Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 5, (May 2015), pp. 703–718
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Szpilrajn theorem for intuitionistic fuzzy orderings

L. Zedam, A. Amroune, B. Davvaz

Received 24 April 2014; Revised 27 September 2014; Accepted 31 October 2014

Abstract. In this paper, we prove that any partial intuitionistic fuzzy ordering defined on an arbitrary non-empty set X can be linearized or can be extended to a linear (total) intuitionistic fuzzy ordering. This is an intuitionistic fuzzy generalization of the Szpilrajn theorem. This result has allowed us to characterize every partial intuitionistic fuzzy ordering by the intuitionistic fuzzy intersection of their linear extensions.

2010 AMS Classification: 03E72

Keywords: Partial intuitionistic fuzzy ordering, Linear intuitionistic fuzzy ordering, Szpilrjan's theorem.

Corresponding Author: B. Davvaz (davvaz@yazd.ac.ir)

1. Introduction

Crisp ordering, in many situation, can be made into a linear (total) ordering by adding some new necessary comparabilities to it. Szpilrajn's theorem [22] (see also [10, 19]) asserts that any partial ordering can be linearized (or is contained in a linear ordering): For any partial ordering $\leq$, there exists a linear ordering $\preceq$ which extends $\leq$ in the sense that, for all $x, y \in X$, $x \leq y \Rightarrow x \preceq y$. In 1971, Zadeh [26] introduced the concept of fuzzy ordering as a generalization of the concept of crisp ordering and generalized the Szpilrajn theorem for this concept. Other fuzzy generalizations or versions of this theorem for various definitions of linearity/completeness can be found in Bodenhofer and Klawonn [4]; Georgescu [11], Gottwald [12] and Höhle and Blanchard [14]. In this paper, we generalize the Szpilrajn theorem to intuitionistic fuzzy orderings. This result has allowed us to characterize every partial intuitionistic fuzzy ordering by the intuitionistic fuzzy intersection of their linear extensions.

This paper is organized as follows. In the second section, we recall some well know definitions and results, also we obtain some examples of intuitionistic fuzzy ordering. In the third section, we first give the key result of the present paper (see Lemma 3.1). By using this result and the crisp Zorn's Lemma, we give the Szpilrajn theorem for intuitionistic fuzzy partial orderings (see Theorem 3.2). This is an intuitionistic fuzzy generalization (or version) of the Szpilrajn theorem. We conclude by a theorem that characterize every partial intuitionistic fuzzy ordering by the intuitionistic fuzzy intersection of their linear extensions (see Theorem 4.1).

2. Preliminaries

Let X be a universe of discourse. Then, a fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ defined by Zadeh [25] is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$. In [1], Atanassov introduced another fuzzy object, called intuitionistic fuzzy subset (briefly IFS) as a generalization of the concept of fuzzy subset, shown as follows $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where the numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy subset A for each $x \in X$. Then, many researchers worked on this subject, for example see [13, 15, 16, 17, 18, 20, 27].

In fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard negation) and thus it is fixed. In intuitionistic fuzzy set theory, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) < 1 - \mu_A(x)$. Certainly fuzzy subsets are intuitionistic fuzzy subsets by setting $\nu_A(x) = 1 - \mu_A(x)$, but not conversely.

We know that an intuitionistic fuzzy relation (shortly IFR) from a universe X to a universe Y is an intuitionistic fuzzy subset in $X \times Y$, that is, is an expression R given by

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid x \in X, y \in Y\},$$

where $\mu_R : X \times Y \to [0,1]$ and $\nu_A : X \times Y \to [0,1]$ satisfy the condition

$$0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1, \quad \text{for every} \quad (x,y) \in X \times Y.$$

Next, we need the following definitions.

Let R and P be two IFRs from a universe X to a universe Y. R is said to be contained in P or we say that P contains R (notation $R \subseteq P$) if and only if for all $(x,y) \in X \times Y$: $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

The intersection and the union of two IFRs R and P from a universe X to a universe Y are defined as the IFR

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid x \in X, y \in Y\},$$
$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid x \in X, y \in Y\}.$$

In general, if A is a set of IFRs from a universe X to a universe Y, the intersection (resp. the union) is defined as

$$\bigcap_{R \in A} R = \{\langle (x,y), \inf_{R \in A} \mu_R(x,y), \sup_{R \in A} \nu_R(x,y)\rangle \ \mid \ x \in X, y \in Y\}$$

$$(\text{resp. } \bigcup_{R \in A} R = \{\langle (x,y), \sup_{R \in A} \mu_R(x,y), \inf_{R \in A} \nu_R(x,y)\rangle \mid x \in X, y \in Y\}).$$

More details and some important properties of the intuitionistic fuzzy relations are studied in [2, 3, 5, 6, 7, 8, 9, 21, 23, 24].

Definition 2.1 ([7]). Let X be a non-empty crisp set and

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\}$$

be an intuitionistic fuzzy relation on X. We will say that R is

(i) *reflexive*, if for every $x \in X$, $\mu_R(x, x) = 1$. Just notice that for every $x \in X$, $\nu_R(x, x) = 0$;
(ii) *antisymmetrical intuitionistic*, if for all $x, y \in X$, $x \neq y$, then $\mu_R(x, y) \neq \mu_R(y, x)$, $\nu_R(x, y) \neq \nu_R(y, x)$ and $\pi_R(x, y) = \pi_R(y, x)$, where $\pi_R(x, y) = 1 - \mu_R(x, y) - \nu_R(x, y)$;
(iii) *perfect antisymmetrical intuitionistic*, if for all $x, y \in X$ with $x \neq y$ and $\mu_R(x, y) > 0$ or ($\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1$), then $\mu_R(y, x) = 0$ and $\nu_R(y, x) = 1$;
(iv) *transitive*, if $R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R$;
(v) *intuitionistic fuzzy ordering*, if it is reflexive, transitive and antisymmetrical intuitionistic.

Notice that in [7], Bustince and Burillo mentioned that the definition of intuitionistic antisymmetry does not recover the fuzzy antisymmetry for the case in which the relation R considered is fuzzy. However, the definition of perfect antisymmetrical intuitionistic does recover the definition of fuzzy antisymmetry given by Zadeh [26] when the considered relation is fuzzy. This note justify the following definition of intuitionistic fuzzy ordering used in this paper.

Definition 2.2. Let X be a non-empty crisp set and

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\}$$

an intuitionistic fuzzy relation on X. Then, R is called an *intuitionistic fuzzy ordering* or a *partial intuitionistic fuzzy ordering* if it is reflexive, transitive and perfect antisymmetrical intuitionistic.

