Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 5, (May 2015), pp. 719–732
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Some algebraic properties of fuzzy mathematical machine and fuzzy automata

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Received 28 May 2014; Revised 1 September 2014; Accepted 14 October 2014

Abstract. The states and inputs play a vital role in state transition of a fuzzy mathematical machine. In this paper the behaviour of a fuzzy mathematical machine for some fuzzy operations on the states and inputs are studied. The behaviour of a fuzzy mathematical machine in terms of fuzzy cover is also studied. Some generalised behavior of the states, input symbols and output symbols of a fuzzy automata is established.

2010 AMS Classification: 03E72, 08A72

uzzy Mathematical Machine, Fuzzy cover, Fuzzy union, Fuzzy Intersection, Fuzzy Automata.

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1. Introduction

The concept of fuzzy set was first introduced by [24] and since then it has wide applications in the field of basic sciences[18] [1] [25], engineering , technology and social sciences. The fuzzy number theory was enriched by [4]. Among some of the important fields in fuzzy set theory, the fuzzy similarity relation and the construction of similarity class is described by [16]. The concept of relation plays an important role in the fuzzy set theory. The fuzzy relation was defined by [24], [22] [19] et al. Fuzzy logic is an integral part of the fuzzy mathematical machine [26]. As far as the automata theory is concerned, the classical automata theory was described by [10]. The automata theory was presented by [9] from the algebraic point of view. It is also studied from the algebraic view point by [11] [5] [12] [20] [17]. The crisp automata theory is used in many interesting fields like the study of mutation of the genes by [13]. Fuzzy automata is a generalised version of the crisp automata. The properties of the fuzzy set theory is used with great success in describing a more naturalistic and conceptually potent version of the crisp automata. The theory of fuzzy automata was studied by fuzzy sub systems [23],fuzzy finite state machines and fuzzy automata was studied by [2] [3], topology [21] [15],tree structure [6] and formal languages [7]. Basic ideas of fuzzy sets is given in Section 2. This section has informations required for the next sections. Section 3 deals with some algebraic results about the states and inputs of a fuzzy mathematical machine. In Section 4 some generalised results regarding the states and inputs of a fuzzy automata is established.

2. Preliminaries

2.1. **Fuzzy Set.** The fuzzy set is used to represent the vague concepts of human understanding in a particular form. The concept of fuzzy set is operationally powerful enough to use in computers. Fuzzy set theory is an extension of the crisp set. In the crisp set theory, a characteristic function μ_A is defined as $\mu_A : U \to \{0, 1\}$ where A is a crisp set defined on the universe U. If $x \in A$, then $\mu_A(x) = 1$, otherwise $\mu_A(x) = 0$. In the fuzzy set theory, the characteristic function is generalised by a membership function. The membership function assigns to every $x \in U$ a real value from the unit interval $[0, 1]$ instead of the two member set $\{0, 1\}$. The fuzzy set A is determined by the set of tuples,
$A = \{x; \mu_A(x)|x \in U\}$. If the universe U is discrete, A can be expressed as, $A = \sum_{x \in U} \mu_A(x)/x$. If U is uncountable or continuous, A can be written as,

$$A = \int_U \frac{\mu_A(x)}{x}.$$

Standard operations on fuzzy sets:
(i) Fuzzy Complement : Let $c(A(x))$ denote a fuzzy complement of $A(x)$. $c(A(x))$ can be interpreted as the degree to which x does not belong to A. Mathematically, $c(a(x)) = 1 - A(x)$.
(ii) Fuzzy Intersection : Let A and B be two fuzzy sets over the universe U. The fuzzy intersection of A and B is denoted as $A \cap B$ and is defined by $(A \cap B)(x) = min(A(x), B(x))$ for every x in U. $A \cap B$ is also defined by a function $i : [0, 1] \times [0, 1] \to [0, 1]$, such that, $(A \cap B)(x) = i[A(x), B(x)], \forall x \in U$.
(iii) Fuzzy Union : The general fuzzy union of two fuzzy sets A and B is specified by a function $u : [0, 1] \times [0, 1] \to [0, 1]$. The function has the membership grade of the element as in the set $A \cup B$. Therefore, $(A \cup B)(x) = u[A(x), B(x)]\ \forall x \in X$.

Definition 2.1. Let S be a fuzzy set defined over the universe U. A family of fuzzy sets $\tau = \{Q\}$ is said to be a fuzzy cover of S if $S = \cup_{Q \in \tau}\{Q\}$. Each Q is a fuzzy subset of S [16].

Definition 2.2. For any fuzzy cover $\tau = \{Q\}$ of S, a fuzzy binary relation R_τ is defined by [16]

$$R_\tau(s_i, s_j) = \vee_{Q \in \tau}\{\mu_Q(s_i) \wedge \mu_Q(s_j)\} \tag{2.1}$$

3. Fuzzy mathematical machine without output

Definition 3.1. A fuzzy sequential machine without output is a four tuple $S =< S, \Sigma_k, \tilde{M}, a >$ where S is the set of internal states, Σ_k is a fixed non-empty set called the input alphabet $\{\sigma_0, \sigma_1, ..., \sigma_{k-1}\}$. $\tilde{M}$ is a function from $S \times \Sigma_k \times S \to [0, 1]$ called the transition function and a is an member of S called the initial state.

A fuzzy word is a fuzzy subset over the input alphabet. The collection of all fuzzy words over the input alphabet is called fuzzy dictionary and is denoted by Σ^*. A finite sequence of fuzzy input words is called a fuzzy input tape. A null tape is denoted by λ. The transition function $\tilde{M}$ is given in the form of matrices, one each for the input symbols $\sigma_0, \sigma_1, ..., \sigma_{k-1}$. The matrices are denoted by $\tilde{M}(\sigma_0), \tilde{M}(\sigma_1), ..., \tilde{M}(\sigma_{k-1})$. $\mu_{\tilde{M}}(s_i, \sigma_k, s_j)$ is a real number indicating the membership of the triplet of $\tilde{M}$ where $s_i, s_j \in S$ and $\sigma_k \in \Sigma_k$.

