Some remarks on pairwise fuzzy Baire spaces

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ABSTRACT. In this paper several characterizations of pairwise fuzzy Baire bitopological spaces are studied. The conditions under which a fuzzy bitopological space becomes a pairwise fuzzy Baire space, are established.

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1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh in his classical paper [15] in the year 1965. This inspired mathematicians to fuzzify Mathematical Structures. The first notion of fuzzy topological space had been defined by C. L. Chang [4] in 1968. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

The concepts of bitopological Baire spaces have been studied extensively in classical topology in [1], [2], [5] and [6]. In 1989, A. Kandil [8] introduced the concept of fuzzy bitopological space as an extension of fuzzy topological space and as a generalization of bitopological space. The concept of Baire space in fuzzy setting was introduced and studied by G. Thangaraj and S. Anjalmose in [11]. The concept of bitopological Baire space in fuzzy setting was introduced and studied by the authors in [12]. The concept of pairwise dense sets was introduced and studied by Biswanath and Bandyopadhyay in [7]. In this paper we study under what conditions a fuzzy bitopological space becomes a pairwise fuzzy Baire space and pairwise fuzzy strongly irresolvable space, pairwise fuzzy submaximal space, pairwise fuzzy P-space are considered for this work.
2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work, by a fuzzy bitopological space (Kandil, 1989), we mean an ordered triple \((X, T_1, T_2)\) where \(T_1\) and \(T_2\) are fuzzy topologies on the non-empty set \(X\). The complement of a fuzzy set \(\lambda\) is defined by \(\lambda'(x) = 1 - \lambda(x)\).

**Definition 2.1.** Let \(\lambda\) and \(\mu\) be any two fuzzy sets in a fuzzy topological space \((X, T)\). Then we define \(\lambda \vee \mu : X \to [0,1]\) as follows : \((\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}\). Also we define \(\lambda \wedge \mu : X \to [0,1]\) as follows : \((\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}\).

For a family \(\{\lambda_i / i \in I\}\) of fuzzy sets in \((X, T)\), the union \(\psi = \bigvee \lambda_i\) and intersection \(\delta = \bigwedge \lambda_i\) are defined respectively as \(\psi(x) = \sup_i \{\lambda_i(x) / x \in X\}\) and \(\delta(x) = \inf_i \{\lambda_i(x) / x \in X\}\).

**Definition 2.2.** Let \((X, T)\) be a fuzzy topological space and \(\lambda\) be any fuzzy set in \((X, T)\). We define the interior and closure of \(\lambda\) as \(\text{int}(\lambda) = \bigvee \{\mu / \mu \leq \lambda, \mu \in T\}\) and \(\text{cl}(\lambda) = \bigwedge \{\mu / \lambda \leq \mu, 1 - \mu \in T\}\).

**Lemma 2.3** ([9]). Let \(\lambda\) be any fuzzy set in a fuzzy topological space \((X, T)\). Then \(1 - \text{cl}(\lambda) = \text{int}(1 - \lambda)\) and \(1 - \text{int}(\lambda) = \text{cl}(1 - \lambda)\).

**Definition 2.4** ([9]). A fuzzy set \(\lambda\) in a fuzzy topological space \((X, T)\) is called a pairwise fuzzy dense set if there exists no fuzzy closed set \(\mu\) in \((X, T)\) such that \(\lambda < \mu < 1\).

**Definition 2.5** ([10]). Let \((X, T_1, T_2)\) be a fuzzy bitopological space. A fuzzy set \(\lambda\) in \((X, T_1, T_2)\) is called a pairwise fuzzy dense set if \(\text{cl}_{T_1}(\text{cl}_{T_2}(\lambda)) = \text{cl}_{T_2}(\text{cl}_{T_1}(\lambda)) = 1\).

**Definition 2.6** ([9]). A fuzzy set \(\lambda\) in a fuzzy topological space \((X, T)\) is called a pairwise fuzzy nowhere dense set if there exists no non-zero fuzzy open set \(\mu\) in \((X, T)\) such that \(\mu < \text{cl}(\lambda)\). That is, \(\text{int}(\lambda) = 0\).