A non-empty set X with a partial intuitionistic fuzzy ordering R defined on it is called partially intuitionistic fuzzy ordered set (for short, IF-poset) and we denote it by (X, μ_R, ν_R).

Notice that any partially ordered set $(X, \leq)$ and generally any fuzzy ordered set (X, r) can be regarded as partially intuitionistic fuzzy ordered sets.

The composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ in the above definition of transitivity means that

$$\{\langle (x, z), \alpha_{y\in X}\{\beta[\mu_R(x, y), \mu_R(y, z)], \lambda_{y\in X}\{\rho[\nu_R(x, y), \nu_R(y, z)]\rangle \mid x, z \in X\},$$

where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition:

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x, y), \mu_R(y, z)] + \lambda_{y\in X}\{\rho[\nu_R(x, y), \nu_R(y, z)] \leq 1,$$

for every $(x, z) \in X^2$. The properties of this composition and the choice of α, β, λ and ρ for which this composition fulfills a maximal number of properties are investigated in [6]. If no other conditions are imposed, in the sequel we will take $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ and $\rho = \max$.

Example 2.3. Let $X = \{a, b, c, d, e\}$. Then, the intuitionistic fuzzy subset R defined on $X \times X$ by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\},$$

where μ_R and ν_R given by the following tables:

$\mu_R(\cdot,\cdot)$	a	b	c	d	e
a	1	0	0	0.55	0.40
b	0	1	0	0.35	0.45
c	0	0	1	0	0.70
d	0	0	0	1	0
e	0	0	0	0	1

$\nu_R(\cdot,\cdot)$	a	b	c	d	e
a	0	1	0.40	0.45	0.25
b	0.30	0	0.20	0.35	0.10
c	1	1	0	0.85	0.15
d	1	1	1	0	1
e	1	1	1	0.90	0

is an intuitionistic fuzzy ordering on X.

Example 2.4. Let $m, n \in \mathbb{N}$. Then, the intuitionistic fuzzy relation R defined by

$$\mu_R(m,n) = \begin{cases} 1 & \text{if } m = n \\ 1 - \frac{m}{n} & \text{if } m < n \\ 0 & \text{if } m > n \end{cases} \text{ and } \nu_R(m,n) = \begin{cases} 0 & \text{if } m = n \\ \frac{m}{2n} & \text{if } m < n \\ 1 & \text{if } m > n, \end{cases}$$

for all $m, n \in \mathbb{N}$, is an intuitionistic fuzzy ordering on $\mathbb{N}$.

Based on the definition of perfect antisymmetrical intuitionistic we define linear or total intuitionistic fuzzy ordering as follows:

Definition 2.5. An intuitionistic fuzzy ordering R is *linear* (or *total*) on X if for every $x, y \in X$, we have $[\mu_R(x, y) > 0$ and $\nu_R(x, y) = 0]$ or $[\mu_R(y, x) > 0$ and $\nu_R(y, x) = 0]$.

An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a *linearly intuitionistic fuzzy ordered set* or an *intuitionistic fuzzy chain.*

Conversely, we obtain the following definition of incomparable elements.

Definition 2.6. Let (X, μ_R, ν_R) be a non-empty intuitionistic fuzzy ordered set and let a, b be two elements of X. We say that a and b are *incomparable* in (X, μ_R, ν_R) if

$$[\mu_R(a, b) = 0 \text{ or } \nu_R(a, b) > 0] \text{ and } [\mu_R(b, a) = 0 \text{ or } \nu_R(b, a) > 0].$$

Definition 2.7. Let X be a non-empty set and R, P be two intuitionistic fuzzy orderings on X. P is called a *linear extension* of R if P is linear and contains R.

3. Szpilrajn theorem for intuitionistic fuzzy orderings

In this section, we shall show that any partial intuitionistic fuzzy ordering R on a non-empty set X can be extended to a linear intuitionistic fuzzy ordering. First, we need to establish the following lemma.

Lemma 3.1. *Let (X, μ_R, ν_R) be a non-empty intuitionistic fuzzy ordered set and let a, b be two incomparable elements in (X, μ_R, ν_R). Then, there exists at least an intuitionistic fuzzy ordering P on X containing R such that a, b are comparable elements in (X, μ_P, ν_P), (i.e., either*

$$[\mu_P(a, b) > 0 \text{ and } \nu_P(a, b) = 0] \text{ or } [\mu_P(b, a) > 0 \text{ and } \nu_P(b, a) = 0]).$$

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and let $a, b \in X$ be two incomparable elements, i.e.,

$$[\mu_R(a, b) = 0 \text{ or } \nu_R(a, b) > 0] \text{ and } [\mu_R(b, a) = 0 \text{ or } \nu_R(b, a) > 0].$$

Let P be the fuzzy relation defined on X by setting

$$P = \{\langle (x, y), \mu_P(x, y), \nu_P(x, y)\rangle \mid x, y \in X\},$$

where

$$\mu_P(x, y) = \max\{\mu_R(x, y), \min(\mu_R(x, a), \mu_R(b, y))\}$$

and

$$\nu_P(x, y) = \min\{\nu_R(x, y), \max(\nu_R(x, a), \nu_R(b, y))\}.$$

Let us verify that P is an intuitionistic fuzzy ordering on X containing R such that either $[\mu_P(a, b) > 0$ and $\nu_P(a, b) = 0]$ or $[\mu_P(b, a) > 0$ and $\nu_P(b, a) = 0]$.

(i) The intuitionistic fuzzy relation P is reflexive. Indeed, for all $x \in X$, we have

$$\mu_P(x, x) = \max\{\mu_R(x, x), \min(\mu_R(x, a), \mu_R(b, x))\} = \mu_R(x, x) = 1.$$

Thus, P is reflexive.

(ii) The intuitionistic fuzzy relation P is perfect antisymmetrical intuitionistic. Indeed, let $x, y \in X$ such that $x \neq y$. We shall show that If $\mu_P(x, y) > 0$ or $[\mu_P(x, y) = 0$ and $\nu_P(x, y) < 1]$, then $\mu_P(y, x) = 0$ and $\nu_P(y, x) = 1$. We can distinguish two cases:

First case. Assume that $\mu_P(x, y) > 0$. Since

$$\mu_P(x, y) = \max\{\mu_R(x, y), \min(\mu_R(x, a), \mu_R(b, y))\},$$

we distinguish also the following two subcases:

(a) $\mu_P(x, y) = \mu_R(x, y) > 0$;
(b) $\mu_P(x, y) = \min(\mu_R(x, a), \mu_R(b, y))\} > 0$.