If a fuzzy machine is taken into account whose states are fuzzy as well as the inputs also belong to the fuzzy dictionary, it is called the most general fuzzy finite state machine($mgffsm$). The transition matrix $\tilde{M}$ of the $mgffsm$ is defined by $\tilde{M} : \mathcal{F}(S) \times \mathcal{F}\ (\Sigma^*) \to \mathcal{F}(S)$ determined from the fuzzy transition matrices $\tilde{M}(\sigma_0), \tilde{M}(\sigma_1), ..., \tilde{M}(\sigma_{k-1})$. The algorithm for the state transition used here is studied by [8]. Some authors [14] studied some properties of fuzzy Mealy machines.

Definition 3.2. Concatenation of two tapes x and y is defined by the tape obtained by writing x followed by y and is defined by xy.

Definition 3.3. The length of an input tape x is denoted by $lg(x)$.

Example 3.4. Let $x = \sigma_0\sigma_1\sigma_2\sigma_3$. Then $lg(x) = 4$.

Definition 3.5. The response function of a fuzzy mathematical machine $S =< S, \Sigma^*, \tilde{M}, a, >$, denoted by $rp_S(x)$ is a function from $S \times \Sigma^* \to S$ defined by $(\forall x) \in \Sigma^*, rp_S(x) = \tilde{M}(a, x)$ [8].

Definition 3.6. $\tilde{M}(\tilde{s}, i) = \bigvee_{j=0}^{k-1} \{\tilde{M}(\tilde{s}, \mu(\sigma_j)/\sigma_j\}$, where

$$\tilde{s} = \sum_{i=0}^{n-1} \mu(s_i)/s_i, i = \sum_{t=0}^{m-1} \mu(\sigma_t)/\sigma_t.$$

Theorem 3.7.

$$\tilde{M}(s, \sigma_0\sigma_1...\sigma_{n-1}, t) = \vee_{t_{n-2}\in S} \vee_{t_{n-3}\in S} ... \vee_{t_0\in S} \{\tilde{M}(s, \sigma_0, t_0) \wedge \tilde{M}(t_0, \sigma_1, t_1) \wedge ... \wedge \tilde{M}(t_{n-2}, \sigma_{n-1}, t)\}$$

Proof. We have,

$$\begin{aligned}\tilde{M}(s, \sigma_0\sigma_1...\sigma_{n-1}, t) &= \vee_{t_{n-2}\in S}\{\tilde{M}(s, \sigma_0\sigma_1...\sigma_{n-2}, t_{n-2}) \wedge \tilde{M}(t_{n-2}, \sigma_{n-1}, t)\} \\ &= \vee_{t_{n-2}\in S}\{\vee_{t_{n-3}\in S}\{\tilde{M}(s, \sigma_0\sigma_1...\sigma_{n-3}, t_{n-3}) \\ &\wedge \tilde{M}(t_{n-3}, \sigma_{n-2}, t_{n-2})\} \wedge \tilde{M}(t_{n-2}, \sigma_{n-1}, t)\} \\ &... \\ &= \vee_{t_{n-2}\in S} \vee_{t_{n-3}\in S} ... \vee_{t_0\in S} \{\tilde{M}(s, \sigma_0, t_0) \wedge \tilde{M}(t_0, \sigma_1, t_1) \wedge \\ &\quad ... \wedge \tilde{M}(t_{n-2}, \sigma_{n-1}, t)\}\end{aligned}$$

□

Observation 3.8.

$$\tilde{M}(s, \sigma_0\sigma_1\sigma_2, t) = \vee_{r\in S} \vee_{r'\in S} \{\tilde{M}(s, \sigma_0, r') \wedge \tilde{M}(r', \sigma_1, r) \wedge \tilde{M}(r, \sigma_2, t)\}.$$

It is to be noted that,

$$\tilde{M}(s,\lambda,t) = 1, \text{ if } s = t$$
$$= 0, \text{ if } s \neq t$$

Theorem 3.9. *In a fuzzy mathematical machine without output* $\tilde{S} =< \tilde{S}, \Sigma_k, \tilde{M}, \tilde{a} >$, $\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma) = \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2,\sigma)$, $\forall \tilde{u}_1, \tilde{u}_2 \in \tilde{S}$ *and* $\forall \sigma \in \Sigma_k$.

Proof. Let $\tilde{u}_1 = \mu_{u_1}(s_0)/s_0 + \mu_{u_1}(s_1)/s_1 + ... + \mu_{u_1}(s_{n-1})/s_{n-1}$ and $\tilde{u}_2 = \mu_{u_2}(s_0)/s_0 + \mu_{u_2}(s_1)/s_1 + ... + \mu_{u_2}(s_{n-1})/s_{n-1}$ $\forall s_0, s_1, ...s_{n-1} \in S$. The membership value of s_0 in $\tilde{M}(\tilde{u}_1,\sigma)$ is given by,

$$\mu_{\tilde{M}(\tilde{u}_1,\sigma)}(s_0) = max\{min(\mu_{u_1}(s_i), \mu(s_i,\sigma,s_0)\}, i = 0,1,...n-1. \tag{3.1}$$

Similarly, the membership value of s_0 in $\tilde{M}(\tilde{u}_2,\sigma)$ is given by,

$$\mu_{\tilde{M}(\tilde{u}_2,\sigma)}(s_0) = max\{min(\mu_{u_2}(s_i), \mu(s_i,\sigma,s_0)\}, i = 0,1,...n-1. \tag{3.2}$$

Taking max over 3.1 and 3.2 we have the membership value of s_0 in $\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma)$ as

$$\begin{aligned}
& max[max\{min(\mu_{u_1}(s_i), \mu(s_i,\sigma,s_0)\}, max\{min(\mu_{u_2}(s_i), \mu(s_i,\sigma,s_0)\}] \\
= \; & max[min(\mu_{u_1}(s_i), \mu(s_i,\sigma.s_0)), min(\mu_{u_2}(s_i), \mu(s_i,\sigma.s_0))] \\
= \; & min[max\{(\mu_{u_1}(s_i), \mu(s_i,\sigma.s_0)), (\mu_{u_2}(s_i), \mu(s_i,\sigma.s_0)) \\
= \; & min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(s_i,\sigma.s_0)
\end{aligned}$$

