol{y} \in A, \boldsymbol{z} \in B\} \tag{3.1}$$

is called the inner product of A and B, where $\langle \boldsymbol{y}, \boldsymbol{z}\rangle$ is the canonical inner product of $\boldsymbol{y}$ and $\boldsymbol{z}$.

(ii) For $A \subset \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^n$,

$$\langle A, \boldsymbol{b}\rangle = \{x \in \mathbb{R} : x = \langle \boldsymbol{y}, \boldsymbol{b}\rangle, \boldsymbol{y} \in A\} \tag{3.2}$$

is called the inner product of A and $\boldsymbol{b}$. Furthermore, $\langle \boldsymbol{b}, A\rangle = \langle A, \boldsymbol{b}\rangle$ is called the inner product of $\boldsymbol{b}$ and A.

The following proposition shows fundamental properties of the inner product for crisp sets. It can be shown easily.

Proposition 3.3. *Let $A, B, C \subset \mathbb{R}^n$, and let $\lambda \in \mathbb{R}$.*

(i) $\langle A, B\rangle = \langle B, A\rangle$.
(ii) $\langle A + B, C\rangle \subset \langle A, C\rangle + \langle B, C\rangle$.
(iii) *It does not always hold that* $\langle A + B, C\rangle \supset \langle A, C\rangle + \langle B, C\rangle$.
(iv) $\langle \lambda A, B\rangle = \lambda\langle A, B\rangle$.
(v) *It does not always hold that* $\langle A, A\rangle \geq \{0\}$.
(vi) $A = \{\mathbf{0}\} \Leftrightarrow \langle A, A\rangle = \{0\}$.
(vii) $A \neq \varnothing \Rightarrow \langle A, \{\mathbf{0}\}\rangle = \{0\}$.

The following proposition shows relationships between orderings and the inner product of crisp sets.

Proposition 3.4. *Let $A, B \subset \mathbb{R}^n$.*

(i) $A \leq B \Rightarrow \langle A, \boldsymbol{d}\rangle \leq \langle B, \boldsymbol{d}\rangle, \boldsymbol{d} \in \mathbb{R}^n_+$.
(ii) $A, B \in \mathcal{BCK}(\mathbb{R}^n)$ *and* $\langle A, \boldsymbol{d}\rangle \leq \langle B, \boldsymbol{d}\rangle, \boldsymbol{d} \in \mathbb{R}^n_+ \Rightarrow A \leq B$.
(iii) $A < B \Rightarrow \langle A, \boldsymbol{d}\rangle < \langle B, \boldsymbol{d}\rangle, \boldsymbol{d} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.
(iv) $A, B \in \mathcal{K}(\mathbb{R}^n)$ *and* $\langle A, \boldsymbol{d}\rangle < \langle B, \boldsymbol{d}\rangle, \boldsymbol{d} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \Rightarrow A < B$.

Proof. (i) and (ii) are special cases of Lemma 4.4 in Kurano et al. [9].

(iii) Assume that $A < B$, and let $\boldsymbol{d} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. Then, it is sufficient to show that (iii-1) for any $x \in \langle A, \boldsymbol{d}\rangle$, there exists $y \in \langle B, \boldsymbol{d}\rangle$ such that $x < y$, and (iii-2) for any $y \in \langle B, \boldsymbol{d}\rangle$, there exists $x \in \langle A, \boldsymbol{d}\rangle$ such that $x < y$. We show only (iii-1). (iii-2) can be shown in the similar way to (iii-1). Let $x \in \langle A, \boldsymbol{d}\rangle$. Then, there exists $\boldsymbol{x}_0 \in A$ such that $x = \langle \boldsymbol{x}_0, \boldsymbol{d}\rangle$. Since $A < B$, there exists $\boldsymbol{y}_0 \in B$ such that $\boldsymbol{x}_0 < \boldsymbol{y}_0$. We set $y = \langle \boldsymbol{y}_0, \boldsymbol{d}\rangle \in \langle B, \boldsymbol{d}\rangle$. Since $\boldsymbol{d} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, we have $x = \langle \boldsymbol{x}_0, \boldsymbol{d}\rangle < \langle \boldsymbol{y}_0, \boldsymbol{d}\rangle = y$.

(iv) Assume that $A \not< B$. Then, there are the following two cases: (iv-1) there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} \not< \boldsymbol{y}$ for any $\boldsymbol{y} \in B$; (iv-2) there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} \not< \boldsymbol{y}$ for any $\boldsymbol{x} \in A$. We show only the case (iv-1). The case (iv-2) can be shown in the similar way to the case (iv-1). In the case (iv-1), since $\boldsymbol{y} \notin \boldsymbol{x} + \mathrm{int}(\mathbb{R}^n_+)$ for any $\boldsymbol{y} \in B$, it follows that $B \cap (\boldsymbol{x} + \mathrm{int}(\mathbb{R}^n_+)) = \varnothing$. Since $B, \boldsymbol{x} + \mathrm{int}(\mathbb{R}^n_+) \in \mathcal{K}(\mathbb{R}^n)$, there exists $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \boldsymbol{a}, \boldsymbol{y}\rangle \leq \langle \boldsymbol{a}, \boldsymbol{x}\rangle + \langle \boldsymbol{a}, \boldsymbol{d}\rangle$ for any $\boldsymbol{y} \in B$ and any $\boldsymbol{d} \in \mathrm{int}(\mathbb{R}^n_+)$ from the separation theorem. Assume that there exists $\boldsymbol{d}_0 \in \mathrm{int}(\mathbb{R}^n_+)$ such that $\langle \boldsymbol{a}, \boldsymbol{d}_0\rangle < 0$. Then, since $\lambda\boldsymbol{d}_0 \in \mathrm{int}(\mathbb{R}^n_+)$ for any $\lambda > 0$, it follows that $\langle \boldsymbol{a}, \boldsymbol{x}\rangle + \langle \boldsymbol{a}, \lambda\boldsymbol{d}_0\rangle = \langle \boldsymbol{a}, \boldsymbol{x}\rangle + \lambda\langle \boldsymbol{a}, \boldsymbol{d}_0\rangle \to -\infty$ as $\lambda \to \infty$, which contradicts that $\langle \boldsymbol{a}, \boldsymbol{y}\rangle \leq \langle \boldsymbol{a}, \boldsymbol{x}\rangle + \langle \boldsymbol{a}, \boldsymbol{d}\rangle$ for any $\boldsymbol{y} \in B$ and any $\boldsymbol{d} \in \mathrm{int}(\mathbb{R}^n_+)$. Thus, since $\langle \boldsymbol{a}, \boldsymbol{d}\rangle \geq 0$ for any $\boldsymbol{d} \in \mathrm{int}(\mathbb{R}^n_+)$, it follows that $\boldsymbol{a} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. For any $\boldsymbol{y} \in B$, when $\boldsymbol{d} \to \mathbf{0}$, $\boldsymbol{d} \in \mathrm{int}(\mathbb{R}^n_+)$ in $\langle \boldsymbol{a}, \boldsymbol{y}\rangle \leq \langle \boldsymbol{a}, \boldsymbol{x}\rangle + \langle \boldsymbol{a}, \boldsymbol{d}\rangle$, we have $\langle \boldsymbol{a}, \boldsymbol{y}\rangle \leq \langle \boldsymbol{a}, \boldsymbol{x}\rangle$, which contradicts that $\langle A, \boldsymbol{a}\rangle < \langle B, \boldsymbol{a}\rangle$. □

For $A_i \subset \mathbb{R}$, $i = 1, 2, \cdots, n$, we set $\prod_{i=1}^n A_i = A_1 \times A_2 \times \cdots \times A_n$ and $\sum_{i=1}^n A_i = A_1 + A_2 + \cdots + A_n$.

The following proposition shows fundamental properties of the product for crisp sets, and relationships between the product and operations, the inner product, orderings for crisp sets. It can be shown easily.

Proposition 3.5. *Let $A_i, B_i \subset \mathbb{R}$, $i = 1, 2, \cdots, n$, and let $\lambda \in \mathbb{R}$.*

(i) $A_i \neq \varnothing,\ i = 1, 2, \cdots, n,\ \prod_{i=1}^n A_i \in \mathcal{K}(\mathbb{R}^n) \Rightarrow A_i \in \mathcal{K}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(ii) $\prod_{i=1}^n A_i \in \mathcal{K}(\mathbb{R}^n) \Leftarrow A_i \in \mathcal{K}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(iii) $A_i \neq \varnothing,\ i = 1, 2, \cdots, n,\ \prod_{i=1}^n A_i \in \mathcal{C}(\mathbb{R}^n) \Rightarrow A_i \in \mathcal{C}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(iv) $\prod_{i=1}^n A_i \in \mathcal{C}(\mathbb{R}^n) \Leftarrow A_i \in \mathcal{C}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(v) $A_i \neq \varnothing,\ i = 1, 2, \cdots, n,\ \prod_{i=1}^n A_i \in \mathcal{BC}(\mathbb{R}^n) \Rightarrow A_i \in \mathcal{BC}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(vi) $\prod_{i=1}^n A_i \in \mathcal{BC}(\mathbb{R}^n) \Leftarrow A_i \in \mathcal{BC}(\mathbb{R}),\ i = 1, 2, \cdots, n.$
(vii) $\prod_{i=1}^n A_i + \prod_{i=1}^n B_i = \prod_{i=1}^n (A_i + B_i).$
(viii) $\lambda \prod_{i=1}^n A_i = \prod_{i=1}^n \lambda A_i.$
(ix) $\langle \prod_{i=1}^n A_i, \prod_{i=1}^n B_i \rangle = \sum_{i=1}^n \langle A_i, B_i \rangle.$
(x) $A_i \neq \varnothing,\ B_i \neq \varnothing,\ i = 1, 2, \cdots, n,\ \prod_{i=1}^n A_i \leq \prod_{i=1}^n B_i \Rightarrow A_i \leq B_i,\ i = 1, 2, \cdots, n.$
(xi) $\prod_{i=1}^n A_i \leq \prod_{i=1}^n B_i \Leftarrow A_i \leq B_i,\ i = 1, 2, \cdots, n.$
(xii) $A_i \neq \varnothing,\ B_i \neq \varnothing,\ i = 1, 2, \cdots, n,\ \prod_{i=1}^n A_i < \prod_{i=1}^n B_i \Rightarrow A_i < B_i,\ i = 1, 2, \cdots, n.$
(xiii) $\prod_{i=1}^n A_i < \prod_{i=1}^n B_i \Leftarrow A_i < B_i,\ i = 1, 2, \cdots, n.$

4. Operations and orderings of fuzzy set

In this section, operations and orderings on $\mathcal{F}(\mathbb{R}^n)$ are defined, and their properties are presented.