(a) If $\mu_P(x, y) = \mu_R(x, y) > 0$, then it follows from the perfect antisymmetrical intuitionistic of R that $\mu_R(y, x) = 0$ and $\nu_R(y, x) = 1$. Hence,

$$\begin{aligned}\mu_P(y, x) &= \max\{\mu_R(y, x), \min(\mu_R(y, a), \mu_R(b, x))\}\\ &= \min\{\mu_R(y, a), \mu_R(b, x)\}.\end{aligned}$$

On the other hand, the transitivity of R implies that

$$\mu_R(b, a) \geq \min(\mu_R(b, y), \mu_R(y, a)).$$

Since $\mu_R(b, a) = 0$, it follows that it holds that $\mu_R(y, a)) = 0$ or $\mu_R(b, y) = 0$.

(a1) If $\mu_R(y, a)) = 0$, then it follows that $\mu_P(y, x) = \min\{\mu_R(y, a), \mu_R(b, x)\} = 0$.

(a2) If $\mu_R(b, y) = 0$, then it follows from the transitivity of R that $\mu_R(b, y) \geq \min(\mu_R(b, x), \mu_R(x, y))$. Since $\mu_R(b, y) = 0$ and $\mu_R(x, y) > 0$, it follows that it holds that $\mu_R(b, x) = 0$. Thus,

$$\mu_P(y, x) = \min\{\mu_R(y, a), \mu_R(b, x)\} = 0.$$

In the same way, we show that

$$\nu_P(y, x) = \min\{\nu_R(y, x), \max(\nu_R(y, a), \nu_R(b, x))\} = 1.$$

Indeed, since $\nu_R(y,x)=1$, we have

$$\begin{aligned}\nu_P(y,x) &= \min\{1, \max(\nu_R(y,a), \nu_R(b,x))\}\\ &= \max(\nu_R(y,a), \nu_R(b,x)).\end{aligned}$$

On the other hand, the transitivity of R implies that

$$\nu_R(y,x) \le \max(\nu_R(y,a), \nu_R(a,x)).$$

Since $\nu_R(y,x)=1$, then it holds that $\nu_R(y,a)=1$ or $\nu_R(a,x)=1$. Thus, $\nu_P(y,x) = \max(\nu_R(y,a), \nu_R(b,x)) = 1$. Therefore,

$$\mu_P(y,x) = 0 \text{ and } \nu_P(y,x) = 1.$$

(b) If $\mu_P(x,y) = \min(\mu_R(x,a), \mu_R(b,y)) > 0$, then it holds that $\mu_R(x,a) > 0$ and $\mu_R(b,y) > 0$. We have also by the transitivity of R that

$$\mu_R(b,a) \ge \min(\mu_R(b,x), \mu_R(x,a)).$$

Since $\mu_R(b,a) = 0$ and $\mu_R(x,a) > 0$, then it holds that $\mu_R(b,x) = 0$. This implies that

$$\mu_P(y,x) = \min\{\mu_R(y,a), \mu_R(b,x)\} = 0.$$

In the same way, we get that

$$\nu_P(y,x) = \min\{\nu_R(y,x), \max(\nu_R(y,a), \nu_R(b,x))\} = 1.$$

Therefore,

$$\mu_P(y,x) = 0 \text{ and } \nu_P(y,x) = 1.$$

Second case. Assume that $[\mu_P(x,y) = 0 \text{ and } \nu_P(x,y) < 1]$. Since

$$\mu_P(x,y) = \max\{\mu_R(x,y), \min(\mu_R(x,a), \mu_R(b,y))\}$$

and

$$\nu_P(x,y) = \min\{\nu_R(x,y), \max(\nu_R(x,a), \nu_R(b,y))\},$$

so we get

$$[\mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1]$$

or

$$[\mu_R(x,y) = 0 \text{ and } \max(\nu_R(x,a), \nu_R(b,y))\}].$$

Now the perfect antisymmetrical intuitionistic of R implies that $[\mu_R(y,x) = 0$ and $\nu_R(y,x) = 1]$. Hence,

$$\begin{aligned}\mu_P(y,x) &= \max\{\mu_R(y,x), \min(\mu_R(y,a), \mu_R(b,x))\}\\ &= \min\{\mu_R(y,a), \mu_R(b,x)\},\end{aligned}$$

and

$$\begin{aligned}\nu_P(y,x) &= \min\{\nu_R(y,x), \max(\nu_R(y,a), \nu_R(b,x))\}\\ &= \max(\nu_R(y,a), \nu_R(b,x))\}.\end{aligned}$$

Now since we have the same data as the first case above, then we take the same subcases and we apply the same proof. Therefore,

$$\mu_P(y,x) = 0 \text{ and } \nu_P(y,x) = 1.$$

(iii) The intuitionistic fuzzy relation P is transitive. Indeed, suppose that $x, y, z \in X$. We shall show that

$$\mu_P(x,z) \ge \sup_{y\in X}[\min\{\mu_P(x,y), \mu_P(y,z)\}],$$

and

$$\nu_P(x,z) \leq \inf_{y\in X}[\max\{\nu_P(x,y), \nu_P(y,z)\}].$$

For the first condition

$$\mu_P(x,z) \geq \sup_{y\in X}[\min\{\mu_P(x,y), \mu_P(y,z)\}],$$

we can distinguish two cases:

First case. If $\mu_P(x,y) = \mu_R(x,y)$ and $\mu_P(y,z) = \mu_R(y,z)$, then

$$\mu_P(x,y) \wedge \mu_P(y,z) = \mu_R(x,y) \wedge \mu_R(y,z).$$

By the transitivity of R we have $\mu_R(x,z) \geq \mu_R(x,y) \wedge \mu_R(y,z)$. Hence, $\mu_R(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z)$. On the other hand, since $\mu_P = \mu_R(x,z) \vee (\mu_R(x,a) \wedge \mu_R(b,z))$, then it holds that $\mu_P(x,z) \geq \mu_R(x,z)$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

Second case. If $\mu_P(x,y) = \mu_R(x,y)$ and $\mu_P(y,z) = \mu_R(y,a) \wedge \mu_R(b,z)$, then $\mu_P(x,y) \wedge \mu_P(y,z) = \mu_R(x,y) \wedge \mu_R(y,a) \wedge \mu_R(b,z)$. On the other hand, as

$$\mu_P(x,z) = \mu_R(x,z) \vee (\mu_R(x,a) \wedge \mu_R(b,z))$$

so $\mu_P(x,z) \geq \mu_R(x,a) \wedge \mu_R(b,z)$. By the transitivity of R we have

$$\mu_R(x,a) \geq \mu_R(x,y) \wedge \mu_R(y,a).$$

Hence, $\mu_P(x,z) \geq \mu_R(x,y) \wedge \mu_R(y,a) \wedge \mu_R(b,z)$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