**Definition 2.7** ([12]). A fuzzy set \(\lambda\) in a fuzzy bitopological space \((X, T_1, T_2)\) is called a pairwise fuzzy nowhere dense set if \(\text{int}_{T_1}(\text{cl}_{T_2}(\lambda)) = \text{int}_{T_2}(\text{cl}_{T_1}(\lambda)) = 0\).

**Definition 2.8** ([12]). A fuzzy set \(\lambda\) in \((X, T_1, T_2)\) is called a pairwise fuzzy open set if \(\lambda \in T_i\) \((i = 1, 2)\) and a pairwise fuzzy closed set if \(1 - \lambda \in T_i\) \((i = 1, 2)\).

**Definition 2.9** ([12]). Let \((X, T_1, T_2)\) be a fuzzy bitopological space. A fuzzy set \(\lambda\) in \((X, T_1, T_2)\) is called a pairwise fuzzy first category set if \(\lambda = \bigvee_{k=1}^{\infty} (\lambda_k)\), where \((\lambda_k)\)'s are pairwise fuzzy nowhere dense sets in \((X, T_1, T_2)\). A fuzzy set which is not of pairwise fuzzy first category, is called a pairwise fuzzy second category set in \((X, T_1, T_2)\).

**Definition 2.10** ([12]). If \(\lambda\) is a pairwise fuzzy first category set in a fuzzy bitopological space \((X, T_1, T_2)\), then the fuzzy set \(1 - \lambda\) is called a pairwise fuzzy residual set in \((X, T_1, T_2)\).

**Definition 2.11** ([11]). A fuzzy topological space \((X, T)\) is called a fuzzy Baire space if \(\text{int}(\bigvee_{k=1}^{\infty} (\lambda_k)) = 0\), where \((\lambda_k)\)'s are fuzzy nowhere dense sets in \((X, T)\).

**Lemma 2.12** ([3]). For a family \(\mathcal{A} = \{\lambda_\alpha\}\) of fuzzy sets of a fuzzy space \(X\), \(\text{cl}(\lambda_\alpha) \leq \text{cl}(\bigvee_{\alpha} \lambda_\alpha)\). In case \(\mathcal{A}\) is a finite set, \(\text{cl}(\lambda_\alpha) = \text{cl}(\bigvee_{\alpha} \lambda_\alpha)\). Also \(\text{int}(\lambda_\alpha) \leq \text{int}(\bigvee_{\alpha} \lambda_\alpha)\).
3. PAIRWISE FUZZY BAIRE SPACES

Definition 3.1 ([12]). A fuzzy bitopological space \((X, T_1, T_2)\) is called a pairwise fuzzy Baire space if \(\text{int}_{T_1}(\bigvee_{k=1}^{\infty} (\lambda_k)) = 0, \ (i = 1, 2)\) where \((\lambda_k)\)'s are pairwise fuzzy nowhere dense sets in \((X, T_1, T_2)\).

Theorem 3.2 ([12]). Let \((X, T_1, T_2)\) be a fuzzy bitopological space. Then the following are equivalent:

(i) \((X, T_1, T_2)\) is a pairwise fuzzy Baire space.
(ii) \(\text{int}_{T_1}(\lambda) = 0, \ (i = 1, 2)\) for every pairwise fuzzy first category set \(\lambda\) in \((X, T_1, T_2)\).
(iii) \(\text{cl}_{T_i}(\mu) = 1, \ (i = 1, 2)\) for every pairwise fuzzy residual set \(\mu\) in \((X, T_1, T_2)\).

Proposition 3.3. If \(\text{int}_{T_i}(\lambda) = 0, \ (i = 1, 2)\) for a fuzzy set \(\lambda\) in a fuzzy bitopological space \((X, T_1, T_2)\), then \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = 0\) and \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = 0\).

Proof. Let \(\text{int}_{T_i}(\lambda) = 0, \ (i = 1, 2)\) in \((X, T_1, T_2)\). Then \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = \text{int}_{T_1}(0) = 0\). Also, \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = \text{int}_{T_2}(0) = 0\). Hence \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = \text{int}_{T_1}\text{int}_{T_2}(\lambda) = 0\).