The following definitions are addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by Zadeh's extension principle.

Definition 4.1. For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, we define $\widetilde{a} + \widetilde{b}, \lambda\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ as

$$(\widetilde{a} + \widetilde{b})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{z}} \min\left\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\right\} \quad \text{and} \quad (\lambda\widetilde{a})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y}) \tag{4.1}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$.

The following proposition shows relationships between operations of fuzzy sets and operations of level sets of the fuzzy sets.

Proposition 4.2. (Kon [7]) *Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda \in \mathbb{R}$. In addition, let $\alpha \in]0, 1]$.*

(i) $[\widetilde{a} + \widetilde{b}]_\alpha \supset [\widetilde{a}]_\alpha + [\widetilde{b}]_\alpha.$
(ii) $\widetilde{a} \in \mathcal{FBC}(\mathbb{R}^n),\ \widetilde{b} \in \mathcal{FC}(\mathbb{R}^n) \Rightarrow [\widetilde{a} + \widetilde{b}]_\alpha \subset [\widetilde{a}]_\alpha + [\widetilde{b}]_\alpha.$
(iii) $[\lambda\widetilde{a}]_\alpha \supset \lambda[\widetilde{a}]_\alpha.$
(iv) $\widetilde{a} \in \mathcal{FBC}(\mathbb{R}^n) \Rightarrow [\lambda\widetilde{a}]_\alpha \subset \lambda[\widetilde{a}]_\alpha.$

The following proposition shows relationships between operations of fuzzy sets and generators of the fuzzy sets.

Proposition 4.3. (Kon [7]) *Let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ and $\widetilde{b} = M(\{T_\alpha\}_{\alpha\in]0,1]})$. In addition, let $\lambda \in \mathbb{R}$.*

(i) $\widetilde{a} + \widetilde{b} = M(\{S_\alpha + T_\alpha\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha+T_\alpha}.$
(ii) $\lambda\widetilde{a} = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\lambda S_\alpha}.$

The following proposition shows a property of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ in some special case.

Proposition 4.4. (Kon [7]) $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, $hgt(\widetilde{a}) = hgt(\widetilde{b}) \Rightarrow 1 \cdot \widetilde{a} + 0 \cdot \widetilde{b} = \widetilde{a}$.

The following proposition shows properties of the convexity, closedness, and compactness with respect to addition and scalar multiplication of fuzzy sets.

Proposition 4.5. (Kon [7]) *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, *and let* $\lambda \in \mathbb{R}$.

(i) $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(\mathbb{R}^n) \Rightarrow \widetilde{a} + \widetilde{b} \in \mathcal{FK}(\mathbb{R}^n)$.
(ii) $\widetilde{a}, \widetilde{b} \in \mathcal{FBC}(\mathbb{R}^n) \Rightarrow \widetilde{a} + \widetilde{b} \in \mathcal{FBC}(\mathbb{R}^n)$.
(iii) $\widetilde{a} \in \mathcal{FBC}(\mathbb{R}^n)$, $\widetilde{b} \in \mathcal{FC}(\mathbb{R}^n) \Rightarrow \widetilde{a} + \widetilde{b} \in \mathcal{FC}(\mathbb{R}^n)$.
(iv) *It does not always hold that* $\widetilde{a} + \widetilde{b} \in \mathcal{FC}(\mathbb{R}^n)$ *even if* $\widetilde{a}, \widetilde{b} \in \mathcal{FC}(\mathbb{R}^n)$.
(v) $\widetilde{a} \in \mathcal{FK}(\mathbb{R}^n) \Rightarrow \lambda\widetilde{a} \in \mathcal{FK}(\mathbb{R}^n)$.
(vi) $\widetilde{a} \in \mathcal{FC}(\mathbb{R}^n) \Rightarrow \lambda\widetilde{a} \in \mathcal{FC}(\mathbb{R}^n)$.
(vii) $\widetilde{a} \in \mathcal{FBC}(\mathbb{R}^n) \Rightarrow \lambda\widetilde{a} \in \mathcal{FBC}(\mathbb{R}^n)$.

We define orders on $\mathcal{F}(\mathbb{R}^n)$ based on orderings of level sets of fuzzy sets.

Definition 4.6. (Kon [7]) Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$.

(i) We write $\widetilde{a} \preceq \widetilde{b}$ or $\widetilde{b} \succeq \widetilde{a}$ if $[\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$.
(ii) We write $\widetilde{a} \prec \widetilde{b}$ or $\widetilde{b} \succ \widetilde{a}$ if $[\widetilde{a}]_\alpha < [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$.

The orders $\preceq$ and $\prec$ in Definition 4.6 are called the fuzzy max order and the strict fuzzy max order, respectively. It can be shown easily that the order $\preceq$ is a pseudo-order on $\mathcal{F}(\mathbb{R}^n)$.

5. Fuzzy inner product

In this section, a fuzzy inner product on $\mathcal{F}(\mathbb{R}^n)$ is defined, and relationships between it and the (strict) fuzzy max order are investigated.

We define the fuzzy inner product on $\mathcal{F}(\mathbb{R}^n)$ based on Zadeh's extension principle.

Definition 5.1. (i) For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, $\langle \widetilde{a}, \widetilde{b} \rangle \in \mathcal{F}(\mathbb{R})$ defined as

$$\langle \widetilde{a}, \widetilde{b} \rangle (x) = \sup_{x = \langle \boldsymbol{y}, \boldsymbol{z} \rangle} \min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\} \tag{5.1}$$

for each $x \in \mathbb{R}$ is called the fuzzy inner product of $\widetilde{a}$ and $\widetilde{b}$.

(ii) For $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\boldsymbol{b} \in \mathbb{R}^n$, $\langle \widetilde{a}, \boldsymbol{b} \rangle \in \mathcal{F}(\mathbb{R})$ defined as

$$\langle \widetilde{a}, \boldsymbol{b} \rangle (x) = \sup_{x = \langle \boldsymbol{y}, \boldsymbol{b} \rangle} \widetilde{a}(\boldsymbol{y}) \tag{5.2}$$

for each $x \in \mathbb{R}$ is called the fuzzy inner product of $\widetilde{a}$ and $\boldsymbol{b}$. Furthermore, $\langle \boldsymbol{b}, \widetilde{a} \rangle = \langle \widetilde{a}, \boldsymbol{b} \rangle$ is called the fuzzy inner product of $\boldsymbol{b}$ and $\widetilde{a}$.

Note that $\langle \widetilde{a}, \boldsymbol{b} \rangle = \langle \widetilde{a}, c_{\{\boldsymbol{b}\}} \rangle$ for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\boldsymbol{b} \in \mathbb{R}^n$. The fuzzy inner product has been defined for fuzzy vectors by Maeda [11], and for fuzzy sets which are closed, convex, normal, and support bounded by Kurano et al. [9]. The fuzzy inner product in Definition 5.1 is an extension of these definitions in the sense that it is the fuzzy inner product for general fuzzy sets.

The following proposition shows a relationship between the fuzzy inner product of fuzzy sets and generators of the fuzzy sets.

Proposition 5.2. *Let* $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$*, and let* $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ *and* $\widetilde{b} = M(\{T_\alpha\}_{\alpha\in]0,1]})$*. Then,*

$$\langle \widetilde{a}, \widetilde{b}\rangle = M(\{\langle S_\alpha, T_\alpha\rangle\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\langle S_\alpha, T_\alpha\rangle}. \tag{5.3}$$

Proof. Let $x \in \mathbb{R}$. We set $\beta = \langle \widetilde{a}, \widetilde{b}\rangle(x) = \sup_{x=\langle \boldsymbol{y},\boldsymbol{z}\rangle} \min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}$. Then, we show that $M(\{\langle S_\alpha, T_\alpha\rangle\}_{\alpha\in]0,1]})(x) = \sup_{\alpha\in]0,1]} \alpha c_{\langle S_\alpha,T_\alpha\rangle}(x) = \beta$.

If $\beta = 1$, then $\alpha c_{\langle S_\alpha,T_\alpha\rangle}(x) \le 1 = \beta$ for any $\alpha \in]0, 1]$. Suppose that $\beta < 1$. It follows that $x \notin \langle S_\gamma, T_\gamma\rangle$ for any $\gamma \in]\beta, 1]$. If there exists $\gamma \in]\beta, 1]$ such that $x \in \langle S_\gamma, T_\gamma\rangle$, then there exist $\boldsymbol{y} \in S_\gamma$ and $\boldsymbol{z} \in T_\gamma$ such that $x = \langle \boldsymbol{y}, \boldsymbol{z}\rangle$, and then $\min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\} = \min\{\sup\{\alpha \in]0, 1] : \boldsymbol{y} \in S_\alpha\}, \sup\{\alpha \in]0, 1] : \boldsymbol{z} \in T_\alpha\}\} \ge \gamma > \beta$, which contradicts the definition of β. Therefore, we have $\alpha c_{\langle S_\alpha,T_\alpha\rangle}(x) \le \beta$ for any $\alpha \in]0, 1]$.