Third case. If $\mu_P(x,y) = \mu_R(x,a) \wedge \mu_R(b,y)$ and $\mu_P(y,z) = \mu_R(y,z)$, then we get that

$$\mu_P(x,y) \wedge \mu_P(y,z) = \mu_R(x,a) \wedge \mu_R(b,y) \wedge \mu_R(y,z).$$

On the other hand, as $\mu_P(x,z) = \mu_R(x,z) \vee (\mu_R(x,a) \wedge \mu_R(b,z))$ so

$$\mu_P(x,z) \geq \mu_R(x,a) \wedge \mu_R(b,z).$$

Also, by the transitivity of R we have $\mu_R(b,z) \geq \mu_R(b,y) \wedge \mu_R(y,z)$. Hence, $\mu_P(x,z) \geq \mu_R(x,a) \wedge \mu_R(b,y) \wedge \mu_R(y,z)$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

Fourth case. If $\mu_P(x,y) = \mu_R(x,a) \wedge \mu_R(b,y)$ and $\mu_P(y,z) = \mu_R(y,a) \wedge \mu_R(b,z)$, then it follows that

$$\mu_P(x,y) \wedge \mu_P(y,z) = \mu_R(x,a) \wedge \mu_R(b,y) \wedge \mu_R(y,a) \wedge \mu_R(b,z).$$

We have by the transitivity of R that $\mu_R(b,a) \geq \mu_R(b,y) \wedge \mu_R(y,a)$. Since $\mu_R(b,a) = 0$ then it holds that $\mu_R(b,y) \wedge \mu_R(y,a) = 0$. Hence, $\mu_P(x,y) \wedge \mu_P(y,z) = 0$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

The same proof by cases has to be done in a completely analogous way to show the second condition of transitivity:

$$\nu_P(x,z) \leq \inf_{y\in X}[\max\{\nu_P(x,y), \nu_P(y,z)\}].$$

Consequently, P is transitive. Therefore, P is an intuitionistic fuzzy ordering on X.

(iv) P is an extension of R. Indeed, for all $(x, y) \in X^2$, we have

$$\mu_P(x, y) = \max\{\mu_R(x, y), \min(\mu_R(x, a), \mu_R(b, y))\} \geq \mu_R(x, y),$$

and

$$\nu_P(x, y) = \nu_R(x, y) \leq \nu_R(x, y).$$

Hence, P is an extension of R on X.

(v) Either ($\mu_P(a, b) > 0$ and $\nu_P(a, b) = 0$) or ($\mu_P(b, a) > 0$ and $\nu_P(b, a) = 0$). Indeed, we have

$$\mu_P(a, b) = \max\{\mu_R(a, b), \min(\mu_R(a, a), \mu_R(b, b))\} = \max\{\mu_R(a, b), 1\} = 1 > 0,$$

and

$$\nu_P(a, b) = \min\{\nu_R(a, b), \max(\nu_R(a, a), \nu_R(b, b))\} = \min\{\nu_R(a, b), 0\} = 0.$$

In the same way, we take

$$\mu_P(x, y) = \max\{\mu_R(x, y), \min(\mu_R(x, b), \mu_R(a, y))\},$$

and

$$\nu_P(x, y) = \min\{\nu_R(x, y), \max(\nu_R(x, b), \nu_R(a, y))\}.$$

Then, we get that

$$\mu_P(b, a) = \max\{\mu_R(b, a), \min(\mu_R(b, b), \mu_R(a, a))\} = 1 > 0,$$

and

$$\nu_P(b, a) = \min\{\nu_R(b, a), \max(\nu_R(b, b), \nu_R(a, a))\} = 0.$$

□

Theorem 3.2 (Szpilrajn theorem for intuitionistic fuzzy orderings)**.** *Let X be a non-empty set. Then, any partial intuitionistic fuzzy ordering on X can be extended to a linear intuitionistic fuzzy ordering.*

Proof. Let X be a non-empty set and let R be an intuitionistic fuzzy ordering on X. Let Σ_R be the set of all intuitionistic fuzzy orderings on X containing R. We know that Σ_R is a non-empty crisp ordered set under intuitionistic fuzzy set inclusion and R its smallest element.

We claim that Σ_R has at least a maximal element and this maximal element is linear.

Claim 1. Every non-empty chain in Σ_R has an upper bound in Σ_R. Indeed, let C be a non-empty chain in Σ_R and let R_C be the intuitionistic fuzzy relation on X defined for all $x, y \in X$ by $R_C = \bigcup_{S \in C} S$, i.e.,

$$\mu_{R_C}(x, y) = \sup_{S \in C} \mu_S(x, y)$$

and

$$\nu_{R_C}(x, y) = \inf_{S \in C} \nu_S(x, y).$$

We have

$$\mu_{R_C}(x, y) + \nu_{R_C}(x, y) = \sup_{S \in C} \mu_S(x, y) + \inf_{S \in C} \nu_S(x, y) \leq 1.$$

Now let us show that R_C is an upper bound of C in Σ_R. In view of the definition of R_C, it suffices to show that $R_C \in \Sigma_R$. That is R_C is an intuitionistic fuzzy ordering on X containing R.

(a) The intuitionistic fuzzy relation R_C is reflexive. Indeed, for all $x \in X$, we have

$$\mu_{R_C}(x,x) = \sup_{S\in C} \mu_S(x,x) = 1.$$

Thus, R_C is reflexive.

(b) The intuitionistic fuzzy relation R_C is perfect antisymmetrical intuitionistic. Indeed, let $x, y \in X$ such that $x \neq y$. We can distinguish the following two cases:

(b1) $\mu_{R_C}(x,y) > 0$;

(b2) $\mu_{R_C}(x,y) = 0$ and $\nu_{R_C}(x,y) < 1$.

(b1) If $\mu_{R_C}(x,y) > 0$, then there exists $S_0 \in C$ such that $\mu_{S_0}(x,y) > 0$. Since S_0 is perfect antisymmetrical intuitionistic, it follows that $\mu_{S_0}(y,x) = 0$ and $\nu_{S_0}(y,x) = 1$.