Proposition 3.4. If \(\text{cl}_{T_i}(\lambda) = 1, \ (i = 1, 2)\) for a fuzzy set \(\lambda\) in a fuzzy bitopological space \((X, T_1, T_2)\), then \(\lambda\) is a pairwise fuzzy dense set in \((X, T_1, T_2)\).

Proof. Let \(\text{cl}_{T_i}(\lambda) = 1, \ (i = 1, 2)\) in \((X, T_1, T_2)\). Then \(\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = \text{cl}_{T_1}(1) = 1\). Also, \(\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = \text{cl}_{T_1}(1) = 1\). Hence we have \(\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = \text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = 1\). Therefore \(\lambda\) is a pairwise fuzzy dense set in \((X, T_1, T_2)\).

Proposition 3.5. If \((X, T_1, T_2)\) is a pairwise fuzzy Baire space, then,

(i) \(\text{int}_{T_1}\text{int}_{T_2}(\lambda) = 0 = \text{int}_{T_2}\text{int}_{T_1}(\lambda)\), for a pairwise fuzzy first category set \(\lambda\) in \((X, T_1, T_2)\).
(ii) \(\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = 1 = \text{cl}_{T_1}\text{cl}_{T_2}(\lambda)\), for a pairwise fuzzy residual set \(\lambda\) in \((X, T_1, T_2)\).

Proof. Proof follows from theorem 3.2, propositions 3.3 and 3.3.

Proposition 3.6. If the fuzzy bitopological space \((X, T_1, T_2)\) is a pairwise fuzzy Baire space, then no pairwise fuzzy closed set other than 1, is a pairwise fuzzy residual set in \((X, T_1, T_2)\).

Proof. Let \(\lambda(\neq 1)\), be a pairwise fuzzy closed set in a pairwise fuzzy Baire space \((X, T_1, T_2)\). Then \(\text{cl}_{T_i}(\lambda) = \lambda, \ (i = 1, 2)\). Suppose \(\lambda\) is a pairwise fuzzy residual set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy Baire space, by theorem 3.2 \(\text{cl}_{T_i}(\lambda) = 1\). This implies that \(\text{cl}_{T_i}(\lambda) \neq \lambda\), a contradiction to \(\lambda\) being a pairwise fuzzy residual set. Hence no pairwise fuzzy closed set other than 1, is a pairwise fuzzy residual set in \((X, T_1, T_2)\).

Theorem 3.7 ([13]). If the fuzzy bitopological space \((X, T_1, T_2)\) is a pairwise fuzzy Baire space, then no non-zero pairwise fuzzy open set is a pairwise fuzzy first category set in \((X, T_1, T_2)\).

Proposition 3.8. If \(\lambda\) is a non-zero pairwise fuzzy open set in a pairwise fuzzy Baire space \((X, T_1, T_2)\), then \(\lambda \neq \bigvee_{k=1}^{\infty} (\lambda_k)\), where \((\lambda_k)\)'s are pairwise fuzzy nowhere dense sets in \((X, T_1, T_2)\).
Proof. Let $\lambda$ be a non-zero pairwise fuzzy open set in a pairwise fuzzy Baire space $(X, T_1, T_2)$. Then, by theorem 3.7, $\lambda$ is not a pairwise fuzzy first category set in $(X, T_1, T_2)$. Hence, $\lambda \neq \bigvee_{k=1}^{\infty} (\lambda_k)$, where $(\lambda_k)$’s are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2)$. \hfill \Box

4. Inter-Relations between Pairwise Fuzzy Strongly Irresolvable Spaces, Pairwise Fuzzy Submaximal Spaces and Pairwise Fuzzy Baire Spaces

Definition 4.1 (13). A fuzzy bitopological space $(X, T_1, T_2)$ is said to be a pairwise fuzzy strongly irresolvable space if $cl_{T_1} int_{T_2}(\lambda) = 1 = cl_{T_2} int_{T_1}(\lambda)$ for each pairwise fuzzy dense set $\lambda$ in $(X, T_1, T_2)$.