If $\beta = 0$, then $\sup_{\alpha\in]0,1]} \alpha c_{\langle S_\alpha,T_\alpha\rangle}(x) = 0$. Suppose that $\beta > 0$. Fix any $\varepsilon \in]0, \beta[$. From the definition of β, there exist $\boldsymbol{y}_0, \boldsymbol{z}_0 \in \mathbb{R}^n$ such that $x = \langle \boldsymbol{y}_0, \boldsymbol{z}_0\rangle$ and $\min\{\widetilde{a}(\boldsymbol{y}_0), \widetilde{b}(\boldsymbol{z}_0)\} > \beta - \frac{\varepsilon}{2}$. From Proposition 2.2, it follows that $\boldsymbol{y}_0 \in [\widetilde{a}]_{\beta-\frac{\varepsilon}{2}+\delta} = \bigcap_{\alpha\in]0,\beta-\frac{\varepsilon}{2}+\delta[} S_\alpha \subset S_{\beta-\frac{\varepsilon}{2}}$ and $\boldsymbol{z}_0 \in [\widetilde{b}]_{\beta-\frac{\varepsilon}{2}+\delta} = \bigcap_{\alpha\in]0,\beta-\frac{\varepsilon}{2}+\delta[} T_\alpha \subset T_{\beta-\frac{\varepsilon}{2}}$ for sufficiently small $\delta > 0$. Therefore, we have $\left(\beta - \frac{\varepsilon}{2}\right) c_{\left\langle S_{\beta-\frac{\varepsilon}{2}}, T_{\beta-\frac{\varepsilon}{2}}\right\rangle}(x) = \beta - \frac{\varepsilon}{2} > \beta - \varepsilon$. □

The following proposition can be obtained from Proposition 5.2.

Proposition 5.3. *Let* $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$*, and let* $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$*. In addition, let* $\boldsymbol{b} \in \mathbb{R}^n$*. Then,*

$$\langle \widetilde{a}, \boldsymbol{b}\rangle = M(\{\langle S_\alpha, \boldsymbol{b}\rangle\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\langle S_\alpha, \boldsymbol{b}\rangle}. \tag{5.4}$$

Example 5.4. Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^2)$ be fuzzy sets defined as $\widetilde{a}(y, z) = \min\{\max\{0, 1 - |y-1|\}, \max\{0, 1-|z-2|\}\}$ and $\widetilde{b}(y, z) = \min\{\max\{0, 1-|y-4|\}, \max\{0, 1-|z-1|\}\}$ for each $(y, z) \in \mathbb{R}^2$. For each $\alpha \in]0, 1]$, since $[\widetilde{a}]_\alpha = [\alpha, 2 - \alpha] \times [1 + \alpha, 3 - \alpha]$ and $[\widetilde{b}]_\alpha = [3+\alpha, 5-\alpha] \times [\alpha, 2-\alpha]$, it follows that $\langle [\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha\rangle = [\alpha(3+\alpha)+(1+\alpha)\alpha, (2-\alpha)(5-\alpha) + (3-\alpha)(2-\alpha)] = [2\alpha^2 + 4\alpha, 2\alpha^2 - 12\alpha + 16]$. From the decomposition theorem (2.1) and Proposition 5.2, we have

$$\langle \widetilde{a}, \widetilde{b}\rangle(x) = \sup_{\alpha\in]0,1]} \alpha c_{\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle}(x) = \begin{cases} \frac{\sqrt{2x+4}}{2} - 1 & \text{if } x \in [0, 6], \\ -\frac{\sqrt{2x+4}}{2} + 3 & \text{if } x \in]6, 16], \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$.

The following proposition shows relationships between level sets of the fuzzy inner product of fuzzy sets and the inner product of level sets of the fuzzy sets.

Proposition 5.5. *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$*, and let* $\alpha \in]0, 1]$*.*

(i) $[\langle \widetilde{a}, \widetilde{b}\rangle]_\alpha \supset \langle [\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha\rangle$.
(ii) $\widetilde{a}, \widetilde{b} \in \mathcal{FBC}(\mathbb{R}^n) \Rightarrow [\langle \widetilde{a}, \widetilde{b}\rangle]_\alpha \subset \langle [\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha\rangle$.

Proof. From the decomposition theorem (2.1) and Propositions 2.1 and 5.2, $[\langle\widetilde{a},\widetilde{b}\rangle]_\alpha = \cap_{\beta\in]0,\alpha[}\langle[\widetilde{a}]_\beta,[\widetilde{b}]_\beta\rangle$. We set $A=[\langle\widetilde{a},\widetilde{b}\rangle]_\alpha$ and $B=\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle$.

(i) Since $B=\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle\subset\langle[\widetilde{a}]_\beta,[\widetilde{b}]_\beta\rangle$ for any $\beta\in]0,\alpha[$, we have $B\subset\cap_{\beta\in]0,\alpha[}\langle[\widetilde{a}]_\beta,[\widetilde{b}]_\beta\rangle=A$.

(ii) Let $x\in A$. Since $x\in\langle[\widetilde{a}]_\beta,[\widetilde{b}]_\beta\rangle$ for any $\beta\in]0,\alpha[$, there exist $\boldsymbol{y}_k\in[\widetilde{a}]_{\alpha-\frac{\alpha}{k+1}}$ and $\boldsymbol{z}_k\in[\widetilde{b}]_{\alpha-\frac{\alpha}{k+1}}$ such that $x=\langle\boldsymbol{y}_k,\boldsymbol{z}_k\rangle$ for each $k\in\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Since $\widetilde{a},\widetilde{b}\in\mathcal{FBC}(\mathbb{R}^n)$, without loss of generality, we assume that there exist $\boldsymbol{y}_0,\boldsymbol{z}_0\in\mathbb{R}^n$ such that $\boldsymbol{y}_k\to\boldsymbol{y}_0$ and $\boldsymbol{z}_k\to\boldsymbol{z}_0$.

Fix any $\beta\in]0,\alpha[$. There exists $k_0\in\mathbb{N}$ such that $\alpha-\frac{\alpha}{k+1}\in]\beta,\alpha[$ for any $k\geq k_0$. Then, $\boldsymbol{y}_k\in[\widetilde{a}]_{\alpha-\frac{\alpha}{k+1}}\subset[\widetilde{a}]_\beta$ and $\boldsymbol{z}_k\in[\widetilde{b}]_{\alpha-\frac{\alpha}{k+1}}\subset[\widetilde{b}]_\beta$ for any $k\geq k_0$. Since $[\widetilde{a}]_\beta,[\widetilde{b}]_\beta\in\mathcal{BC}(\mathbb{R}^n)$, it follows that $\boldsymbol{y}_k\to\boldsymbol{y}_0\in[\widetilde{a}]_\beta$ and $\boldsymbol{z}_k\to\boldsymbol{z}_0\in[\widetilde{b}]_\beta$.

Thus, $\boldsymbol{y}_0\in[\widetilde{a}]_\beta$ and $\boldsymbol{z}_0\in[\widetilde{b}]_\beta$ for any $\beta\in]0,\alpha[$. From the decomposition theorem (2.1) and Proposition 2.2, it follows that $\boldsymbol{y}_0\in\cap_{\beta\in]0,\alpha[}[\widetilde{a}]_\beta=[\widetilde{a}]_\alpha$ and $\boldsymbol{z}_0\in\cap_{\beta\in]0,\alpha[}[\widetilde{b}]_\beta=[\widetilde{b}]_\alpha$. Therefore, we have $x=\langle\boldsymbol{y}_k,\boldsymbol{z}_k\rangle\to x=\langle\boldsymbol{y}_0,\boldsymbol{z}_0\rangle\in\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle=B$. □

The following proposition can be obtained from Proposition 5.5.

Proposition 5.6. *Let $\widetilde{a}\in\mathcal{F}(\mathbb{R}^n)$, and let $\boldsymbol{b}\in\mathbb{R}^n$. In addition, let $\alpha\in]0,1]$.*

(i) $[\langle\widetilde{a},\boldsymbol{b}\rangle]_\alpha\supset\langle[\widetilde{a}]_\alpha,\boldsymbol{b}\rangle$.

(ii) $\widetilde{a}\in\mathcal{FBC}(\mathbb{R}^n)\Rightarrow[\langle\widetilde{a},\boldsymbol{b}\rangle]_\alpha\subset\langle[\widetilde{a}]_\alpha,\boldsymbol{b}\rangle$.

The following proposition shows that the fuzzy inner product in Definition 5.1 is not an inner product, but the fuzzy inner product has nearly properties of inner products.

Proposition 5.7. *Let $\widetilde{a},\widetilde{b},\widetilde{c}\in\mathcal{F}(\mathbb{R}^n)$, and let $\lambda\in\mathbb{R}$.*

(i) $\langle\widetilde{a},\widetilde{b}\rangle=\langle\widetilde{b},\widetilde{a}\rangle$.

(ii) $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle\leq\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle$.

(iii) *It does not always hold that* $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle\geq\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle$.

(iv) $\langle\lambda\widetilde{a},\widetilde{b}\rangle=\lambda\langle\widetilde{a},\widetilde{b}\rangle$.

(v) *It does not always hold that* $\langle\widetilde{a},\widetilde{a}\rangle\succeq\widetilde{0}$.

(vi) $\widetilde{a}=\widetilde{\boldsymbol{0}}\Leftrightarrow\langle\widetilde{a},\widetilde{a}\rangle=\widetilde{0}$.

(vii) $hgt(\widetilde{a})=1\Rightarrow\langle\widetilde{a},\widetilde{\boldsymbol{0}}\rangle=\widetilde{0}$.