Also we get, $\mu_S(x,y) > 0$ for all $S \in C$ satisfying $S_0 \subseteq S$. Hence, the perfect antisymmetrical intuitionistic of S implies that

$$\mu_S(y,x) = 0 \text{ and } \nu_S(y,x) = 1, \text{ for all } S \in C \text{ satisfying } S_0 \subseteq S.$$

On the other hand, since $\mu_{S_0}(y,x) = 0$ and $\nu_{S_0}(y,x) = 1$, we get that

$$\mu_S(y,x) = 0 \text{ and } \nu_S(y,x) = 1, \text{ for all } S \in C \text{ satisfying } S \subseteq S_0.$$

Combining the foregoing, we obtain $\mu_S(y,x) = 0$ and $\nu_S(y,x) = 1$ for all $S \in C$. Thus,

$$\mu_{R_C}(y,x) = \sup_{S\in C} \mu_S(y,x) = 0$$

and

$$\nu_{R_C}(y,x) = \inf_{S\in C} \nu_S(y,x) = 1 \quad .$$

(b2) If $\mu_{R_C}(x,y) = 0$ and $\nu_{R_C}(x,y) < 1$, then $\mu_S(x,y) = 0$ for all $S \in C$ and there exists $S_0 \in C$ such that $\nu_{S_0}(x,y) < 1$.

Also, we have $\nu_S(x,y) < 1$ for all $S \in C$ satisfying $S_0 \subseteq S$. Since $\mu_S(x,y) = 0$ and $\nu_S(x,y) < 1$ for all $S \in C$ satisfying $S_0 \subseteq S$, then by the perfect antisymmetrical intuitionistic of S we obtain

$$\mu_S(y,x) = 0 \text{ and } \nu_S(y,x) = 1, \text{ for all } S \in C \text{ satisfying } S_0 \subseteq S.$$

On the other hand, since $S_0 \subseteq S_0$, it follows that $\nu_{S_0}(y,x) = 1$. This implies that $\nu_S(y,x) = 1$ for all $S \in C$ satisfying $S \subseteq S_0$.

Combining the foregoing, we obtain $\mu_S(y,x) = 0$ and $\nu_S(y,x) = 1$ for all $S \in C$.

Thus,

$$\mu_{R_C}(y,x) = \sup_{S\in C} \mu_S(y,x) = 0$$

and

$$\nu_{R_C}(y,x) = \inf_{S\in C} \nu_S(y,x) = 1. \quad .$$

Consequently, R is perfect antisymmetrical intuitionistic.

(c) The intuitionistic fuzzy relation R_C is transitive. Indeed, from the definition of transitivity we need to show that for all $x, z \in X$,

$$\mu_{R_C}(x,z) \geq \sup_{y\in X} [\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\}],$$

and

$$\nu_{R_C}(x,z) \leq \inf_{y\in X} [\max\{\nu_{R_C}(x,y), \nu_{R_C}(y,z)\}].$$

Let us show the first condition

$$\mu_{R_C}(x,z) \geq \sup_{y\in X}\left[\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\}\right].$$

Let $x, y, z \in X$, we have

$$\mu_{R_C}(x,y) = \sup_{S\in C} \mu_S(x,y);$$
$$\mu_{R_C}(y,z) = \sup_{S\in C} \mu_S(y,z);$$

and

$$\mu_{R_C}(x,z) = \sup_{S\in C} \mu_S(x,z).$$

Let $\varepsilon > 0$ be given. Then, there exist $S_0, S_1 \in C$ such that

$$\mu_{R_C}(x,y) - \varepsilon < \mu_{S_0}(x,y) \leq \mu_{R_C}(x,y)$$

and

$$\mu_{R_C}(y,z) - \varepsilon < \mu_{S_1}(y,z) \leq \mu_{R_C}(y,z).$$

As C is a chain and $S_0, S_1 \in C$, we have $S_1 \subseteq S_0$ or $S_0 \subseteq S_1$.

First case. Assume that $S_1 \subseteq S_0$. Then, it follows that

$$\mu_{R_C}(x,y) < \mu_{S_0}(x,y) + \varepsilon$$

and

$$\mu_{R_C}(y,z) < \mu_{S_1}(y,z) + \varepsilon \leq \mu_{S_0}(y,z) + \varepsilon.$$

This implies that

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \min\{\mu_{S_0}(x,y) + \varepsilon, \mu_{S_0}(y,z) + \varepsilon\}.$$

Hence, we obtain

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \min\{\mu_{S_0}(x,y), \mu_{S_0}(y,z)\} + \varepsilon.$$

On the other hand, we know that S_0 is transitive. Then, it holds that

$$\min\{\mu_{S_0}(x,y), \mu_{S_0}(y,z)\} \leq \mu_{S_0}(x,z).$$

Therefore, we deduce that

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \mu_{S_0}(x,z) + \varepsilon.$$

Now since $\mu_{R_C}(x,z) = \sup_{S\in C} \mu_S(x,z)$, we get $\mu_{S_0}(x,z) \leq \mu_{R_C}(x,z)$. So, we obtain

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \mu_{R_C}(x,z) + \varepsilon, \text{ for every } \varepsilon > 0.$$

Thus, $\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} \leq \mu_{R_C}(x,z)$, for every $y \in X$.

Second case. Assume that $S_0 \subseteq S_1$. Then, we have

$$\mu_{R_C}(x,y) < \mu_{S_0}(x,y) + \varepsilon \leq \mu_{S_1}(x,y) + \varepsilon.$$

and

$$\mu_{R_C}(y,z) < \mu_{S_1}(y,z) + \varepsilon.$$

This implies that

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \min\{\mu_{S_1}(x,y) + \varepsilon, \mu_{S_1}(y,z) + \varepsilon\}.$$

Hence, we have

$$\min\{\mu_{R_C}(x,y), \mu_{R_C}(y,z)\} < \min\{\mu_{S_1}(x,y), \mu_{S_1}(y,z)\} + \varepsilon.$$

On the other hand, we know that S_1 is transitive, so it holds that

$$\min\{\mu_{S_1}(x,y),\mu_{S_1}(y,z)\} \leq \mu_{S_1}(x,z).$$

Therefore, we deduce that

$$\min\{\mu_{R_C}(x,y),\mu_{R_C}(y,z)\} < \mu_{S_1}(x,z)+\varepsilon.$$

Now, since $\mu_{R_C}(x,z)=\sup\limits_{S\in C}\mu_S(x,z)$, we get $\mu_{S_1}(x,z)\leq \mu_{R_C}(x,z)$. Thus, we obtain

$$\min\{\mu_{R_C}(x,y),\mu_{R_C}(y,z)\} < \mu_{R_C}(x,z)+\varepsilon, \text{for every} \varepsilon > 0.$$

Thus, $\min\{\mu_{R_C}(x,y),\mu_{R_C}(y,z)\} \leq \mu_{R_C}(x,z)$, for every $y\in X$.

The same proof by cases has to be done in a completely analogous way to show the second condition of transitivity

$$\nu_{R_C}(x,z) \leq \inf_{y\in X}\left[\max\{\nu_{R_C}(x,y),\nu_{R_C}(y,z)\}\right].$$

Thus, R_C is intuitionistic fuzzy transitive. Therefore, R_C is an intuitionistic fuzzy ordering on X.