Proposition 4.2. Let $(X, T_1, T_2)$ be a pairwise fuzzy strongly irresolvable space. If $\lambda$ is a pairwise fuzzy dense set in $(X, T_1, T_2)$, then $1 - \lambda$ is a pairwise fuzzy nowhere dense set.

Proof. Let $\lambda$ be a pairwise fuzzy dense set in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy strongly irresolvable space, $cl_{T_1} int_{T_2}(\lambda) = 1 = cl_{T_2} int_{T_1}(\lambda)$. This implies that $1 - cl_{T_1} int_{T_2}(\lambda) = 0 = 1 - cl_{T_2} int_{T_1}(\lambda)$. Therefore $int_{T_1} cl_{T_2}(1 - \lambda) = 0 = int_{T_2} cl_{T_1}(1 - \lambda)$ and hence $1 - \lambda$ is a pairwise fuzzy nowhere dense set. \hfill \Box

Theorem 4.3 (12). If $\lambda$ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space $(X, T_1, T_2)$, then $1 - \lambda$ is a pairwise fuzzy dense set in $(X, T_1, T_2)$.

Proposition 4.4. Let $(X, T_1, T_2)$ be a pairwise fuzzy strongly irresolvable space. Then $\lambda$ is a pairwise fuzzy dense set in $(X, T_1, T_2)$ if and only if $1 - \lambda$ is a pairwise fuzzy nowhere dense set.

Proof. Proof follows from proposition 4.2 and theorem 4.3. \hfill \Box

Theorem 4.5 (13). If $cl_{T_i}(\bigwedge_{k=1}^{\infty} (\lambda_k)) = 1$ $(i = 1, 2)$, where $(\lambda_k)$’s are pairwise fuzzy dense sets in a pairwise fuzzy strongly irresolvable space $(X, T_1, T_2)$, then $(X, T_1, T_2)$ is a pairwise fuzzy Baire space.

Proposition 4.6. If $(\lambda_k)$’s are pairwise fuzzy dense sets in a pairwise fuzzy strongly irresolvable and pairwise fuzzy Baire space $(X, T_1, T_2)$, then $cl_{T_i}(\bigwedge_{k=1}^{\infty} (\lambda_k)) = 1$, $(i = 1, 2)$.

Proof. Let $(\mu_k)$’s $(k = 1$ to $\infty)$ be pairwise fuzzy nowhere dense sets in a pairwise fuzzy strongly irresolvable and pairwise fuzzy Baire space $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy Baire space, $int_{T_i}(\bigvee_{k=1}^{\infty} (\mu_k)) = 0$. Then $1 - int_{T_i}(\bigvee_{k=1}^{\infty} (\mu_k)) = 1$. This implies that $cl_{T_i} (\bigwedge_{k=1}^{\infty} (1 - \mu_k)) = 1 \rightarrow (1)$. Since $(\mu_k)$’s are pairwise fuzzy nowhere dense sets in a pairwise fuzzy strongly irresolvable space $(X, T_1, T_2)$, by proposition 4.3 $(1 - \mu_k)$’s are pairwise fuzzy dense sets in $(X, T_1, T_2)$. Let $\lambda_k = 1 - \mu_k$. Then we have from $(1)$, $cl_{T_i}(\bigwedge_{k=1}^{\infty} (\lambda_k)) = 1$, $(i = 1, 2)$ where $(\lambda_k)$’s are pairwise fuzzy dense sets in $(X, T_1, T_2)$. \hfill \Box

Proposition 4.7. If $(X, T_1, T_2)$ is a pairwise fuzzy strongly irresolvable space, then $(X, T_1, T_2)$ is a pairwise fuzzy Baire space if and only if $cl_{T_i}(\bigwedge_{k=1}^{\infty} (\lambda_k)) = 1$, $(i = 1, 2)$.

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Proof. Proof follows from proposition 4.6 and theorem 4.5 □

Proposition 4.8. If \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable space and \(\lambda = \bigwedge_{k=1}^{\infty} (\lambda_k)\) is a pairwise fuzzy dense set in \((X, T_1, T_2)\), then \(1 - \lambda\) is a pairwise fuzzy first category set in \((X, T_1, T_2)\).