Proof. (i) Let $x\in\mathbb{R}$. Then, we have $\langle\widetilde{a},\widetilde{b}\rangle(x)=\sup_{x=\langle\boldsymbol{y},\boldsymbol{z}\rangle}\min\{\widetilde{a}(\boldsymbol{y}),\widetilde{b}(\boldsymbol{z})\}=\sup_{x=\langle\boldsymbol{z},\boldsymbol{y}\rangle}\min\{\widetilde{b}(\boldsymbol{z}),\widetilde{a}(\boldsymbol{y})\}=\langle\widetilde{b},\widetilde{a}\rangle(x)$.

(ii) From the decomposition theorem (2.1) and Propositions 4.3 (i) and 5.2, it follows that $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle=M(\{\langle[\widetilde{a}]_\alpha+[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle\}_{\alpha\in]0,1]})$ and $\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle=M(\{\langle[\widetilde{a}]_\alpha,[\widetilde{c}]_\alpha\rangle+\langle[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle\}_{\alpha\in]0,1]})$. From Proposition 3.3 (ii), it follows that $\langle[\widetilde{a}]_\alpha+[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle\subset\langle[\widetilde{a}]_\alpha,[\widetilde{c}]_\alpha\rangle+\langle[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle$ for any $\alpha\in]0,1]$. Therefore, we have $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle\leq\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle$ from Proposition 2.1.

(iii) From Proposition 3.3 (iii), there exist $A,B,C\subset\mathbb{R}^n$ such that $\langle A+B,C\rangle\not\supset\langle A,C\rangle+\langle B,C\rangle$. We set $\widetilde{a}=c_A$, $\widetilde{b}=c_B$, and $\widetilde{c}=c_C$. From the decomposition theorem (2.1) and Propositions 4.3 (i) and 5.2, it follows that $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle=$

$M(\{\langle[\widetilde{a}]_\alpha+[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle\}_{\alpha\in]0,1]})=M(\{\langle A+B,C\rangle\}_{\alpha\in]0,1]})=c_{\langle A+B,C\rangle}$ and $\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle$ $=M(\{\langle[\widetilde{a}]_\alpha,[\widetilde{c}]_\alpha\rangle+\langle[\widetilde{b}]_\alpha,[\widetilde{c}]_\alpha\rangle\}_{\alpha\in]0,1]})=M(\{\langle A,C\rangle+\langle B,C\rangle\}_{\alpha\in]0,1]})=c_{\langle A,C\rangle+\langle B,C\rangle}$. Since $\langle A+B,C\rangle\not\supset\langle A,C\rangle+\langle B,C\rangle$, there exists $x_0\in\langle A,C\rangle+\langle B,C\rangle$ such that $x_0\notin\langle A+B,C\rangle$. Therefore, we have $\langle\widetilde{a}+\widetilde{b},\widetilde{c}\rangle(x_0)=c_{\langle A+B,C\rangle}(x_0)=0\not\geq 1=c_{\langle A,C\rangle+\langle B,C\rangle}(x_0)=(\langle\widetilde{a},\widetilde{c}\rangle+\langle\widetilde{b},\widetilde{c}\rangle)(x_0)$.

(iv) From the decomposition theorem (2.1) and Propositions 3.3 (iv), 4.3 (ii), and 5.2, we have $\langle\lambda\widetilde{a},\widetilde{b}\rangle=M(\{\langle\lambda[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle\}_{\alpha\in]0,1]})=M(\{\lambda\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle\}_{\alpha\in]0,1]})=\lambda M(\{\langle[\widetilde{a}]_\alpha,[\widetilde{b}]_\alpha\rangle\}_{\alpha\in]0,1]})=\lambda\langle\widetilde{a},\widetilde{b}\rangle$.

(v) From Proposition 3.3 (v), there exists $A\subset\mathbb{R}^n$ such that $\langle A,A\rangle\not\geq\{0\}$. We set $\widetilde{a}=c_A$. From the decomposition theorem (2.1) and Proposition 5.2, it follows that $\langle\widetilde{a},\widetilde{a}\rangle=M(\{[\widetilde{a}]_\alpha,[\widetilde{a}]_\alpha\rangle\}_{\alpha\in]0,1]})=M(\{\langle A,A\rangle\}_{\alpha\in]0,1]})=c_{\langle A,A\rangle}$. Since $[\langle\widetilde{a},\widetilde{a}\rangle]_\alpha=\langle A,A\rangle\not\geq\{0\}=[\widetilde{0}]_\alpha$ for any $\alpha\in]0,1]$, we have $\langle\widetilde{a},\widetilde{a}\rangle\not\succeq\widetilde{0}$.

(vi) First, we show the necessity. From the decomposition theorem (2.1) and Proposition 5.2, we have $\langle\widetilde{a},\widetilde{a}\rangle=\langle\widetilde{\mathbf{0}},\widetilde{\mathbf{0}}\rangle=M(\{\langle[\widetilde{\mathbf{0}}]_\alpha,[\widetilde{\mathbf{0}}]_\alpha\rangle\}_{\alpha\in]0,1]})=M(\{\{0\}\}_{\alpha\in]0,1]})$ $=\widetilde{0}$.

Next, we show the sufficiency. Suppose that $\widetilde{a}\neq\widetilde{\mathbf{0}}$. Then, there are the following two cases: (vi-1) there exists $\boldsymbol{x}_0\in\mathbb{R}^n$ such that $\boldsymbol{x}_0\neq\mathbf{0}$ and $\widetilde{a}(\boldsymbol{x}_0)>0$; (vi-2) $\widetilde{a}(\mathbf{0})<1$, and there does not exist $\boldsymbol{x}_0\in\mathbb{R}^n$ such that $\boldsymbol{x}_0\neq\mathbf{0}$ and $\widetilde{a}(\boldsymbol{x}_0)>0$. Suppose the case (vi-1). We set $y_0=\langle\boldsymbol{x}_0,\boldsymbol{x}_0\rangle>0$. Then, we have $\langle\widetilde{a},\widetilde{a}\rangle(y_0)=\sup_{y_0=\langle\boldsymbol{y},\boldsymbol{z}\rangle}\min\{\widetilde{a}(\boldsymbol{y}),\widetilde{a}(\boldsymbol{z})\}\geq\min\{\widetilde{a}(\boldsymbol{x}_0),\widetilde{a}(\boldsymbol{x}_0)\}=\widetilde{a}(\boldsymbol{x}_0)>0=\widetilde{0}(y_0)$. Suppose the case (vi-2). From the decomposition theorem (2.1) and Proposition 5.2, we have $\langle\widetilde{a},\widetilde{a}\rangle(0)=\sup\{\alpha\in]0,1]:0\in\langle[\widetilde{a}]_\alpha,[\widetilde{a}]_\alpha\rangle\}=\widetilde{a}(\mathbf{0})<1=\widetilde{0}(0)$.

(vii) Since $[\widetilde{a}]_\alpha\neq\varnothing$ and $[\widetilde{\mathbf{0}}]_\alpha=\{\mathbf{0}\}$ for any $\alpha\in]0,1[$, it follows that $\langle[\widetilde{a}]_\alpha,[\widetilde{\mathbf{0}}]_\alpha\rangle=\{0\}$ for any $\alpha\in]0,1[$ from Proposition 3.3 (vii). From Propositions 2.2 and 5.2, it follows that $[\langle\widetilde{a},\widetilde{\mathbf{0}}\rangle]_\alpha=\{0\}$ for any $\alpha\in]0,1]$. Therefore, we have $\langle\widetilde{a},\widetilde{\mathbf{0}}\rangle=\widetilde{0}$ from the decomposition theorem (2.1). □

The following proposition shows characterizations of the (strict) fuzzy max order based on the fuzzy inner product.

Proposition 5.8. *Let* $\widetilde{a},\widetilde{b}\in\mathcal{F}(\mathbb{R}^n)$.

(i) $\widetilde{a},\widetilde{b}\in\mathcal{FBC}(\mathbb{R}^n)$, $\widetilde{a}\preceq\widetilde{b}\Rightarrow\langle\widetilde{a},\boldsymbol{d}\rangle\preceq\langle\widetilde{b},\boldsymbol{d}\rangle$, $\boldsymbol{d}\in\mathbb{R}^n_+$.
(ii) $\widetilde{a},\widetilde{b}\in\mathcal{FBCK}(\mathbb{R}^n)$ *and* $\langle\widetilde{a},\boldsymbol{d}\rangle\preceq\langle\widetilde{b},\boldsymbol{d}\rangle$, $\boldsymbol{d}\in\mathbb{R}^n_+\Rightarrow\widetilde{a}\preceq\widetilde{b}$.
(iii) $\widetilde{a},\widetilde{b}\in\mathcal{FBC}(\mathbb{R}^n)$, $\widetilde{a}\prec\widetilde{b}\Rightarrow\langle\widetilde{a},\boldsymbol{d}\rangle\prec\langle\widetilde{b},\boldsymbol{d}\rangle$, $\boldsymbol{d}\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}$.
(iv) $\widetilde{a},\widetilde{b}\in\mathcal{FBCK}(\mathbb{R}^n)$ *and* $\langle\widetilde{a},\boldsymbol{d}\rangle\prec\langle\widetilde{b},\boldsymbol{d}\rangle$, $\boldsymbol{d}\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}\Rightarrow\widetilde{a}\prec\widetilde{b}$.

Proof. We show only (i) and (ii). (iii) and (iv) can be shown in the similar ways to (i) and (ii), respectively.

(i) Fix any $\alpha\in]0,1]$ and any $\boldsymbol{d}\in\mathbb{R}^n_+$. Since $[\widetilde{a}]_\alpha\leq[\widetilde{b}]_\alpha$, it follows that $\langle[\widetilde{a}]_\alpha,\boldsymbol{d}\rangle\leq\langle[\widetilde{b}]_\alpha,\boldsymbol{d}\rangle$ from Proposition 3.4 (i), and that $[\langle\widetilde{a},\boldsymbol{d}\rangle]_\alpha\leq[\langle\widetilde{b},\boldsymbol{d}\rangle]_\alpha$ from Proposition 5.6. Therefore, we have $\langle\widetilde{a},\boldsymbol{d}\rangle\preceq\langle\widetilde{b},\boldsymbol{d}\rangle$ for any $\boldsymbol{d}\in\mathbb{R}^n_+$ by the arbitrariness of $\alpha\in]0,1]$ and $\boldsymbol{d}\in\mathbb{R}^n_+$.