Finally, we show that R_C containing R. Since $R \leq S$ for all $S \in C$, it holds that $R \leq \bigcup\limits_{S\in C} S = R_C$. Therefore, $R_C \in \Sigma_R$.

Claim 2. Σ_R has at least a maximal element. Indeed, from Claim 1, every non-empty chain in Σ_R has an upper bound in Σ_R. Then, Zorn's Lemma implies that Σ_R has at least a maximal element, M, say.

Claim 3. The maximal element M is linear. Indeed, by absurd assume that M is not linear. This implies that there exists two incomparable elements a,b in (X,μ_M,ν_M). Then, by Lemma 3.1, there exist an intuitionistic fuzzy ordering M' on X which extend M and satisfies that either

$$[\mu_{M'}(a,b)>0 \text{ and } \nu_{M'}(a,b)=0] \text{ or } [\mu_{M'}(b,a)>0 \text{ and } \nu_{M'}(b,a)=0].$$

Since $R\subseteq M \subseteq M'$, so $M' \in \Sigma_R$. As M is a maximal element in Σ_R, so we get $M = M'$. That is a contradiction with the fact that

$$[\mu_M(a,b)=0 \text{ or } \nu_M(a,b)>0] \text{ and } [\mu_M(b,a)=0 \text{ or } \nu_M(b,a)>0].$$

Therefore, M is a linear intuitionistic fuzzy ordering on X which extends R. □

Remark 3.3. If X is finite non-empty set, then Theorem 3.2 can be followed by a finite number of applications of Lemma 3.1 (constructive answer).

4. Characterization of intuitionistic fuzzy orderings by theirs linear extensions

The following theorem characterizes any partial intuitionistic fuzzy ordering by the intuitionistic fuzzy intersection of their linear extensions.

Theorem 4.1. *Any partial intuitionistic fuzzy ordering on a non-empty set X is the intuitionistic fuzzy intersection of their linear extensions.*

In order to prove Theorem 4.1, we need to show the following lemma.

Lemma 4.2. *Let (X, μ_R, ν_R) be a non-empty IF-poset and let a, b be elements of X such that $[\mu_R(a, b) > 0$ and $\nu_R(a, b) = 0]$. Then, there exists at least a linear intuitionistic fuzzy ordering P on X containing R such that $[\mu_P(a, b) = \mu_R(a, b)$ and $\nu_P(a, b) = \nu_R(a, b)]$.*

Proof. Let (X, μ_R, ν_R) be a non-empty IF-poset and let a, b be elements of X such that $[\mu_R(a, b) > 0$ and $\nu_R(a, b) = 0]$. We can distinguish two cases:

First case. If R is linear, then we take $P = R$ which is linear intuitionistic fuzzy ordering on X containing R and satisfies $[\mu_P(a, b) = \mu_R(a, b)$ and $\nu_P(a, b) = \nu_R(a, b)]$. Hence, Lemma 4.2 is proved.

Second case. If R is not linear, then it follows from Theorem 3.1 that there exists a linear intuitionistic fuzzy ordering S on X containing R, i.e., $[\mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)]$ for all $x, y \in X$. This implies that $[\mu_R(a, b) \leq \mu_S(a, b)$ and $\nu_R(a, b) \geq \nu_S(a, b)]$. Since $\nu_R(a, b) = 0$, then it holds that $\nu_R(a, b) = \nu_S(a, b)$. We can distinguish the following two subcases:

(i) $\mu_R(a, b) = \mu_S(a, b)$;
(ii) $\mu_R(a, b) < \mu_S(a, b)$.

(i) If $\mu_R(a, b) = \mu_S(a, b)$, then we take $P = S$. Hence, Lemma 4.2 is proved.

(ii) If $\mu_R(a, b) < \mu_S(a, b)$, then we set $0 < \beta = \mu_R(a, b) < 1$ and we define the intuitionistic fuzzy relation P on X as

$$\mu_P(x, y) = \begin{cases} \mu_S(x, y) & \text{if } \mu_R(x, y) > \beta \\ \beta \wedge \mu_S(x, y) & \text{if } \mu_R(x, y) \leq \beta \end{cases}$$

and

$$\nu_P(x, y) = \nu_S(x, y).$$

It is clear that P containing R. In addition, since $\mu_R(a, b) = \beta \leq \beta$, then it holds that

$$\mu_P(a, b) = \beta \wedge \mu_S(a, b) = \mu_R(a, b) \wedge \mu_S(a, b) = \mu_R(a, b)$$

and

$$\nu_P(a, b) = \nu_S(a, b) = \nu_R(a, b).$$

Now to complete the proof we need to show that the intuitionistic fuzzy relation P defined on X as:

$$\mu_P(x, y) = \begin{cases} \mu_S(x, y) & \text{if } \mu_R(x, y) > \beta \\ \beta \wedge \mu_S(x, y) & \text{if } \mu_R(x, y) \leq \beta \end{cases}$$

and

$$\nu_P(x, y) = \nu_S(x, y)$$

is a linear intuitionistic fuzzy ordering.

(1) P is reflexive. Indeed, since $\mu_R(x, x) = 1 > \beta$ for all $x \in X$, then it holds that $\mu_P(x, x) = \mu_S(x, x) = 1$. Thus, P is reflexive.

(2) P is perfect antisymmetrical intuitionistic. Indeed, let $x, y \in X$ such that $x \neq y$. We can distinguish the following two cases:

(i) $\mu_P(x, y) > 0$;
(ii) $\mu_P(x, y) = 0$ and $\nu_P(x, y) < 1$.

(i) If $\mu_P(x,y) > 0$, then it follows that $\mu_S(x,y) > 0$. Since S is a perfect antisymmetrical intuitionistic, it holds that $\mu_S(y,x) = 0$ and $\nu_S(y,x) = 1$. This implies that

$$\begin{aligned}\mu_P(y,x) &= \begin{cases} \mu_S(y,x) & \text{if } \mu_R(y,x) > \beta \\ \beta \wedge \mu_S(y,x) & \text{if } \mu_R(y,x) \leq \beta \end{cases} \\ &= \begin{cases} 0 & \text{if } \mu_R(y,x) > \beta \\ \beta \wedge 0 & \text{if } \mu_R(y,x) \leq \beta \end{cases} \\ &= 0,\end{aligned}$$

and

$$\nu_P(y,x) = \nu_S(y,x) = 1.$$

Thus, P is perfect antisymmetrical intuitionistic.