Proof. Let \(\lambda = \bigwedge_{k=1}^{\infty} (\lambda_k)\) be a pairwise fuzzy dense set in \((X, T_1, T_2)\). Then \(\text{cl}_{T_1} \text{cl}_{T_2} (\bigwedge_{k=1}^{\infty} (\lambda_k)) = 1 = \text{cl}_{T_2} \text{cl}_{T_1} (\bigwedge_{k=1}^{\infty} (\lambda_k))\). But \(\text{cl}_{T_1} (\bigwedge_{k=1}^{\infty} (\lambda_k)) \leq \bigwedge_{k=1}^{\infty} (\text{cl}_{T_1} (\lambda_k))\) and hence \(\text{cl}_{T_2} \text{cl}_{T_1} (\bigwedge_{k=1}^{\infty} (\lambda_k)) \leq \text{cl}_{T_2} \text{cl}_{T_1} (\lambda_k)\). Thus \(1 \leq \text{cl}_{T_2} \text{cl}_{T_1} (\lambda_k)\). But \(\lambda_k\) is a pairwise fuzzy strongly irresolvable space, by proposition 4.12. Therefore, we have \(1 - \lambda = \bigvee_{k=1}^{\infty} (1 - \lambda_k)\), where \((1 - \lambda_k)'s\) are pairwise fuzzy nowhere dense sets. Therefore, we have \(1 - \lambda\) is a pairwise fuzzy first category set in \((X, T_1, T_2)\). □

Proposition 4.9. If \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable space and \(\lambda = \bigwedge_{k=1}^{\infty} (\lambda_k)\) is a pairwise fuzzy dense set in \((X, T_1, T_2)\), then \(\lambda\) is a pairwise fuzzy residual set in \((X, T_1, T_2)\).

Proof. Let \(\lambda = \bigwedge_{k=1}^{\infty} (\lambda_k)\) be a pairwise fuzzy dense set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable space, by proposition 4.8 \(1 - \lambda\) is a pairwise fuzzy first category set in \((X, T_1, T_2)\). Therefore \(\lambda\) is a pairwise fuzzy residual set in \((X, T_1, T_2)\). □

Definition 4.10 ([13]). A fuzzy bitopological space \((X, T_1, T_2)\) is called a pairwise fuzzy submaximal space if each pairwise fuzzy dense set in \((X, T_1, T_2)\), is a pairwise fuzzy open set in \((X, T_1, T_2)\). That is., if \(\lambda\) is a pairwise fuzzy dense set in a fuzzy bitopological space \((X, T_1, T_2)\), then \(\lambda \in T_i\) \((i = 1, 2)\).

Proposition 4.11. If \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space and \(\lambda\) is a pairwise fuzzy first category set, then \(1 - \lambda\) is a pairwise fuzzy Gδ-set in \((X, T_1, T_2)\).

Proof. Let \(\lambda\) be a pairwise fuzzy first category set in \((X, T_1, T_2)\). Then \(\lambda = \bigvee_{k=1}^{\infty} (\lambda_k)\), where \((\lambda_k)'s\) are pairwise fuzzy nowhere dense sets in \((X, T_1, T_2)\). By theorem 4.3 \((1 - \lambda_k)'s\) are pairwise fuzzy dense sets in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space, \((1 - \lambda_k)'s\) are pairwise fuzzy open sets in \((X, T_1, T_2)\). Also \(1 - \lambda = 1 - \bigvee_{k=1}^{\infty} (\lambda_k) = \bigwedge_{k=1}^{\infty} (1 - \lambda_k)\), where \((1 - \lambda_k)'s\) are pairwise fuzzy open sets in \((X, T_1, T_2)\). Therefore \(1 - \lambda\) is a pairwise fuzzy Gδ-set in \((X, T_1, T_2)\). □

Proposition 4.12. If \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space, then every pairwise fuzzy first category set is a pairwise fuzzy Fα-set in \((X, T_1, T_2)\).