Similarly, the membership value of s_1 in

$$\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma) = min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(s_i,\sigma.s_1)$$

...

and the membership value of s_{n-1} in

$$\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma) = min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(s_i,\sigma.s_{n-1})$$

Therefore,

$$\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma) = \sum_{r=0}^{n-1} min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(s_i,\sigma,s_r)]/s_r \tag{3.3}$$

$i = 0,1,...n-1$

Again,

$$\begin{aligned}
\tilde{u}_1 \cup \tilde{u}_2 = \; & max\{\mu_{u_1}(s_0), \mu_{u_2}(s_0)\}/s_0 + max\{\mu_{u_1}(s_1), \mu_{u_2}(s_1)\}/s_1 + ... \\
& + max\{\mu_{u_1}(s_{n-1}), \mu_{u_2}(s_{n-1})\}/s_{n-1}.
\end{aligned}$$

Therefore,

$$\tilde{M}(\tilde{u}_1 \cup \tilde{u}_2,\sigma) = \sum_{r=0}^{n-1} min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(s_i,\sigma,s_r)]/s_r \tag{3.4}$$

$i = 0,1,...n-1$. Comparing 3.3 and 3.4 we have,

$$\tilde{M}(\tilde{u}_1,\sigma) \cup \tilde{M}(\tilde{u}_2,\sigma) = \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2,\sigma)$$

□

Theorem 3.10. *In a fuzzy mathematical machine without output* $\tilde{S} =< \tilde{S}, \Sigma, \tilde{M}, \tilde{a} >$, $\tilde{M}(\tilde{u}_1, i) \cup \tilde{M}(\tilde{u}_2, i) = \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2, i),\ \forall \tilde{u}_1, \tilde{u}_2 \in \tilde{S}$ *and* $\forall i \in \Sigma$.

Theorem 3.11. *In a fuzzy mathematical machine without output* $\tilde{S} =< \tilde{S}, \Sigma^*, \tilde{M}, \tilde{a} >$, $\tilde{M}(\tilde{u}_1, x) \cup \tilde{M}(\tilde{u}_2, x) = \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2, x),\ \forall \tilde{u}_1, \tilde{u}_2 \in \tilde{S}$ *and* $\forall x \in \Sigma^*$.

Theorem 3.12. *In a fuzzy mathematical machine without output* $\tilde{S} =< \tilde{S}, \Sigma_k, \tilde{M}, \tilde{a} >$, $\tilde{M}(\tilde{u}_1, \sigma) \cap \tilde{M}(\tilde{u}_2, \sigma) \neq \tilde{M}(\tilde{u}_1 \cap \tilde{u}_2, \sigma),\ \forall \tilde{u}_1, \tilde{u}_2 \in \tilde{S}$ *and* $\forall \sigma \in \Sigma_k$.

Proof. We have the membership value of s_r in $\tilde{M}(\tilde{u}_1, \sigma)$ as,

$$\mu_{\tilde{M}(\tilde{u}_1,\sigma)}(s_r) = max\{min(\mu_{u_1}(s_i), \mu(s_i, \sigma, s_r)\}, i = 0, 1, ..., n-1. \tag{3.5}$$

and the membership value of s_r in $\tilde{M}(\tilde{u}_2, \sigma)$ as

$$\mu_{\tilde{M}(\tilde{u}_2,\sigma)}(s_r) = max\{min(\mu_{u_2}(s_i), \mu(s_i, \sigma, s_r)\}, i = 0, 1, ..., n-1. \tag{3.6}$$

Therefore,

$$\tilde{M}(\tilde{u}_1, \sigma) \cap \tilde{M}(\tilde{u}_2, \sigma) = \sum_{r=0}^{n-1} [max\{min(\mu_{u_1}(s_i), \mu(s_i, \sigma, s_r))\}, \\ max\{min(\mu_{u_2}(s_i), \mu(s_i, \sigma, s_r))\}]/s_r \tag{3.7}$$

$i = 0, 1, ..., n-1$. Also the membership of s_r in $\tilde{u_1} \cap \tilde{u_2}$ is

$$\mu_{(\tilde{u}_1 \cap \tilde{u}_2)}(s_r) = min(\mu_{\tilde{u}_1}(s_r), \mu_{\tilde{u}_2}(s_r)) \tag{3.8}$$

Therefore,

$$\tilde{M}(\tilde{u}_1 \cap \tilde{u}_2, \sigma) = \sum_{r=0}^{n-1} [min\{min(\mu_{u_1}(s_r), \mu_{u_2}(s_r)), \mu(s_i, \sigma, s_r)\}]/s_r \tag{3.9}$$

$i = 0, 1, ..., n-1$. Comparing 3.7 and 3.9 we have the result. □

Example 3.13. Let $\tilde{u_0}$ and $\tilde{u_1}$ be two fuzzy states defined on the universe of discourse $S = \{s_0, s_1, s_2\}$. Let $\tilde{u_0} = 0.1/s_0 + 0.8/s_1 + 0.3/s_2$ and $\tilde{u_1} = 0.9/s_0 + 0.2/s_1 + 0.1/s_2$. Therefore, $\tilde{u}_0 \cup \tilde{u}_1 = 0.9/s_0 + 0.8/s_1 + 0.3/s_2$ and $\tilde{u}_0 \cap \tilde{u}_1 = 0.1/s_0 + 0.2/s_1 + 0.1/s_2$. Let $x = i_0 i_1$, where $i_0 = 0.8/\sigma_0 + 0.4/\sigma_1$, and $i_1 = 0.4/\sigma_0 + 0.7/\sigma_1$.