(ii) If $\mu_P(x,y) = 0$ and $\nu_P(x,y) < 1$, then from the fact that $\beta > 0$ we deduce that $\mu_S(x,y) = 0$ and $\nu_S(x,y) = \nu_P(x,y) < 1$. Since S is a perfect antisymmetrical intuitionistic, then it holds that $\mu_S(y,x) = 0$ and $\nu_S(y,x) = 1$. This implies also that $\mu_P(y,x) = 0$ and $\nu_P(y,x) = 1$. Thus, P is perfect antisymmetrical intuitionistic.

(3) P is transitive. Indeed, Let $x,y,z \in X$, we have four cases to study.

First case. If $\mu_P(x,y) = \mu_S(x,y)$ and $\mu_P(y,z) = \mu_S(y,z)$, then $\mu_R(x,y) > \beta$ and $\mu_R(y,z) > \beta$. Since R is transitive, we have $\mu_R(x,z) \geq \mu_R(x,y) \wedge \mu_R(y,z)$. This implies that $\mu_R(x,z) > \beta$. Hence, $\mu_P(x,z) = \mu_S(x,z)$. Now, from the transitivity of S we have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

Second case. If $\mu_P(x,y) = \mu_S(x,y)$ and $\mu_P(y,z) = \beta \wedge \mu_S(y,z)$, then $\mu_R(x,y) > \beta$ and $\mu_R(y,z) \leq \beta$. So, $\mu_R(x,y) \wedge \mu_R(y,z) \leq \beta$. Now we have two subcases to consider:

(a) If $\mu_R(x,z) > \beta$, then it holds that $\mu_P(x,z) = \mu_S(x,z)$. We have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$ (since S is transitive). Hence, we get that $\mu_S(x,z) \geq \mu_S(x,y) \wedge (\beta \wedge \mu_S(y,z))$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

(b) If $\mu_R(x,z) \leq \beta$, then $\mu_P(x,z) = \beta \wedge \mu_S(x,z)$. We have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$ (since S is transitive). Hence, we get also that $\beta \wedge \mu_S(x,z) \geq \mu_S(x,y) \wedge (\beta \wedge \mu_S(y,z))$. Thus,

$$\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z).$$

Third case. If $\mu_P(x,y) = \beta \wedge \mu_S(x,y)$ and $\mu_P(y,z) = \mu_S(y,z)$ then it follows that $\mu_R(x,y) \leq \beta$ and $\mu_R(y,z) > \beta$. So, $\mu_R(x,y) \wedge \mu_R(y,z) \leq \beta$. In this case, we have also two subcases to consider:

(a) If $\mu_R(x,z) > \beta$, then $\mu_P(x,z) = \mu_S(x,z)$. Since by the fuzzy transitivity of S we have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$, so we get also that $\mu_S(x,z) \geq (\beta \wedge \mu_S(x,y)) \wedge \mu_S(y,z)$. Thus, $\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z)$.

(b) If $\mu_R(x,z) \leq \beta$ then $\mu_P(x,z) = \beta \wedge \mu_S(x,z)$. Since by the fuzzy transitivity of S we have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$, so we get also that $\beta \wedge \mu_S(x,z) \geq (\beta \wedge \mu_S(x,y)) \wedge \mu_S(y,z)$. Thus, $\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z)$.

Fourth case. If $\mu_P(x,y) = \beta \wedge \mu_S(x,y)$ and $\mu_P(y,z) = \beta \wedge \mu_S(y,z)$, then we get $\mu_R(x,y) \leq \beta$ and $\mu_R(y,z) \leq \beta$. So, $\mu_R(x,y) \wedge \mu_R(y,z) \leq \beta$. In this case we have also two subcases to consider:

(a) If $\mu_R(x,z) > \beta$, then $\mu_P(x,z) = \mu_S(x,z)$. Since by the fuzzy transitivity of S, we have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$, so we get also that $\mu_S(x,z) \geq (\beta \wedge \mu_S(x,y)) \wedge (\beta \wedge \mu_S(y,z))$. Thus, $\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z)$.

(b) If $\mu_R(x,z) \leq \beta$, then $\mu_P(x,z) = \beta \wedge \mu_S(x,z)$. Since by the fuzzy transitivity of S we have $\mu_S(x,z) \geq \mu_S(x,y) \wedge \mu_S(y,z)$, so we get also that $\beta \wedge \mu_S(x,z) \geq (\beta \wedge \mu_S(x,y)) \wedge (\beta \wedge \mu_S(y,z))$. Thus, $\mu_P(x,z) \geq \mu_P(x,y) \wedge \mu_P(y,z)$.

As consequence of the above fourth cases, we get that

$$\mu_P(x,z) \geq \min\{\mu_P(x,y), \mu_P(y,z)\},$$

for all $y \in X$. Thus, P is transitive. Therefore, P is an intuitionistic fuzzy ordering on X.

Moreover, let $x, y \in X$, such that $x \neq y$. Then, since S is a linear intuitionistic fuzzy ordering we get that

$$[\mu_S(x,y) > 0 \text{ and } \nu_S(x,y) = 0] \text{ or } [\mu_S(y,x) > 0 \text{ and } \nu_S(y,x) = 0].$$

Also, as $\beta > 0$ so we obtain that $\beta \wedge \mu_S(x,y) > 0$ or $\beta \wedge \mu_S(y,x) > 0$.

Hence, we get that

$$[\mu_P(x,y) > 0 \text{ and } \nu_P(x,y) = 0] \text{ or } [\mu_P(y,x) > 0 \text{ and } \nu_P(y,x) = 0].$$

Thus, P is linear. Therefore, Lemma 4.2 is proved. □

Remark 4.3. The linear intuitionistic fuzzy ordering P always satisfies the following condition:

$$R \subseteq P \subseteq S.$$

Now we are able to give the proof of Theorem 4.1.

Proof. (`The proof of Theorem 4.1.`) Let R be an intuitionistic fuzzy ordering on X and L_R be the set of all linear intuitionistic fuzzy orderings on X containing R. Theorem 4.1 can be shown by setting $S_0 = \bigcap_{S \in L_R} S$ and proving that $R = S_0$. We distinguish the following cases:

(i) If R linear, then the equality $R = S_0$ follows trivially from the fact that $R \in L_R$.

(ii) If R isn't linear. The first inclusion $R \subseteq S_0$ follows trivially from the fact that $R \subseteq S$, for all $S \in L_R$.