Proof. Let \(\lambda\) be a pairwise fuzzy first category set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space, by proposition 4.11 \(1 - \lambda\) is a pairwise fuzzy Gδ-set in \((X, T_1, T_2)\) and hence \(\lambda\) is a pairwise fuzzy Fα-set in \((X, T_1, T_2)\). □

Theorem 4.13 ([13]). If \((X, T_1, T_2)\) is a pairwise fuzzy Baire space, then every pairwise fuzzy residual set is a pairwise fuzzy dense set in \((X, T_1, T_2)\).
Definition 4.14 ([14]). A fuzzy bitopological space \((X, T_1, T_2)\) is called a **pairwise fuzzy** \(P\)-**space** if every non-zero pairwise fuzzy \(G_δ\)-set in \((X, T_1, T_2)\), is a pairwise fuzzy open set in \((X, T_1, T_2)\). That is, if \((X, T_1, T_2)\) is a pairwise fuzzy \(P\)-space if \(\lambda \in T_i\) \((i = 1, 2)\) for \(\lambda = \wedge_{k=1}^∞(λ_k)\), where \(λ_k\)'s are pairwise fuzzy open sets in \((X, T_1, T_2)\).

**Proposition 4.15.** If \((X, T_1, T_2)\) is a pairwise fuzzy submaximal and pairwise fuzzy \(P\)-**space**, then every pairwise fuzzy residual set is a pairwise fuzzy open set in \((X, T_1, T_2)\).

**Proof.** Let \(λ\) be a pairwise fuzzy residual set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space, by proposition 4.12 \(λ\) is a pairwise fuzzy \(G_δ\)-set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy \(P\)-space, \(λ\) is a pairwise fuzzy open set in \((X, T_1, T_2)\). Hence every pairwise fuzzy residual set is a pairwise fuzzy open set in a pairwise fuzzy submaximal and pairwise fuzzy \(P\)-space. \(\square\)

Definition 4.16 ([13]). A fuzzy bitopological space \((X, T_1, T_2)\) is called a **pairwise fuzzy nodec** space if every non-zero pairwise fuzzy nowhere dense set in \((X, T_1, T_2)\), is a pairwise fuzzy closed set in \((X, T_1, T_2)\). That is., if \(λ\) is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space \((X, T_1, T_2)\), then \(1 − λ \in T_i\) \((i = 1, 2)\).

**Proposition 4.17.** If \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space, then \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space.

**Proof.** Let \((X, T_1, T_2)\) be a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space. Let \(λ\) be a pairwise fuzzy dense set in \((X, T_1, T_2)\). Since \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable space, by proposition 4.2 \(1 − λ\) is a pairwise fuzzy nowhere dense set. Since \((X, T_1, T_2)\) is a pairwise fuzzy nodec space, \(1 − λ\) is a pairwise fuzzy closed set. Then \(λ\) is a pairwise fuzzy open set in \((X, T_1, T_2)\). Hence every pairwise fuzzy dense set is pairwise fuzzy open set in \((X, T_1, T_2)\). Therefore \((X, T_1, T_2)\) is a pairwise fuzzy submaximal space. \(\square\)

**Proposition 4.18.** If \((X, T_1, T_2)\) is a pairwise fuzzy strongly irresolvable and pairwise fuzzy second category space, then \(\wedge_{k=1}^∞(λ_k) \neq 0\) where \((λ_k)\)'s are pairwise fuzzy dense sets in \((X, T_1, T_2)\).