Let the fuzzy transition matrices $\tilde{M}(\sigma_0)$ and $\tilde{M}(\sigma_1)$ be given by the Table 1 and Table 2.

TABLE 1.

σ_0	s_0	s_1	s_2
s_0	0.1	0.2	1.0
s_1	0.8	0.2	0.3
s_2	0.0	0.9	0.2

The corresponding transitions are given by, $\tilde{M}(\tilde{u}_0, x) = 0.3/s_0 + 0.3/s_1 + 0.4/s_2$, and $\tilde{M}(\tilde{u}_1, x) = 0.3/s_0 + 0.4/s_1 + 0.3/s_2$.
We have,

$$\begin{aligned} \tilde{M}(\tilde{u}_0 \cup \tilde{u}_1, x) &= 0.3/s_0 + 0.4/s_1 + 0.4/s_2 \\ &= \tilde{M}(\tilde{u}_0, x) \cup \tilde{M}(\tilde{u}_1, x) \end{aligned}$$

Table 2.

σ_1	s_0	s_1	s_2
s_0	1.0	0.2	0.1
s_1	0.3	0.9	0.2
s_2	0.1	0.3	0.8

and,

$$\begin{aligned}\tilde{M}(\tilde{u}_0 \cap \tilde{u}_1, x) &= 0.2/s_0 + 0.2/s_1 + 0.2/s_2 \\ &\neq \tilde{M}(\tilde{u}_0, x) \cap \tilde{M}(\tilde{u}_1, x)\end{aligned}$$

Corollary 3.14. $\tilde{M}(\tilde{u}_1, x) \cup \tilde{M}(\tilde{u}_2, x) \cup ...\tilde{M}(\tilde{u}_k, x) = \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2 ... \cup \tilde{u}_k, x)$.

Definition 3.15. The mapping $\phi : [0,1] \times S \to [0,1] \times S$ is called fuzzy homomorphism if it satisfies

(i) $\phi\{\mu(s_i)/s_i\} = \mu(s_i)/\phi(s_i)\ \forall s_i \in \tilde{S}$ [Preservation of degree of easeness]
(ii) $\phi(\sum\{\mu(s_i)/s_i\}) = \sum \mu(s_i)/\phi(s_i)\ \forall s_i \in \tilde{S}$
(iii) $defuzzy[M\{\phi(u), \sigma\}] = defuzzy\phi[\ M(u, \sigma)]$

Theorem 3.16. *In a fuzzy mathematical machine without output* $\tilde{S} =< \tilde{S}, \Sigma_k, \tilde{M}, \tilde{a} >$, $M\{\phi(\tilde{u}_1 \cup \tilde{u}_2), \sigma\} = M\{\phi(\tilde{u}_1), \sigma\} \cup M\{\phi(\tilde{u}_2), \sigma\}, \tilde{u_1}, \tilde{u_2} \in \tilde{S}, \sigma \in \Sigma_k$

Proof. Let $\tilde{u}_1 = \sum_i \mu_{u_1}(s_i)/s_i,\ \tilde{u}_2 = \sum_i \mu_{u_2}(s_i)/s_i$. Therefore

$$\tilde{u}_1 \cup \tilde{u}_2 = \sum max(\mu_{u_1}(s_i), \mu_{u_2}(s_i))/s_i.$$

Also, $\phi(\tilde{u}_1 \cup \tilde{u}_2) = \phi(\sum max(\mu_{u_1}(s_i), \mu_{u_2}(s_i))/s_i) = \sum max(\mu_{u_1}(s_i), \mu_{u_2}(s_i))/\phi(s_i)$. Therefore,

$$\begin{aligned}&M\{\phi(\tilde{u}_1 \cup \tilde{u}_2), \sigma\} \\ &= (\sum_{r=0}^{n-1} min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(\phi(s_i), \sigma, \phi(s_r))]/\phi(s_r),\end{aligned} \tag{3.10}$$

$i = 0, 1, ..., n-1$. By Theorem 3.9 we have,

$$\begin{aligned}&M\{\phi(\tilde{u}_1), \sigma\} \cup M\{\phi(\tilde{u}_2), \sigma\} \\ &= (\sum_{r=0}^{n-1} min[max\{\mu_{u_1}(s_i), \mu_{u_2}(s_i)\}, \mu(\phi(s_i), \sigma, \phi(s_r)]/\phi(s_r)\end{aligned} \tag{3.11}$$

$i = 0, 1, ..., n-1$. So, we have by 3.10 and 3.11,

$$M\{\phi(\tilde{u}_1 \cup \tilde{u}_2), \sigma\} = M\{\phi(\tilde{u}_1), \sigma\} \cup M\{\phi(\tilde{u}_2), \sigma\}$$ □

Example 3.17. Let

$$[0,1] \times S = \{\mu(s_0)/s_0, \mu(s_1)/s_1, \mu(s_2)/s_2, \mu(s_3)/s_3, \mu(s_4)/s_4, \mu(s_5)/s_5\},$$

where $\mu(s_i)/s_i \in [0,1]$. Let

$$\begin{array}{lllll}
\phi\{\mu(s_0)/s_0\} & = & \mu(s_0)/\phi(s_0) & = & \mu(s_0)/s_1 \\
\phi\{\mu(s_1)/s_1\} & = & \mu(s_1)/\phi(s_1) & = & \mu(s_1)/s_3 \\
\phi\{\mu(s_2)/s_2\} & = & \mu(s_2)/\phi(s_2) & = & \mu(s_2)/s_4 \\
\phi\{\mu(s_3)/s_3\} & = & \mu(s_3)/\phi(s_3) & = & \mu(s_3)/s_0 \\
\phi\{\mu(s_4)/s_4\} & = & \mu(s_4)/\phi(s_4) & = & \mu(s_4)/s_2 \\
\phi\{\mu(s_5)/s_5\} & = & \mu(s_5)/\phi(s_5) & = & \mu(s_5)/s_5.
\end{array}$$

Let,

$\tilde{u}_1 = 0.3/s_0 + 0.4/s_1 + 1.0/s_2 + 0.2/s_3 + 0.4/s_4 + 0.3/s_5$,

$\tilde{u}_2 = 0.2/s_0 + 0.9/s_1 + 0.1/s_2 + 0.2/s_3 + 0.2/s_4 + 0.1/s_5$.