(ii) Fix any $\alpha \in]0,1]$. Since $\langle[\widetilde{a}]_\alpha, \boldsymbol{d}\rangle = [\langle\widetilde{a}, \boldsymbol{d}\rangle]_\alpha \leq [\langle\widetilde{b}, \boldsymbol{d}\rangle]_\alpha = \langle[\widetilde{b}]_\alpha, \boldsymbol{d}\rangle$ for any $\boldsymbol{d} \in \mathbb{R}^n_+$ from Proposition 5.6, it follows that $[\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha$ from Proposition 3.4 (ii). Therefore, we have $\widetilde{a} \preceq \widetilde{b}$ by the arbitrariness of $\alpha \in]0,1]$. □

6. Fuzzy product space

In this section, we consider a fuzzy product space of n $\mathcal{F}(\mathbb{R})$'s which is a subclass of $\mathcal{F}(\mathbb{R}^n)$, and investigate properties of operations, the (strict) fuzzy max order, and the fuzzy inner product on the fuzzy product space.

We define the fuzzy product space.

Definition 6.1. For $\widetilde{a}_i \in \mathcal{F}(\mathbb{R})$, $i = 1, 2, \cdots, n$, $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}(\mathbb{R}^n)$ defined as

$$\widetilde{\boldsymbol{a}}(\boldsymbol{x}) = \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i) \tag{6.1}$$

for each $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is called the fuzzy product set of $\widetilde{a}_i$, $i = 1, 2, \cdots, n$. Moreover,

$$\mathcal{F}^n(\mathbb{R}) = \{(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) : \widetilde{a}_i \in \mathcal{F}(\mathbb{R}), i = 1, 2, \cdots, n\} \subset \mathcal{F}(\mathbb{R}^n) \tag{6.2}$$

is called the fuzzy product space of n $\mathcal{F}(\mathbb{R})$'s.

The following proposition shows a relationship between level sets of fuzzy product sets and level sets of fuzzy sets which construct the fuzzy product sets.

Proposition 6.2. *Let $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}^n(\mathbb{R})$, and let $\alpha \in]0,1]$. Then,*

$$[\widetilde{\boldsymbol{a}}]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha. \tag{6.3}$$

Proof.

$$\begin{aligned}
[\widetilde{\boldsymbol{a}}]_\alpha &= \left\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i) \geq \alpha\right\} \\
&= \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \widetilde{a}_i(x_i) \geq \alpha, i = 1, 2, \cdots, n\} \\
&= \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i \in [\widetilde{a}_i]_\alpha, i = 1, 2, \cdots, n\} \\
&= \prod_{i=1}^n [\widetilde{a}_i]_\alpha.
\end{aligned}$$

□

The following proposition shows a relationship between fuzzy product sets and generators of fuzzy sets which construct the fuzzy product sets.

Proposition 6.3. *Let $\{S_{i\alpha}\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R})$, $i = 1, 2, \cdots, n$, and let $\widetilde{a}_i = M(\{S_{i\alpha}\}_{\alpha\in]0,1]})$, $i = 1, 2, \cdots, n$. In addition, let $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}^n(\mathbb{R})$. Then, $\widetilde{\boldsymbol{a}} = M(\{\prod_{i=1}^n S_{i\alpha}\}_{\alpha\in]0,1]})$.*

Proof. It follows that $[\widetilde{\boldsymbol{a}}]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha = \prod_{i=1}^n \left(\cap_{\beta\in]0,\alpha[} S_{i\beta}\right) \supset \prod_{i=1}^n S_{i\alpha}$ for any $\alpha \in]0,1]$ from Propositions 2.2 and 6.2, and that $\widetilde{\boldsymbol{a}} \geq M(\{\prod_{i=1}^n S_{i\alpha}\}_{\alpha\in]0,1]})$ from the decomposition theorem (2.1) and Proposition 2.1. Suppose that there exists $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ such that $\widetilde{\boldsymbol{a}}(\boldsymbol{x}) > M(\{\prod_{i=1}^n S_{i\alpha}\}_{\alpha\in]0,1]})(\boldsymbol{x})$. We set

$\beta = M(\{\prod_{i=1}^n S_{i\alpha}\}_{\alpha\in]0,1]})(\boldsymbol{x}) = \sup\{\alpha \in]0,1] : \boldsymbol{x} \in \prod_{i=1}^n S_{i\alpha}\}$. Since $\widetilde{\boldsymbol{a}}(\boldsymbol{x}) = \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i) > \beta$, it follows that $\min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i) > \beta + 2\varepsilon$ for sufficiently small $\varepsilon > 0$. For each $i \in \{1,2,\cdots,n\}$, since $\widetilde{a}_i(x_i) > \beta + 2\varepsilon$, it follows that $x_i \in [\widetilde{a}_i]_{\beta+2\varepsilon} = \cap_{\gamma\in]0,\beta+2\varepsilon[} S_{i\gamma} \subset S_{i,\beta+\varepsilon}$ from Proposition 2.2. Therefore, we have $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n S_{i,\beta+\varepsilon}$, which contradicts the definition of β. □

The following proposition shows properties of the convexity, closedness, and compactness with respect to fuzzy product sets.

Proposition 6.4. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}^n(\mathbb{R})$.

(i) $hgt(\widetilde{a}_1) = \cdots = hgt(\widetilde{a}_n)$, $\widetilde{\boldsymbol{a}} \in \mathcal{FK}(\mathbb{R}^n) \Rightarrow \widetilde{a}_i \in \mathcal{FK}(\mathbb{R})$, $i = 1,2,\cdots,n$.
(ii) $\widetilde{\boldsymbol{a}} \in \mathcal{FK}(\mathbb{R}^n) \Leftarrow \widetilde{a}_i \in \mathcal{FK}(\mathbb{R})$, $i = 1,2,\cdots,n$.
(iii) $hgt(\widetilde{a}_1) = \cdots = hgt(\widetilde{a}_n)$, $\widetilde{\boldsymbol{a}} \in \mathcal{FC}(\mathbb{R}^n) \Rightarrow \widetilde{a}_i \in \mathcal{FC}(\mathbb{R})$, $i = 1,2,\cdots,n$.
(iv) $\widetilde{\boldsymbol{a}} \in \mathcal{FC}(\mathbb{R}^n) \Leftarrow \widetilde{a}_i \in \mathcal{FC}(\mathbb{R})$, $i = 1,2,\cdots,n$.
(v) $hgt(\widetilde{a}_1) = \cdots = hgt(\widetilde{a}_n)$, $\widetilde{\boldsymbol{a}} \in \mathcal{FBC}(\mathbb{R}^n) \Rightarrow \widetilde{a}_i \in \mathcal{FBC}(\mathbb{R})$, $i = 1,2,\cdots,n$.
(vi) $\widetilde{\boldsymbol{a}} \in \mathcal{FBC}(\mathbb{R}^n) \Leftarrow \widetilde{a}_i \in \mathcal{FBC}(\mathbb{R})$, $i = 1,2,\cdots,n$.

Proof. We show only (i) and (ii). (iii) and (v) can be shown in the similar way to (i). (iv) and (vi) can be shown in the similar way to (ii).

(i) We set $\alpha_0 = \text{hgt}(\widetilde{a}_1) = \cdots = \text{hgt}(\widetilde{a}_n)$. If $\alpha_0 = 0$, then $\widetilde{a}_i \in \mathcal{FK}(\mathbb{R})$ for any $i \in \{1,2,\cdots,n\}$. Assume that $\alpha_0 > 0$. If $\alpha \in]\alpha_0, 1]$, then $[\widetilde{a}_i]_\alpha = \varnothing$ for any $i \in \{1,2,\cdots,n\}$. Fix any $\alpha \in]0,\alpha_0[$. Then, $[\widetilde{a}_i]_\alpha \neq \varnothing$ for any $i \in \{1,2,\cdots,n\}$, and $[\widetilde{\boldsymbol{a}}]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}(\mathbb{R}^n)$ from Proposition 6.2 and the assumption. From Proposition 3.5 (i), it follows that $[\widetilde{a}_i]_\alpha \in \mathcal{K}(\mathbb{R})$ for any $i \in \{1,2,\cdots,n\}$. Thus, it follows that $[\widetilde{a}_i]_{\alpha_0} = \cap_{\beta\in]0,\alpha_0[} [\widetilde{a}_i]_\beta \in \mathcal{K}(\mathbb{R})$ for any $i \in \{1,2,\cdots,n\}$ from the decomposition theorem (2.1) and Proposition 2.2. Therefore, we have $\widetilde{a}_i \in \mathcal{FK}(\mathbb{R})$ for any $i \in \{1,2,\cdots,n\}$.

(ii) Fix any $\alpha \in]0,1]$. Since $[\widetilde{a}_i]_\alpha \in \mathcal{K}(\mathbb{R})$ for any $i \in \{1,2,\cdots,n\}$, it follows that $[\widetilde{\boldsymbol{a}}]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}(\mathbb{R}^n)$ from Propositions 3.5 (ii) and 6.2. Therefore, we have $\widetilde{\boldsymbol{a}} \in \mathcal{FK}(\mathbb{R}^n)$ by the arbitrariness of $\alpha \in]0,1]$. □

The following proposition shows properties of the (strict) fuzzy max order on $\mathcal{F}^n(\mathbb{R})$.

Proposition 6.5. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n), \widetilde{\boldsymbol{b}} = (\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_n) \in \mathcal{F}^n(\mathbb{R})$.