**Proof.** Let \((X, T_1, T_2)\) be a pairwise fuzzy second category space. Let us assume that \(\wedge_{k=1}^∞(λ_k) = 0\). Since \((λ_k)\)'s are pairwise fuzzy dense sets in \((X, T_1, T_2)\), by proposition 4.2 \((1 − λ_k)\)'s are pairwise fuzzy nowhere dense sets in \((X, T_i, T_2)\). Now \(1 − \wedge_{k=1}^∞(λ_k) = 1\), implies that \(\vee_{k=1}^∞(1 − λ_k) = 1\) and \((1 − λ_k)\)'s are pairwise fuzzy nowhere dense sets in \((X, T_1, T_2)\). Hence \((X, T_1, T_2)\) is a pairwise fuzzy first category space which is a contradiction. Therefore \(\wedge_{k=1}^∞(λ_k) \neq 0\) where \((λ_k)\)'s are pairwise fuzzy dense sets in \((X, T_1, T_2)\). \(\square\)

**Proposition 4.19.** If \(λ\) is a pairwise fuzzy first category set in a pairwise fuzzy submaximal and pairwise fuzzy \(P\)-**space** \((X, T_1, T_2)\), then \(λ\) is not a pairwise fuzzy dense set in \((X, T_1, T_2)\).

**Proof.** Let \(λ\) be a pairwise fuzzy first category set in a pairwise fuzzy submaximal and pairwise fuzzy \(P\)-space \((X, T_1, T_2)\). Hence \(1 − λ\) is a pairwise fuzzy residual set in \((X, T_1, T_2)\). Then, by proposition 4.15 \(1 − λ\) is a pairwise fuzzy open set in \((X, T_1, T_2)\). Therefore \(λ\) is not a pairwise fuzzy dense set in \((X, T_1, T_2)\).
(X, T_1, T_2), so λ is a pairwise fuzzy closed set. Hence cl_{T_1}(\lambda) = cl_{T_2}(\lambda) = \lambda implies that cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = \lambda \neq 1 and therefore λ is not a pairwise fuzzy dense set in (X, T_1, T_2).

\[\square\]

**Remark 4.20.** If int_{T_1}int_{T_2}(\lambda) = 0 and int_{T_2}int_{T_1}(\lambda) = 0 do not imply int_{T_1}(\lambda) = 0 and int_{T_2}(\lambda) = 0 in a fuzzy bitopological space (X, T_1, T_2).

**Proposition 4.21.** If int_{T_1}int_{T_2}(\lambda) = 0 and int_{T_2}int_{T_1}(\lambda) = 0 for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then int_{T_1}(\lambda) = 0 and int_{T_2}(\lambda) = 0 in (X, T_1, T_2).

**Proof.** Let int_{T_1}int_{T_2}(\lambda) = 0 and int_{T_2}int_{T_1}(\lambda) = 0 for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2). Then 1 - int_{T_1}int_{T_2}(\lambda) = 1 and 1 - int_{T_2}int_{T_1}(\lambda) = 1 imply that cl_{T_1}cl_{T_2}(1 - \lambda) = 1 and cl_{T_2}cl_{T_1}(1 - \lambda) = 1. That is, 1 - λ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set 1 - λ in (X, T_1, T_2), we have cl_{T_1}cl_{T_2}(1 - \lambda) = 1 and cl_{T_2}cl_{T_1}(1 - \lambda) = 1. Then we have, int_{T_1}cl_{T_2}(\lambda) = 0 and int_{T_2}cl_{T_1}(\lambda) = 0. Hence int_{T_1}(\lambda) \leq int_{T_1}cl_{T_2}(\lambda) = 0 and int_{T_2}(\lambda) \leq int_{T_2}cl_{T_1}(\lambda) = 0 implies that int_{T_1}(\lambda) \leq 0 and int_{T_2}(\lambda) \leq 0. That is, int_{T_1}(\lambda) = 0 and int_{T_2}(\lambda) = 0.

\[\square\]

**Proposition 4.22.** If cl_{T_1}cl_{T_2}(\lambda) = 1 and cl_{T_2}cl_{T_1}(\lambda) = 1 for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then cl_{T_1}(\lambda) = 1 and cl_{T_2}(\lambda) = 1 in (X, T_1, T_2).