Therefore,

$\phi(\tilde{u}_1) = 0.2/s_0 + 0.3/s_1 + 0.4/s_2 + 0.4/s_3 + 1.0/s_4 + 0.3/s_5$,

$\phi(\tilde{u}_2) = 0.2/s_0 + 0.2/s_1 + 0.2/s_2 + 0.9/s_3 + 0.1/s_4 + 0.1/s_5$.

Let the transition matrix of the fuzzy mathematical machine be given by Table 3,

Table 3.

σ	s_0	s_1	s_2	s_3	s_4	s_5
s_0	1.0	1.0	0.2	0.4	0.3	0.1
s_1	0.2	0.3	0.4	0.8	0.2	0.3
s_2	0.0	0.4	0.2	0.1	1.0	0.4
s_3	0.9	0.2	0.1	0.3	0.3	0.2
s_4	0.1	0.2	0.8	0.2	0.0	0.1
s_5	0.0	0.2	0.3	0.4	0.2	1.0

$\tilde{M}(\tilde{u}_1, \sigma) = 0.2/s_0 + 0.4/s_1 + 0.4/s_2 + 0.4/s_3 + 1.0/s_4 + 0.4/s_5$,

$\tilde{M}(\tilde{u}_2, \sigma) = 0.2/s_0 + 0.3/s_1 + 0.4/s_2 + 0.8/s_3 + 0.2/s_4 + 0.3/s_5$,

Here,

$$\begin{array}{lllll}
defuzzy\phi\{\tilde{M}(\tilde{u}_1,\sigma)\} & = & s_2 & = & defuzzy[\tilde{M}\{\phi(\tilde{u}_1),\sigma\}] \\
defuzzy\phi\{\tilde{M}(\tilde{u}_2,\sigma)\} & = & s_0 & = & defuzzy[\tilde{M}\{\phi(\tilde{u}_2),\sigma\}].
\end{array}$$

Therefore ϕ is a homomorphism. Also,

$\tilde{u}_1 \cup \tilde{u}_2 = 0.3/s_0 + 0.9/s_1 + 1.0/s_2 + 0.2/s_3 + 0.4/s_4 + 0.3/s_5$,

$\phi(\tilde{u}_1 \cup \tilde{u}_2) = 0.2/s_0 + 0.3/s_1 + 0.4/s_2 + 0.9/s_3 + 1.0/s_4 + 0.3/s_5$,

$$M\{\phi(\tilde{u}_1 \cup \tilde{u}_2), \sigma\} = 0.9/s_0 + 0.4/s_1 + 0.8/s_2 + 0.3/s_3 + 0.4/s_4 + 0.4/s_5, \tag{3.12}$$

$$M\{\phi(\tilde{u}_1), \sigma\} = 0.4/s_0 + 0.4/s_1 + 0.8/s_2 + 0.3/s_3 + 0.4/s_4 + 0.4/s_5, \tag{3.13}$$

$$M\{\phi(\tilde{u}_2), \sigma\} = 0.9/s_0 + 0.2/s_1 + 0.2/s_2 + 0.3/s_3 + 0.3/s_4 + 0.2/s_5, \tag{3.14}$$

From 3.12, 3.13 and 3.14 it follows that,

$\tilde{M}\{\phi(\tilde{u}_1 \cup \tilde{u}_2), \sigma\} = \tilde{M}\{\phi(\tilde{u}_1), \sigma\} \cup \tilde{M}\{\phi(\tilde{u}_2), \sigma\}$

Theorem 3.18. *For a fuzzy mathematical machine* $\tilde{S} =< \tilde{S}, \Sigma_k, \tilde{M}, \tilde{a} >$, *let* $\tau = \{\tilde{P}\}$ *be a fuzzy cover of* $\tilde{a}$. *If* $\tilde{M}(\tilde{a}, \sigma) = \tilde{u}$, *where* $\tilde{u} \in \tilde{S}$ *then for any* $\sigma \in \Sigma_k$, $\cup_{\tilde{P}\in\tau}\{\tilde{M}(\tilde{P}, \sigma)\} = \tilde{u}$.

Proof. Let $\tilde{a} = \cup_{p\in\tau}\{\tilde{P}\} = \cup\{\tilde{P}_1, \tilde{P}_2, ...\tilde{P}_k\}$(say). Then $\tilde{a} = \tilde{P}_1 \cup \tilde{P}_2 \cup ... \cup \tilde{P}_k$. Therefore,

$$\begin{aligned} \cup_{\tilde{P}\in\tau}\{\tilde{M}(P,\sigma)\} &= \tilde{M}(\tilde{P}_1,\sigma) \cup \tilde{M}(\tilde{P}_2,\sigma) ... \cup \tilde{M}(\tilde{P}_k,\sigma) \\ &= \tilde{M}(\tilde{P}_1 \cup \tilde{P}_2 \cup ... \cup \tilde{P}_k, \sigma) \\ &= \tilde{M}(\cup_k \tilde{P}_k, \sigma) \\ &= \tilde{M}(\tilde{a}, \sigma) \\ &= \tilde{u}. \end{aligned}$$

□

Theorem 3.19. *For a fuzzy mathematical machine $\tilde{S} =< \tilde{S}, \Sigma, \tilde{M}, \tilde{a} >$, let $\tau = \{\tilde{P}\}$ be a fuzzy cover of $\tilde{a}$. If $\tilde{M}(\tilde{a}, i) = \tilde{u}$, where $\tilde{u} \in \tilde{S}$ then for any $i \in \Sigma$, $\cup_{\tilde{P}\in\tau}\{\tilde{M}(\tilde{P}, i)\} = \tilde{u}$.*