Next, we show that $S_0 \subseteq R$. That means that

$$\mu_{S_0}(x,y) = \inf_{S \in L_R} \mu_S(x,y) \leq \mu_R(x,y)$$

and

$$\nu_{S_0}(x,y) = \sup_{S \in L_R} \nu_S(x,y) \geq \nu_R(x,y).$$

Let $a, b \in X$. In order to simplify the proof, we distinguish two subcases:

First subcase. If a and b are incomparable in (X, μ_R, ν_R), then

$$[\mu_R(a,b) = 0 \text{ or } \nu_R(a,b) > 0] \text{ and } [\mu_R(b,a) = 0 \text{ or } \nu_R(b,a) > 0].$$

Hence, Lemma 3.1 and Theorem 3.2 allow to find two linear intuitionistic fuzzy orderings P, Q on X containing R such that

$$[\mu_P(a,b) > 0 \text{ and } \nu_P(a,b) = 0]$$

and

$$[\mu_Q(b,a) > 0 \text{ and } \nu_Q(b,a) = 0].$$

Moreover, the perfect antisymmetrical intuitionistic of P and Q implies that

$$\mu_P(b,a) = 0 \text{ and } \nu_P(b,a) = 1$$

and

$$\mu_Q(a,b) = 0 \text{ and } \nu_Q(a,b) = 1.$$

Thus,

$$\mu_{S_0}(a,b) = \inf_{S \in L_R} \mu_S(a,b) = 0 \leq \mu_R(a,b)$$

and

$$\nu_{S_0}(a,b) = \sup_{S \in L_R} \nu_S(a,b) = 1 \geq \nu_R(a,b).$$

Therefore, $S_0 \subseteq R$.

Second subcase. If a and b are comparable in (X, μ_R, ν_R), then

$$[\mu_R(a,b) > 0 \text{ and } \nu_R(a,b) = 0] \text{ or } [\mu_R(b,a) > 0 \text{ and } \nu_R(b,a) = 0].$$

Assume, without loss of generality, that $[\mu_R(a,b) > 0$ and $\nu_R(a,b) = 0]$. Then, by Lemma 4.2, there exists a linear intuitionistic fuzzy ordering P on X containing R such that

$$[\mu_P(a,b) = \mu_R(a,b) \text{ and } \nu_P(a,b) = \nu_R(a,b)].$$

Thus,

$$\mu_{S_0}(a,b) = \inf_{S \in L_R} \mu_S(a,b) = \mu_R(a,b)$$

and

$$\nu_{S_0}(a,b) = \sup_{S \in L_R} \nu_S(a,b) = \nu_R(a,b).$$

Analogously, we can show that $\mu_{S_0}(a,b) = \mu_R(a,b)$ and $\nu_{S_0}(a,b) = \nu_R(a,b)$ follows from the converse assumption $\mu_R(b,a) > 0$ and $\nu_R(b,a) = 0]$. Therefore, $S_0 \subseteq R$. □

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] K. T. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33 (1989) 37–45.

[3] K. T. Atanassov, Intuitionistic Fuzzy Sets, Springer-Verlag, Heidelberg, 1999.

[4] U. Bodenhofer and F. Klawonn, A formal study of linearity axioms for fuzzy orderings, Fuzzy Sets and Systems 145 (2004) 323–354.

[5] T. Buhaescu, Some observations on intuitionistic fuzzy relations, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989), 111–118, Preprint, 89-6, Univ. "Babes-Bolyai", Cluj-Napoca, 1989.

[6] P. Burillo and H. Bustince, Intuitionistic fuzzy relations (Part I), Mathware Soft Comput. 2(1) (1995) 5–38.

[7] H. Bustince and P. Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems 78 (1996) 293–303.

[8] F. Chiclana, F. Herrera and E. Herrera-Viedma, Integrating multiplicative preference relations in a multi-purpose decision-making based on fuzzy preference relations, Fuzzy Sets and Systems 122 (2001) 277–291.

[9] G. Deschrijver and E. E. Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy Sets and Systems 136 (2003) 333–361.
[10] B. Dushnik and E. W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600–610.
[11] I. Georgescu, Fuzzy Szpilrajn theorem through indicators, Int. J. Computers Communications & Control III(4) (2008) 336–342.
[12] S. Gottwald, Fuzzy Sets and Fuzzy Logic, Vieweg, Braunschweig, 1993.
[13] H. Hedayati, S. Azizpour and B. Davvaz, Prime (semiprime) bi-hyperideals of semihypergroups based on intuitionistic fuzzy points, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 75(3) (2013) 45–58.
[14] U. Höhle and N. Blanchard, Partial ordering in L-underdeterminate sets, Inform. Sci. 35 (1985) 133–144.
[15] A. Mukherjee, A. K. Das and A. Saha, Interval valued intuitionistic fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 6 (2013) 689–703.
[16] S. Padmapriya, M. K. Uma and E. Roja, Intuitionistic fuzzy *lim* extremally disconnected spaces, Ann. Fuzzy Math. Inform. 6 (2013) 743–753.
[17] P. Rajarajeswari and P. Dhanalakshmi, Intuitionistic fuzzy soft matrix theory and its application in medical diagnosis, Ann. Fuzzy Math. Inform. 7 (2014) 765–772.
[18] S. Rahman, H. K. Saikia and B. Davvaz, On the definition of Atanassov's intuitionistic fuzzy subrings and ideals, Bull. Malays. Math. Sci. Soc. 36(2) (2013) 401–41.
[19] B. Schroder, Ordered sets, Birkhauser Boston, 2002.
[20] R. Srinivasan and N. Palaniappan, Some operators on intuitionistic fuzzy sets of root type, Ann. Fuzzy Math. Inform. 4 (2012) 377–383.
[21] E. Szmidt and J. Kacprzyk, Using intuitionistic fuzzy sets in group decision making, Control & Cybernetics 31 (2002) 1037–1053.
[22] J. Szpilrajn, Sur I'extension de I'ordre partiel, Fund. Math. 16 (1930) 386–389.
[23] Z. S. Xu and Q. L. Da, A least deviation method to obtain a priority vector of a fuzzy preference relation, European J. Oper. Res. 164 (2005) 206–216.
[24] Z. S. Xu, Intuitionistic preference relations and their application in group decision making, Inform. Sci. 177 (2007) 2363–2379.
[25] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[26] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[27] J. Zhan, B. Davvaz and P. Corsini, On intuitionistic (S,T)-fuzzy H_v-submodules of H_v-modules, Southeast Asian Bull. Math. 36 (2012) 589–601.

L. Zedam (l.zedam@yahoo.fr)
Department of Mathematics, Faculty of Mathematics and Informatics, M'sila University, P.O.Box 166 Ichbilia, M'sila 28105, Algeria

A. Amroune (aamrounedz@yahoo.fr)
Department of Mathematics, Faculty of Mathematics and Informatics, M'sila University, P.O.Box 166 Ichbilia, M'sila 28105, Algeria

B. Davvaz (davvaz@yazd.ac.ir)
Department of Mathematics, Yazd University, Yazd, Iran