(i) $I(\widetilde{a}_1) = \cdots = I(\widetilde{a}_n) = I(\widetilde{b}_1) = \cdots = I(\widetilde{b}_n)$, $\widetilde{\boldsymbol{a}} \preceq \widetilde{\boldsymbol{b}} \Rightarrow \widetilde{a}_i \preceq \widetilde{b}_i$, $i = 1,2,\cdots,n$.
(ii) $\widetilde{\boldsymbol{a}} \preceq \widetilde{\boldsymbol{b}} \Leftarrow \widetilde{a}_i \preceq \widetilde{b}_i$, $i = 1,2,\cdots,n$.
(iii) $I(\widetilde{a}_1) = \cdots = I(\widetilde{a}_n) = I(\widetilde{b}_1) = \cdots = I(\widetilde{b}_n)$, $\widetilde{\boldsymbol{a}} \prec \widetilde{\boldsymbol{b}} \Rightarrow \widetilde{a}_i \prec \widetilde{b}_i$, $i = 1,2,\cdots,n$.
(iv) $\widetilde{\boldsymbol{a}} \prec \widetilde{\boldsymbol{b}} \Leftarrow \widetilde{a}_i \prec \widetilde{b}_i$, $i = 1,2,\cdots,n$.

Proof. We show only (i) and (ii). (iii) and (iv) can be shown in the similar ways to (i) and (ii), respectively.

(i) We set $I = I(\widetilde{a}_1) = \cdots = I(\widetilde{a}_n) = I(\widetilde{b}_1) = \cdots = I(\widetilde{b}_n)$, and fix any $\alpha \in]0,1]$. If $\alpha \notin I$, then $[\widetilde{a}_i]_\alpha = \varnothing \leq \varnothing = [\widetilde{b}_i]_\alpha$ for any $i \in \{1,2,\cdots,n\}$. If $\alpha \in I$, then $[\widetilde{a}_i]_\alpha \leq [\widetilde{b}_i]_\alpha$ for any $i \in \{1,2,\cdots,n\}$ from Proposition 3.5 (x) since $\prod_{i=1}^n [\widetilde{a}_i]_\alpha = [\widetilde{\boldsymbol{a}}]_\alpha \leq [\widetilde{\boldsymbol{b}}]_\alpha = \prod_{i=1}^n [\widetilde{b}_i]_\alpha$ from Proposition 6.2. Therefore, we have $\widetilde{a}_i \preceq \widetilde{b}_i$ for any $i \in \{1,2,\cdots,n\}$ by the arbitrariness of $\alpha \in]0,1]$.

(ii) Fix any $\alpha \in]0,1]$. Since $[\widetilde{a}_i]_\alpha \leq [\widetilde{b}_i]_\alpha$ for any $i \in \{1,2,\cdots,n\}$, it follows that $[\widetilde{\boldsymbol{a}}]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha \leq \prod_{i=1}^n [\widetilde{b}_i]_\alpha = [\widetilde{\boldsymbol{b}}]_\alpha$ from Propositions 3.5 (xi) and 6.2. Therefore, we have $\widetilde{\boldsymbol{a}} \preceq \widetilde{\boldsymbol{b}}$ by the arbitrariness of $\alpha \in]0,1]$. □

The following three propositions show properties of the fuzzy inner product on $\mathcal{F}^n(\mathbb{R})$.

Proposition 6.6. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n), \widetilde{\boldsymbol{b}} = (\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_n) \in \mathcal{F}^n(\mathbb{R})$. *Then,*

$$\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}} \rangle = \sum_{i=1}^n \langle \widetilde{a}_i, \widetilde{b}_i \rangle. \tag{6.4}$$

Proof.

$$\begin{aligned}
\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}} \rangle &= M(\{\langle [\widetilde{\boldsymbol{a}}]_\alpha, [\widetilde{\boldsymbol{b}}]_\alpha \rangle\}_{\alpha \in]0,1]}) \\
&\qquad \text{(from the decomposition theorem (2.1) and Proposition 5.2)} \\
&= M\left(\left\{\left\langle \prod_{i=1}^n [\widetilde{a}_i]_\alpha, \prod_{i=1}^n [\widetilde{b}_i]_\alpha \right\rangle\right\}_{\alpha \in]0,1]}\right) \quad \text{(from Proposition 6.2)} \\
&= M\left(\left\{\sum_{i=1}^n \langle [\widetilde{a}_i]_\alpha, [\widetilde{b}_i]_\alpha \rangle\right\}_{\alpha \in]0,1]}\right) \quad \text{(from Proposition 3.5 (ix))} \\
&= \sum_{i=1}^n M(\{\langle [\widetilde{a}_i]_\alpha, [\widetilde{b}_i]_\alpha \rangle\}_{\alpha \in]0,1]}) \quad \text{(from Proposition 4.3 (i))} \\
&= \sum_{i=1}^n \langle \widetilde{a}_i, \widetilde{b}_i \rangle \\
&\qquad \text{(from the decomposition theorem (2.1) and Proposition 5.2).}
\end{aligned}$$

□

The following proposition can be shown in the similar way to Proposition 6.6.

Proposition 6.7. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}^n(\mathbb{R})$, *and let* $\boldsymbol{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$. *Then,*

$$\langle \widetilde{\boldsymbol{a}}, \boldsymbol{b} \rangle = \sum_{i=1}^n b_i \widetilde{a}_i. \tag{6.5}$$

The following proposition can be obtained from Propositions 4.4 and 6.7.

Proposition 6.8. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}^n(\mathbb{R})$. *Assume that* $hgt(\widetilde{a}_1) = \cdots = hgt(\widetilde{a}_n)$. *Then,*

$$\langle \widetilde{\boldsymbol{a}}, \boldsymbol{e}_i \rangle = \widetilde{a}_i, \quad i = 1, 2, \cdots, n. \tag{6.6}$$

The following proposition shows properties of operations and the equality on $\mathcal{F}^n(\mathbb{R})$.

Proposition 6.9. *Let* $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n), \widetilde{\boldsymbol{b}} = (\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_n) \in \mathcal{F}^n(\mathbb{R})$, *and let* $\lambda \in \mathbb{R}$.

(i) $\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}}=(\widetilde{a}_1+\widetilde{b}_1,\widetilde{a}_2+\widetilde{b}_2,\cdots,\widetilde{a}_n+\widetilde{b}_n)$.
(ii) $\lambda\widetilde{\boldsymbol{a}}=(\lambda\widetilde{a}_1,\lambda\widetilde{a}_2,\cdots,\lambda\widetilde{a}_n)$.
(iii) $hgt(\widetilde{a}_1)=\cdots=hgt(\widetilde{a}_n)$, $hgt(\widetilde{b}_1)=\cdots=hgt(\widetilde{b}_n)$, $\widetilde{\boldsymbol{a}}=\widetilde{\boldsymbol{b}}\Rightarrow\widetilde{a}_i=\widetilde{b}_i$, $i=1,2,\cdots,n$.
(iv) $\widetilde{\boldsymbol{a}}=\widetilde{\boldsymbol{b}}\Leftarrow\widetilde{a}_i=\widetilde{b}_i$, $i=1,2,\cdots,n$.

Proof. We show only (i). (ii) can be shown in the similar way to (i). (iii) can be obtained from Proposition 6.8. (iv) is trivial.

$$\begin{aligned}\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}} &= M\left(\left\{\prod_{i=1}^{n}[\widetilde{a}_i]_\alpha\right\}_{\alpha\in]0,1]}\right)+M\left(\left\{\prod_{i=1}^{n}[\widetilde{b}_i]_\alpha\right\}_{\alpha\in]0,1]}\right)\\ &\qquad\text{(from the decomposition theorem (2.1) and Proposition 6.2)}\\ &= M\left(\left\{\prod_{i=1}^{n}[\widetilde{a}_i]_\alpha+\prod_{i=1}^{n}[\widetilde{b}_i]_\alpha\right\}_{\alpha\in]0,1]}\right)\quad\text{(from Proposition 4.3 (i))}\\ &= M\left(\left\{\prod_{i=1}^{n}([\widetilde{a}_i]_\alpha+[\widetilde{b}_i]_\alpha)\right\}_{\alpha\in]0,1]}\right)\quad\text{(from Proposition 3.5 (vii))}\\ &= (M(\{[\widetilde{a}_1]_\alpha+[\widetilde{b}_1]_\alpha\}_{\alpha\in]0,1]}),\cdots,M(\{[\widetilde{a}_n]_\alpha+[\widetilde{b}_n]_\alpha\}_{\alpha\in]0,1]}))\\ &\qquad\text{(from Proposition 6.3)}\\ &= (\widetilde{a}_1+\widetilde{b}_1,\cdots,\widetilde{a}_n+\widetilde{b}_n)\\ &\qquad\text{(from the decomposition theorem (2.1) and Proposition 4.3 (i)).}\end{aligned}$$

□

Example 6.10. Let $\widetilde{a}_i,\widetilde{b}_i\in\mathcal{F}(\mathbb{R})$, $i=1,2$ be fuzzy sets defined as $\widetilde{a}_1(x)=\max\{0,1-|x-1|\}$, $\widetilde{a}_2(x)=\max\{0,1-|x-2|\}$, $\widetilde{b}_1(x)=\max\{0,1-|x-4|\}$, and $\widetilde{b}_2(x)=\max\{0,1-|x-1|\}$ for each $x\in\mathbb{R}$. We set $\widetilde{\boldsymbol{a}}=(\widetilde{a}_1,\widetilde{a}_2),\widetilde{\boldsymbol{b}}=(\widetilde{b}_1,\widetilde{b}_2)\in\mathcal{F}^2(\mathbb{R})$. Then, $\widetilde{\boldsymbol{a}}$ and $\widetilde{\boldsymbol{b}}$ are the fuzzy sets $\widetilde{a}$ and $\widetilde{b}$ defined in Example 5.4, respectively. We consider $\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}}$ and $2\widetilde{\boldsymbol{a}}$. Since $(\widetilde{a}_1+\widetilde{b}_1)(x)=\max\left\{0,1-\frac{|x-5|}{2}\right\}$, $(\widetilde{a}_2+\widetilde{b}_2)(x)=\max\left\{0,1-\frac{|x-3|}{2}\right\}$, $(2\widetilde{a}_1)(x)=\max\left\{0,1-\frac{|x-2|}{2}\right\}$, and $(2\widetilde{a}_2)(x)=\max\left\{0,1-\frac{|x-4|}{2}\right\}$ for each $x\in\mathbb{R}$, we have

$$\begin{aligned}(\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}})(x,y) &= \min\left\{\max\left\{0,1-\frac{|x-5|}{2}\right\},\max\left\{0,1-\frac{|y-3|}{2}\right\}\right\},\\ (2\widetilde{\boldsymbol{a}})(x,y) &= \min\left\{\max\left\{0,1-\frac{|x-2|}{2}\right\},\max\left\{0,1-\frac{|y-4|}{2}\right\}\right\}\end{aligned}$$

for each $(x,y)\in\mathbb{R}^2$ from Proposition 6.9 (i), (ii).