TABLE 4. $M(\sigma_0)$

σ_0	s_0	s_1	s_2
s_0	0.1	1.0	0.3
s_1	0.8	0.2	0.1
s_2	0.3	0.1	0.9

TABLE 5. $M(\sigma_1)$

σ_1	s_0	s_1	s_2
s_0	0.9	0.2	0.3
s_1	0.1	0.7	0.2
s_2	0.0	0.1	0.8

Example 3.20. Let $S =< S, \Sigma_k, M, a >, S = \{s_0, s_1, s_2\}, \Sigma_k = \{\sigma_0, \sigma_1\}, \tilde{a} = 0.1/s_0 + 0.8/s_1 + 0.2/s_2$, and $i = 0.2/\sigma_0 + 0.9/\sigma_1$. Let $\tau = \{\tilde{P}_1, \tilde{P}_2, \tilde{P}_3\}$ be a set of fuzzy covers of $\tilde{a}$ such that $\tilde{P}_1 = 0.0/s_0 + 0.8/s_1 + 0.1/s_2, \tilde{P}_2 = 0.1/s_0 + 0.7/s_1 + 0.2/s_2$, and $\tilde{P}_3 = 0.1/s_0 + 0.6/s_1 + 0.2/s_2$. Let us consider the transition matrices as given in Table 4 and Table 5. We have, $M(a, i) = 0.2/s_0 + 0.7/s_1 + 0.2/s_2$, $M(\tilde{P}_1, i) = 0.2/s_0 + 0.7/s_1 + 0.2/s_2$, $M(\tilde{P}_2, i) = 0.2/s_0 + 0.7/s_1 + 0.2/s_2$, $M(\tilde{P}_3, i) = 0.2/s_0 + 0.6/s_1 + 0.2/s_2$. It follows that, $M(a, i) = M(\tilde{P}_1, i) \cup M(\tilde{P}_2, i) \cup M(\tilde{P}_3, i)$.

Theorem 3.21. *For a fuzzy mathematical machine $\tilde{S} =< \tilde{S}, \Sigma, \tilde{M}, \tilde{a} >$, let $\tau = \{\tilde{P}\}$ be a fuzzy cover of $\tilde{a}$. If $\tilde{M}(\tilde{a}, x) = \tilde{u}$, where $\tilde{u} \in \tilde{S}$ then for any $x \in \Sigma^*$, $\cup_{\tilde{P}\in\tau}\{\tilde{M}(\tilde{P}, x)\} = \tilde{u}$.*

Observation 3.22. $\tilde{M} : \tilde{S} \times \Sigma \to \tilde{S}$ is a bijective mapping. Therefore, $\tilde{M}(\tilde{u}_1, \sigma) = \tilde{u}'_1$ and $\tilde{M}(\tilde{u}_2, \sigma) = \tilde{u}'_2 \Rightarrow \tilde{M}(\tilde{u}_1 \cup \tilde{u}_2, \sigma) = \tilde{u}'_1 \cup \tilde{u}'_2 = \tilde{M}(\tilde{u}_1, \sigma) \cup \tilde{M}(\tilde{u}_2, \sigma), \forall \tilde{u}, \tilde{u}' \in \tilde{S}$

4. Fuzzy mathematical machine with output

Definition 4.1. A fuzzy mathematical machine with output or a fuzzy automata is a six tuple $< S, \Sigma_k, \Delta_l, \tilde{M}, Z, a >$ where $< S, \Sigma_k, \tilde{M}, a >$ is a fuzzy mathematical machine without output,Δ_l is the set of output symbols $\{\delta_0, \delta_1, ..., \delta_{l-1}\}$,and Z : $S \times \Sigma_k \times \Delta \to [0, 1]$.

$Z(s, \sigma, \delta) = 1, \text{if } \sigma = \epsilon$
$0 \leq Z(s, \sigma, \delta) < 1$, otherwise, where $\sigma \in \Sigma_k$.

Definition 4.2. A fuzzy recognition device is a five tuple $< S, \Sigma_k, \tilde{M}, a, F >$ where $< S, \Sigma_k, \tilde{M}, a >$ is a fuzzy mathematical machine without output and $F \subseteq S$ called the set of final states.

Definition 4.3. $Z(s, \sigma_0\sigma_1, \delta) = \bigvee_{t\in S}\{\tilde{M}(s, \sigma_0, t) \wedge Z(t, \sigma_1, \delta)\}$, where $s \in S$ and $\sigma_0, \sigma_1 \in \Sigma_k$.

Definition 4.4. The domain of definition is extended from $S\times\Sigma_k\times\Delta \to [0, 1]$ to $S\times \Sigma\times\Delta \to [0, 1]$ for non-fuzzy inputs by $Z(s, i, \delta) = \bigvee_t\{\tilde{M}(s, \sigma_j, t)\wedge Z(t, \sigma_j, \delta)\}$, where $i = \sum \sigma_j$ and $j = 0, 1, ...n - 1$.

Definition 4.5. The domain of definition is extended from $S\times\Sigma_k\times\Delta \to [0, 1]$ to $S\times \Sigma \times \Delta \to [0, 1]$ for fuzzy inputs by $Z(s, i, \delta) = \bigvee_t\{\tilde{M}(s, \sigma_j, t) \wedge Z(t, \sigma_j, \delta) \wedge \mu_i(\sigma_j)\}$, where $i = \sum \mu_i(\sigma_j)/\sigma_j$ and $j = 0, 1, ...n - 1$.

Definition 4.6. The domain of definition is extended from $S\times\Sigma\times\Delta \to [0, 1]$ to $S\times \Sigma^* \times \Delta \to [0, 1]$ by $Z(s, x, \delta) = \bigvee_t\{\tilde{M}(s, i_j, t) \wedge Z(t, i_j, \delta)\},\ j = 0, 1, ...p - 1,\ x = i_0i_1...i_{p-1}$.