**Proof.** Let cl_{T_1}cl_{T_2}(\lambda) = 1 and cl_{T_2}cl_{T_1}(\lambda) = 1 for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2). Hence 1 - cl_{T_1}cl_{T_2}(\lambda) = 0 and 1 - cl_{T_2}cl_{T_1}(\lambda) = 0. This implies that int_{T_1}int_{T_2}(1 - \lambda) = 0 and int_{T_2}int_{T_1}(1 - \lambda) = 0. Now, by proposition 4.21 int_{T_1}(1 - \lambda) = 0 and int_{T_2}(1 - \lambda) = 0 and hence 1 - cl_{T_1}(\lambda) = 0 and 1 - cl_{T_2}(\lambda) = 0. Therefore cl_{T_1}(\lambda) = 1 and cl_{T_2}(\lambda) = 1 in (X, T_1, T_2).

\[\square\]

**Proposition 4.23.** If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, then

(i). int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda), for a fuzzy set λ in (X, T_1, T_2) implies that int_{T_i}(\lambda) = 1, (i = 1, 2).

(ii). cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda), for a fuzzy set λ in (X, T_1, T_2) implies that cl_{T_i}(\lambda) = 1, (i = 1, 2).

**Proof.** Proof follows from propositions 4.21 and 4.22.

\[\square\]

**Proposition 4.24.** If (X, T_1, T_2) is a pairwise fuzzy submaximal space, then every pairwise fuzzy residual set is a pairwise fuzzy Gδ-set in (X, T_1, T_2).

**Proof.** Let λ be a pairwise fuzzy residual set in (X, T_1, T_2). Then 1 - λ is a pairwise fuzzy first category set (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 4.12 1 - λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Therefore λ is a pairwise fuzzy Gδ-set in (X, T_1, T_2).

\[\square\]

**Theorem 4.25** ([13]). If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space, then (X, T_1, T_2) is not a pairwise Baire space.

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Proposition 4.26. If every pairwise fuzzy $G_{\delta}$-set is fuzzy pairwise dense in a pairwise fuzzy submaximal and pairwise fuzzy strongly irresolvable space $(X, T_1, T_2)$, then $(X, T_1, T_2)$ is a pairwise fuzzy Baire space.

Proof. Let $(X, T_1, T_2)$ be a pairwise fuzzy submaximal space in which every pairwise fuzzy $G_{\delta}$ set is pairwise fuzzy dense in $(X, T_1, T_2)$. Let $\lambda$ be a pairwise fuzzy residual set in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy submaximal space, by proposition 4.24, $\lambda$ is a pairwise fuzzy $G_{\delta}$-set in $(X, T_1, T_2)$. By hypothesis, $\lambda$ is a pairwise fuzzy dense set in $(X, T_1, T_2)$. That is, $\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = 1 = \text{cl}_{T_2}\text{cl}_{T_1}(\lambda)$. Then, by proposition 4.22, $\text{cl}_{T_1}(\lambda) = 1$ and $\text{cl}_{T_2}(\lambda) = 1$ in $(X, T_1, T_2)$. Hence, by theorem 3.2, $(X, T_1, T_2)$ is a pairwise fuzzy Baire space.

Theorem 4.27. If the fuzzy bitopological space $(X, T_1, T_2)$ is a pairwise fuzzy nodec space, then $(X, T_1, T_2)$ is not a pairwise fuzzy Baire space.

Proposition 4.28. If every pairwise fuzzy $G_{\delta}$-set is a pairwise fuzzy dense set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space $(X, T_1, T_2)$, then $(X, T_1, T_2)$ is a pairwise fuzzy Baire space.

Proof. Let $\lambda$ be a pairwise fuzzy $G_{\delta}$-set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space $(X, T_1, T_2)$ such that $\text{cl}_{T_1}\text{cl}_{T_2}(\lambda) = 1 = \text{cl}_{T_2}\text{cl}_{T_1}(\lambda)$. By proposition 4.17, $(X, T_1, T_2)$ is a pairwise fuzzy submaximal space. Again, by proposition 4.26, $(X, T_1, T_2)$ is a pairwise fuzzy Baire space.

5. Conclusions

In this paper several characterizations of pairwise fuzzy Baire spaces are studied. The conditions under which a fuzzy bitopological space becomes a pairwise fuzzy Baire space are investigated. For this work, pairwise fuzzy strongly irresolvable space, pairwise fuzzy submaximal space, pairwise fuzzy $P$-space are considered.

References


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