The following proposition can be shown easily.

Proposition 6.11. *For* $\widetilde{\mathbf{0}}\in\mathcal{F}(\mathbb{R}^n)$ *and* $(\widetilde{0},\widetilde{0},\cdots,\widetilde{0})\in\mathcal{F}^n(\mathbb{R})$, $\widetilde{\mathbf{0}}=(\widetilde{0},\widetilde{0},\cdots,\widetilde{0})$.

The following proposition shows that $\mathcal{F}^n(\mathbb{R})$ is not a vector space, but $\mathcal{F}^n(\mathbb{R})$ has nearly properties of vector spaces. From Propositions 6.9 and 6.11, it is a special case of Proposition 4.4 in Kon [7].

Proposition 6.12. *Let* $\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}} \in \mathcal{F}^n(\mathbb{R})$*, and let* $\lambda, \mu \in \mathbb{R}$*.*

(i) $\widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}} = \widetilde{\boldsymbol{b}} + \widetilde{\boldsymbol{a}}$.
(ii) $(\widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}}) + \widetilde{\boldsymbol{c}} = \widetilde{\boldsymbol{a}} + (\widetilde{\boldsymbol{b}} + \widetilde{\boldsymbol{c}})$.
(iii) $\widetilde{\boldsymbol{0}} + \widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{a}}$.
(iv) *There does not always exist* $\widetilde{\boldsymbol{d}} \in \mathcal{F}^n(\mathbb{R})$ *such that* $\widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{d}} = \widetilde{\boldsymbol{0}}$.
(v) *It does not always hold that* $(\lambda + \mu)\widetilde{\boldsymbol{a}} = \lambda\widetilde{\boldsymbol{a}} + \mu\widetilde{\boldsymbol{a}}$.
(vi) $\lambda(\widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}}) = \lambda\widetilde{\boldsymbol{a}} + \lambda\widetilde{\boldsymbol{b}}$.
(vii) $(\lambda\mu)\widetilde{\boldsymbol{a}} = \lambda(\mu\widetilde{\boldsymbol{a}})$.
(viii) $1\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{a}}$.

7. Fuzzy set-valued convex mapping

In this section, the definition of fuzzy set-valued convex mappings is presented, and its properties are investigated.

We define fuzzy set-valued convex mappings.

Definition 7.1. (Kon [7]) Let $\widetilde{F} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m)$.

(i) $\widetilde{F}$ is called a fuzzy set-valued convex mapping if

$$\widetilde{F}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \preceq \lambda\widetilde{F}(\boldsymbol{x}) + (1-\lambda)\widetilde{F}(\boldsymbol{y}) \tag{7.1}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in]0,1[$.

(ii) $\widetilde{F}$ is called a fuzzy set-valued strictly convex mapping if

$$\widetilde{F}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \prec \lambda\widetilde{F}(\boldsymbol{x}) + (1-\lambda)\widetilde{F}(\boldsymbol{y}) \tag{7.2}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \boldsymbol{y}$ and any $\lambda \in]0,1[$.

Let $\mathcal{FM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m))$ be the set of all fuzzy set-valued mappings from $\mathbb{R}^n$ to $\mathcal{F}(\mathbb{R}^m)$, and let $\mathcal{FKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m))$ be the set of all fuzzy set-valued convex mappings from $\mathbb{R}^n$ to $\mathcal{F}(\mathbb{R}^m)$. In addition, let $\mathcal{FSKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m))$ be the set of all fuzzy set-valued strictly convex mappings from $\mathbb{R}^n$ to $\mathcal{F}(\mathbb{R}^m)$.

A fuzzy set-valued mapping $\widetilde{F} \in \mathcal{FM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m))$ is said to be convex-valued, closed-valued, or compact-valued if $\widetilde{F}(\boldsymbol{x}) \in \mathcal{FK}(\mathbb{R}^m)$, $\widetilde{F}(\boldsymbol{x}) \in \mathcal{FC}(\mathbb{R}^m)$, or $\widetilde{F}(\boldsymbol{x}) \in \mathcal{FBC}(\mathbb{R}^m)$ for any $\boldsymbol{x} \in \mathbb{R}^n$, respectively.

Example 7.2. Let $\widetilde{F}, \widetilde{G} \in \mathcal{FM}(\mathbb{R} \to \mathcal{F}(\mathbb{R}^2))$ be fuzzy set-valued mappings defined as $\widetilde{F}(x)(y,z) = \min\{\max\{0, 1 - |y - (x + x^2)|\}, \max\{0, 1 - |z - (-x + x^2)|\}\}$ and $\widetilde{G}(x)(y,z) = \min\{\max\{0, 1 - |y - (x + |x|)|\}, \max\{0, 1 - |z - (-x + |x|)|\}\}$ for each $x \in \mathbb{R}$ and each $(y,z) \in \mathbb{R}^2$. Let $x_1, x_2 \in \mathbb{R}$, and let $\lambda \in]0,1[$. Since $[\widetilde{F}(x)]_\alpha = [x + x^2 - (1-\alpha), x + x^2 + (1-\alpha)] \times [-x + x^2 - (1-\alpha), -x + x^2 + (1-\alpha)]$ and $[\widetilde{G}(x)]_\alpha = [x + |x| - (1-\alpha), x + |x| + (1-\alpha)] \times [-x + |x| - (1-\alpha), -x + |x| + (1-\alpha)]$

for each $x \in \mathbb{R}$ and each $\alpha \in]0,1]$, it follows that

$$\begin{aligned}
&\left[\widetilde{F}(\lambda x_1 + (1-\lambda)x_2)\right]_\alpha \\
&= \left[\lambda x_1 + (1-\lambda)x_2 + (\lambda x_1 + (1-\lambda)x_2)^2 - (1-\alpha),\right. \\
&\quad \left.\lambda x_1 + (1-\lambda)x_2 + (\lambda x_1 + (1-\lambda)x_2)^2 + (1-\alpha)\right] \\
&\times \left[-(\lambda x_1 + (1-\lambda)x_2) + (\lambda x_1 + (1-\lambda)x_2)^2 - (1-\alpha),\right. \\
&\quad \left.-(\lambda x_1 + (1-\lambda)x_2) + (\lambda x_1 + (1-\lambda)x_2)^2 + (1-\alpha)\right], \\
&\left[\widetilde{G}(\lambda x_1 + (1-\lambda)x_2)\right]_\alpha \\
&= [\lambda x_1 + (1-\lambda)x_2 + |\lambda x_1 + (1-\lambda)x_2| - (1-\alpha), \\
&\quad \lambda x_1 + (1-\lambda)x_2 + |\lambda x_1 + (1-\lambda)x_2| + (1-\alpha)] \\
&\times [-(\lambda x_1 + (1-\lambda)x_2) + |\lambda x_1 + (1-\lambda)x_2| - (1-\alpha), \\
&\quad -(\lambda x_1 + (1-\lambda)x_2) + |\lambda x_1 + (1-\lambda)x_2| + (1-\alpha)]
\end{aligned}$$

and

$$\begin{aligned}
&\left[\lambda\widetilde{F}(x_1) + (1-\lambda)\widetilde{F}(x_2)\right]_\alpha \\
&= \left[\lambda x_1 + (1-\lambda)x_2 + \lambda x_1^2 + (1-\lambda)x_2^2 - (1-\alpha),\right. \\
&\quad \left.\lambda x_1 + (1-\lambda)x_2 + \lambda x_1^2 + (1-\lambda)x_2^2 + (1-\alpha)\right] \\
&\times \left[-(\lambda x_1 + (1-\lambda)x_2) + \lambda x_1^2 + (1-\lambda)x_2^2 - (1-\alpha),\right. \\
&\quad \left.-(\lambda x_1 + (1-\lambda)x_2) + \lambda x_1^2 + (1-\lambda)x_2^2 + (1-\alpha)\right], \\
&\left[\lambda\widetilde{G}(x_1) + (1-\lambda)\widetilde{G}(x_2)\right]_\alpha \\
&= [\lambda x_1 + (1-\lambda)x_2 + \lambda|x_1| + (1-\lambda)|x_2| - (1-\alpha), \\
&\quad \lambda x_1 + (1-\lambda)x_2 + \lambda|x_1| + (1-\lambda)|x_2| + (1-\alpha)] \\
&\times [-(\lambda x_1 + (1-\lambda)x_2) + \lambda|x_1| + (1-\lambda)|x_2| - (1-\alpha), \\
&\quad -(\lambda x_1 + (1-\lambda)x_2) + \lambda|x_1| + (1-\lambda)|x_2| + (1-\alpha)]
\end{aligned}$$