Theorem 4.7. $Z(s, x\sigma, \delta) = \bigvee_t\{\tilde{M}(s, \sigma, t)\wedge Z(t, x, \delta)\}$, $\sigma \in \Sigma_k,\ x \in \Sigma,\ and\ s, t \in S$.

Proof. Let $x = \epsilon$. Therefore $lg(x) = 0$, and so

$$\begin{aligned} Z(s, x\sigma, \delta) &= \bigvee_t\{\tilde{M}(s, \sigma, t) \wedge Z(t, \epsilon, \delta)\} \\ &= Z(s, \sigma, \delta) \end{aligned}$$

This completes the proof for the base case.
Let $lg(x) = n - 1,\ n > 2$.
Let $y = x\sigma',\ \sigma' \in \Sigma_k$.
Therefore,

$$\begin{aligned}
Z(s,\sigma y,\delta) &= Z(s,\sigma x\sigma') \\
&= \bigvee_r \{\tilde{M}(s,\sigma x,r) \wedge Z(r,\sigma',\delta)\} \\
&= \bigvee_r \{\bigvee_t \{\tilde{M}(s,\sigma,t) \wedge \tilde{M}(t,x,r)\} \wedge Z(r,\sigma',\delta)\} \\
&= \bigvee_r \{\bigvee_t \{\tilde{M}(s,\sigma,t) \wedge \tilde{M}(t,x,r) \wedge Z(r,\sigma',\delta)\} \\
&= \bigvee_t \{\bigvee_r \{\tilde{M}(s,\sigma,t) \wedge \tilde{M}(t,x,r) \wedge Z(r,\sigma',\delta)\} \\
&= \bigvee_t \{\tilde{M}(s,\sigma,t) \wedge \bigvee_r \{\tilde{M}(t,x,r) \wedge Z(r,\sigma',\delta)\}\} \\
&= \bigvee_t \{\tilde{M}(s,\sigma,t) \wedge Z(t,x\sigma',\delta)\} \\
&= \bigvee_t \{\tilde{M}(s,\sigma,t) \wedge Z(t,y,\delta)\}
\end{aligned}$$

□

Theorem 4.8. $\forall x, y \in \Sigma^*$, $\mu_Z(s,xy,\delta) = \bigvee_{u\in S} \{\mu_{\tilde{M}}(s,x,u) \wedge \mu_Z(u,y,\delta)\}$.

Definition 4.9. $Z(s,\sigma_0\sigma_1,\delta_0\delta_1) = \bigvee_{s'\in S} \{Z(s,\sigma_0,\delta_0) \wedge \tilde{M}(s,\sigma_0,s') \wedge Z(s',\sigma_1,\delta_1)\}$, where $s \in S$; $\sigma_0,\sigma_1 \in \Sigma_k$; and $\delta_0,\delta_1 \in \Delta_l$.

Theorem 4.10. *$Z(s,x,y) = 0$ if $lg(x) \neq lg(y)$, where $x = \sum \mu(\sigma_i)/\sigma_i$ and $y = \sum \mu(\delta_i)/\delta_i$.*

Proof. There are two cases , (i) $lg(x) > lg(y)$ and (ii) $lg(x) < lg(y)$.

Let us consider the first case, that is let $lg(x) > lg(y)$.
If $lg(x) = 0$, there is no y such that $lg(y) < 0$. So, the result is true.
If $lg(x) = 1$ then $lg(y) = 0$, that is $y = \lambda$. Now $Z(s,x,y) = 0$. So the result is true for the base case. Let the result be true for $lg(x) = n-1$,where $n > 2$.
Therefore, $lg(y) < n-1$. Let $x' = x\sigma$ and $y' = y\delta$, where $lg(x) = n-1$, $lg(\sigma) = 1$, $lg(y) < n-1$, $lg(\delta) = 1$.
Therefore $lg(x') = n$, $lg(y') < n$ and

$$\begin{aligned}
Z(s,x',y') = Z(s,x\sigma,y\delta) &= \bigvee_{s'\in S} \{Z(s,x,y) \wedge \tilde{M}(s,\sigma,s') \wedge Z(s',\sigma,\delta)\} \\
&= \bigvee_{s'\in S} \{0 \wedge \tilde{M}(s,\sigma,s') \wedge 0\} = 0.
\end{aligned}$$

Thus the result is true for $lg(x') = n$.
Similarly the result can be proved for the second case, that is for $lg(x) < lg(y)$
Clearly, for a given sequence of inputs $\sigma_0,\sigma_1,...,\sigma_{n-1}$, the fuzzy automata, rather the fuzzy mathematical machine will produce a sequence of outputs $\delta_0,\delta_1,...,\delta_{n-1}$. □

Remark 4.11. $0 \leq Z(s,x,y) < 1$, if $lg(x) = lg(y)$.

Let us generalise Definition 4.9. Let us consider n states $s_0,s_1,...s_{n-1}$ instead of a single state s. Then,

$$(4.1)\qquad Z(s_0,\sigma_0\sigma_1,\delta_0\delta_1)=\bigvee_{i=0}^{n-1}\{Z(s_0,\sigma_0,\delta_0)\wedge\tilde{M}(s_0,\sigma_0,s_i)\wedge Z(s_i,\sigma_1,\delta_1)\}$$

$$(4.2)\qquad Z(s_1,\sigma_0\sigma_1,\delta_0\delta_1)=\bigvee_{i=0}^{n-1}\{Z(s_1,\sigma_0,\delta_0)\wedge\tilde{M}(s_1,\sigma_0,s_i)\wedge Z(s_i,\sigma_1,\delta_1)\}$$

...

$$(4.3)\qquad Z(s_{n-1},\sigma_0\sigma_1,\delta_0\delta_1)=\bigvee_{i=0}^{n-1}\{Z(s_{n-1},\sigma_0,\delta_0)\wedge\tilde{M}(s_{n-1},\sigma_0,s_i)\wedge Z(s_i,\sigma_1,\delta_1)\}$$

In a compact form we can write the above equations as

(4.4)

$$Z(s_j,\sigma_0\sigma_1,\delta_0\delta_1)=\bigvee_{i=0}^{n-1}\{Z(s_j,\sigma_0,\delta_0)\wedge\tilde{M}(s_j,\sigma_0,s_i)\wedge Z(s_i,\sigma_1,\delta_1)\}, j=0,1,...n-1.$$

We have then the following definition

Definition 4.12. $Z(s_0s_1...s_{n-1},\sigma_0\sigma_1,\delta_0\delta_1)=\bigwedge_{j=0}^{n-1}[\bigvee_{i=0}^{n-1}\{Z(s_j,\sigma_0,\delta_0)\wedge\tilde{M}(s_j,\sigma_0,s_i)\wedge Z(s_i,\sigma_1,\delta_1)\}]$.

Let us consider a fuzzy set defined over the universe of discourse $\{s_0,s_1,...,s_{n-1}\}$. Let $\tilde{s}=\sum_i \mu(s_i)/s_i$. We have then the generalised definition as

Definition 4.13. $Z(\tilde{s},\sigma_0\sigma_1,\delta_0\delta_1)=\bigwedge_{j=0}^{n-1}[\bigvee_{i=0}^{n-1}\{(\mu_Z(s_j)\wedge Z(s_j,\sigma_0,\delta_0))\wedge(\mu_{\tilde{M}}(s_j)\wedge \tilde{M}(s_j,\sigma_0,s_i)\wedge\mu_{\tilde{M}}(s_i))\wedge(\mu_Z(s_i)\wedge Z(s_i,\sigma_1,\delta_1))\}]$.

Example 4.14. Let us consider the output function $Z(\tilde{s},\sigma_0\sigma_1,\delta_0\delta_1)$ where $\tilde{s}=0.1/s_0+0.8/s_1+0.3/s_2$ and the let the fuzzy functions $\tilde{M}$ and Z be represented respectively by the couple of transition matrices $\tilde{M}(\sigma_0)$, $\tilde{M}(\sigma_1)$, and $Z(\sigma_0), Z(\sigma_1)$ as given in the Table 6,Table 7,Table 8,and Table 9.

Table 6.

σ_0	s_0	s_1	s_2
s_0	0.0	0.1	0.8
s_1	0.9	0.2	0.3
s_2	0.3	0.7	0.1

We have then $Z(0.1/s_0+0.8/s_1+0.3/s_2,\sigma_0\sigma_1,\delta_0\delta_1)=0.1$.

Table 7.

σ_1	s_0	s_1	s_2
s_0	0.2	1.0	0.3
s_1	0.8	0.2	0.1
s_2	0.0	0.2	1.0

Table 8.

σ_0	δ_0	δ_1
s_0	0.1	0.8
s_1	0.2	0.3
s_2	0.8	0.2

Table 9.

σ_1	δ_0	δ_1
s_0	0.9	0.1
s_1	0.3	0.8
s_2	0.2	0.3

We now extend the number of inputs and the number of output symbols from 2 to n, keeping the single non-fuzzy state in the fuzzy output function Z. We have then the following definition.

Definition 4.15.

$$Z(s,\sigma_0\sigma_1...\sigma_{n-1},\delta_0\delta_1...\delta_{n-1}) = \bigvee_{s,s_0,...s_n\in S}\{Z(s,\sigma_0,\delta)) \wedge \tilde{M}(s,\sigma_0,s_1) \wedge Z(s_1,\sigma_1,\delta_1) \wedge \tilde{M}(s_1,\sigma_1,s_2) \wedge Z(s_2,\sigma_2,\delta_2) \wedge ... \wedge Z(s_n,\sigma_n,\delta_n).$$

Pictorially it can be represented as in the Figure 1.

Definition 4.16. $Z(s_0s_1...s_{n-1},\sigma_0\sigma_1...\sigma_{n-1},\delta_0\delta_1...\delta_{n-1}) = \bigwedge_{j=0}^{n-1}[\bigvee_{i=0}^{n-1}\{Z(s_j,\sigma_0,\delta_0)$ $\wedge_{k=0}^{n-2}\{\tilde{M}(s_k,\sigma_k,s_{k+1}) \wedge Z(s_{k+1},\sigma_{k+1},\delta_{k+1})\}\}]$.

We now try to develop the definition of a most general output function of a fuzzy automata. The states, inputs and outputs of this output functions are all fuzzy sets.

Definition 4.17.

$$Z(\tilde{s},\tilde{\sigma},\tilde{\delta}) = \bigwedge_{j=0}^{n-1}[\bigvee_{i=0}^{n-1}\{(\mu_Z(s_j) \wedge Z(s_j,\sigma_0,\delta_0)) \wedge_{k=0}^{n-2}\{(\mu_{\tilde{M}}(s_k) \wedge \tilde{M}(s_k,\sigma_k,s_{k+1}) \wedge \mu_{\tilde{M}}(s_{k+1})) \wedge (\mu_Z(s_{k+1}) \wedge Z(s_{k+1},\sigma_{k+1},\delta_{k+1}))\}\}],$$

where $\tilde{s} = \sum_{i=0}^{n-1}\mu(s_i)/s_i, \tilde{\sigma} = \sum_{i=0}^{n-1}\mu(\sigma_i)/\sigma_i,$ and $\tilde{\delta} = \sum_{i=0}^{n-1}\mu(\delta_i)/\delta_i$.

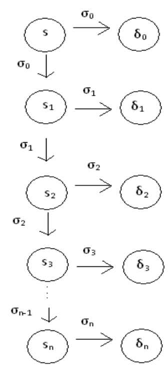

Figure 1. Output function with Multiple input and output symbols

5. Conclusion

In this paper we have studied the properties of fuzzy mathematical machines and fuzzy automata. The states, inputs and outputs are studied under the regular operations such as union and intersection. We have tried to move one step close towards the generalisation of the behavior of the fuzzy mathematical machine and fuzzy automata since more general theory helps to build up more flexible machines for the practical purposes. Undoubtedly, much more work remains in this context. We hope that the future works will extend the horizon of this field.

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