for each $\alpha \in]0,1]$. Thus, it follows that $[\widetilde{F}(\lambda x_1 + (1-\lambda)x_2)]_\alpha \leq [\lambda\widetilde{F}(x_1) + (1-\lambda)\widetilde{F}(x_2)]_\alpha$ and $[\widetilde{G}(\lambda x_1 + (1-\lambda)x_2)]_\alpha \leq [\lambda\widetilde{G}(x_1) + (1-\lambda)\widetilde{G}(x_2)]_\alpha$ for any $\alpha \in]0,1]$, and that $[\widetilde{F}(\lambda x_1 + (1-\lambda)x_2)]_\alpha < [\lambda\widetilde{F}(x_1) + (1-\lambda)\widetilde{F}(x_2)]_\alpha$ for any $\alpha \in]0,1]$ if $x_1 \neq x_2$. These relations imply that $\widetilde{F}(\lambda x_1+(1-\lambda)x_2) \preceq \lambda\widetilde{F}(x_1)+(1-\lambda)\widetilde{F}(x_2)$ and $\widetilde{G}(\lambda x_1+(1-\lambda)x_2) \preceq \lambda\widetilde{G}(x_1)+(1-\lambda)\widetilde{G}(x_2)$, and that $\widetilde{F}(\lambda x_1+(1-\lambda)x_2) \prec \lambda\widetilde{F}(x_1)+(1-\lambda)\widetilde{F}(x_2)$ if $x_1 \neq x_2$. On the other hand, we can see that $[\widetilde{G}(\lambda x_1+(1-\lambda)x_2)]_\alpha \not< [\lambda\widetilde{G}(x_1) + (1-\lambda)\widetilde{G}(x_2)]_\alpha$ for any $\alpha \in]0,1]$ if $x_1, x_2 \geq 0$, $x_1 \neq x_2$ or $x_1, x_2 \leq 0$, $x_1 \neq x_2$. It means that $\widetilde{G}(\lambda x_1 + (1-\lambda)x_2) \not\prec \lambda\widetilde{G}(x_1) + (1-\lambda)\widetilde{G}(x_2)$ if $x_1, x_2 \geq 0$, $x_1 \neq x_2$ or $x_1, x_2 \leq 0$, $x_1 \neq x_2$. Therefore, we have $\widetilde{F}, \widetilde{G} \in \mathcal{FKM}(\mathbb{R} \to \mathcal{F}(\mathbb{R}^2))$, $\widetilde{F} \in \mathcal{FSKM}(\mathbb{R} \to \mathcal{F}(\mathbb{R}^2))$, and $\widetilde{G} \notin \mathcal{FSKM}(\mathbb{R} \to \mathcal{F}(\mathbb{R}^2))$.

The following proposition shows properties of fuzzy product set-valued mappings.

Proposition 7.3. *Let* $\widetilde{F}_i \in \mathcal{FM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}))$, $i = 1, 2, \cdots, m$, *and let* $\widetilde{\boldsymbol{F}} = (\widetilde{F}_1, \widetilde{F}_2, \cdots, \widetilde{F}_m) \in \mathcal{FM}(\mathbb{R}^n \to \mathcal{F}^m(\mathbb{R}))$ *be a fuzzy set-valued mapping defined as* $\widetilde{\boldsymbol{F}}(\boldsymbol{x}) = (\widetilde{F}_1(\boldsymbol{x}), \widetilde{F}_2(\boldsymbol{x}), \cdots, \widetilde{F}_m(\boldsymbol{x}))$ *for each* $\boldsymbol{x} \in \mathbb{R}^n$.

(i) *Assume that* $hgt(\widetilde{F}_1(\boldsymbol{x})) = \cdots = hgt(\widetilde{F}_m(\boldsymbol{x}))$ *for any* $\boldsymbol{x} \in \mathbb{R}^n$. *If* $\widetilde{\boldsymbol{F}}$ *is convex-valued, then* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are also convex-valued.*

(ii) *If* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are convex-valued, then* $\widetilde{\boldsymbol{F}}$ *is also convex-valued.*

(iii) *Assume that* $hgt(\widetilde{F}_1(\boldsymbol{x})) = \cdots = hgt(\widetilde{F}_m(\boldsymbol{x}))$ *for any* $\boldsymbol{x} \in \mathbb{R}^n$. *If* $\widetilde{\boldsymbol{F}}$ *is closed-valued, then* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are also closed-valued.*

(iv) *If* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are closed-valued, then* $\widetilde{\boldsymbol{F}}$ *is also closed-valued.*

(v) *Assume that* $hgt(\widetilde{F}_1(\boldsymbol{x})) = \cdots = hgt(\widetilde{F}_m(\boldsymbol{x}))$ *for any* $\boldsymbol{x} \in \mathbb{R}^n$. *If* $\widetilde{\boldsymbol{F}}$ *is compact-valued, then* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are also compact-valued.*

(vi) *If* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are compact-valued, then* $\widetilde{\boldsymbol{F}}$ *is also compact-valued.*

(vii) *Assume that* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are compact-valued, and that* $I(\widetilde{F}_1(\boldsymbol{x})) = \cdots = I(\widetilde{F}_m(\boldsymbol{x})) = I(\widetilde{F}_1(\boldsymbol{y})) = \cdots = I(\widetilde{F}_m(\boldsymbol{y}))$ *for any* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. *Then,*

$$\widetilde{\boldsymbol{F}} \in \mathcal{FKM}(\mathbb{R}^n \to \mathcal{F}^m(\mathbb{R})) \Rightarrow \widetilde{F}_i \in \mathcal{FKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R})), i = 1, 2, \cdots, m.$$

(viii) $\widetilde{F}_i \in \mathcal{FKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R}))$, $i = 1, 2, \cdots, m \Rightarrow \widetilde{\boldsymbol{F}} \in \mathcal{FKM}(\mathbb{R}^n \to \mathcal{F}^m(\mathbb{R}))$.

(ix) *Assume that* $\widetilde{F}_i$, $i = 1, 2, \cdots, m$ *are compact-valued, and that* $I(\widetilde{F}_1(\boldsymbol{x})) = \cdots = I(\widetilde{F}_m(\boldsymbol{x})) = I(\widetilde{F}_1(\boldsymbol{y})) = \cdots = I(\widetilde{F}_m(\boldsymbol{y}))$ *for any* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. *Then,*

$$\widetilde{\boldsymbol{F}} \in \mathcal{FSKM}(\mathbb{R}^n \to \mathcal{F}^m(\mathbb{R})) \Rightarrow \widetilde{F}_i \in \mathcal{FSKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R})), i = 1, 2, \cdots, m.$$

(x) $\widetilde{F}_i \in \mathcal{FSKM}(\mathbb{R}^n \to \mathcal{F}(\mathbb{R})), i = 1, 2, \cdots, m \Rightarrow \widetilde{\boldsymbol{F}} \in \mathcal{FSKM}(\mathbb{R}^n \to \mathcal{F}^m(\mathbb{R}))$.

Proof. (i)–(vi) follow from Proposition 6.4 (i)–(vi), respectively. We show only (vii) and (viii). (ix) and (x) can be shown in the similar ways to (vii) and (viii), respectively.

(vii) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and let $\lambda \in]0, 1[$. We set $I = I(\widetilde{F}_1(\boldsymbol{x})) = \cdots = I(\widetilde{F}_m(\boldsymbol{x}))$. Since $\widetilde{\boldsymbol{F}}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \preceq \lambda\widetilde{\boldsymbol{F}}(\boldsymbol{x}) + (1-\lambda)\widetilde{\boldsymbol{F}}(\boldsymbol{y})$, it follows that $(\widetilde{F}_1(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}), \cdots, \widetilde{F}_m(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})) \preceq (\lambda\widetilde{F}_1(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_1(\boldsymbol{y}), \cdots, \lambda\widetilde{F}_m(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_m(\boldsymbol{y}))$ from Proposition 6.9 (i), (ii). For each $i \in \{1, 2, \cdots, m\}$, it follows that $I(\widetilde{F}_i(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})) = I$ from the assumption, and that $I(\lambda\widetilde{F}_i(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_i(\boldsymbol{y})) = \{\alpha \in]0, 1] : \lambda[\widetilde{F}_i(\boldsymbol{x})]_\alpha + (1-\lambda)[\widetilde{F}_i(\boldsymbol{y})]_\alpha \neq \varnothing\} = I$ from Propositions 4.2 and 4.5 (vii). Therefore, we have $\widetilde{F}_i(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \preceq \lambda\widetilde{F}_i(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_i(\boldsymbol{y})$ for any $i \in \{1, 2, \cdots, m\}$ from Proposition 6.5 (i).

(viii) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and let $\lambda \in]0, 1[$. Since $\widetilde{F}_i(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \preceq \lambda\widetilde{F}_i(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_i(\boldsymbol{y})$ for any $i \in \{1, 2, \cdots, m\}$, we have $\widetilde{\boldsymbol{F}}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) = (\widetilde{F}_1(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}), \cdots, \widetilde{F}_m(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})) \preceq (\lambda\widetilde{F}_1(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_1(\boldsymbol{y}), \cdots, \lambda\widetilde{F}_m(\boldsymbol{x}) + (1-\lambda)\widetilde{F}_m(\boldsymbol{y})) = \lambda\widetilde{\boldsymbol{F}}(\boldsymbol{x}) + (1-\lambda)\widetilde{\boldsymbol{F}}(\boldsymbol{y})$ from Propositions 6.5 (ii) and 6.9 (i), (ii). □

8. Conclusions

We dealt with general fuzzy sets. First, the fuzzy inner product was defined based on Zadeh's extension principle, and the (strict) fuzzy max order was characterized by the fuzzy inner product. Next, in the fuzzy product space, properties of addition and scalar multiplication, the (strict) fuzzy max order, and the fuzzy inner product were investigated. Finally, the definition of fuzzy set-valued (strictly) convex mappings was presented, and its properties based on the fuzzy product space were investigated. The obtained results can be expected to be useful for analyzing fuzzy mathematical models using general fuzzy sets.

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