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Some remarks on pairwise fuzzy Baire spaces

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Abstract. In this paper several characterizations of pairwise fuzzy Baire bitopological spaces are studied. The conditions underwhich a fuzzy bitopological space becomes a pairwise fuzzy Baire space, are established.

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1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh in his classical paper [15] in the year 1965. This inspired mathematicians to fuzzify Mathematical Structures. The first notion of fuzzy topological space had been defined by C. L. Chang [4] in 1968. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concepts of bitopological Baire spaces have been studied extensively in classical topology in [1], [2], [5] and [6]. In 1989, A. Kandil [8] introduced the concept of fuzzy bitopological space as an extension of fuzzy topological space and as a generalization of bitopological space. The concept of Baire space in fuzzy setting was introduced and studied by G. Thangaraj and S. Anjalmose in [11]. The concept of bitopological Baire space in fuzzy setting was introduced and studied by the authors in [12]. The concept of pairwise dense sets was introduced and studied by Biswanath and Bandyopadhyay in [7]. In this paper we study under what conditions a fuzzy bitopological space becomes a pairwise fuzzy Baire space and pairwise fuzzy strongly irresolvable space, pairwise fuzzy submaximal space, pairwise fuzzy P-space are considered for this work.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work, by a fuzzy bitopological space (Kandil, 1989), we mean an ordered triple (X, T_1, T_2) where T_1 and T_2 are fuzzy topologies on the non-empty set X. The complement λ' of a fuzzy set λ is defined by $\lambda'(x) = 1 - \lambda(x)$.

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define $\lambda \vee \mu$: X → [0,1] as follows : $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the *union* $\psi = \vee_i \lambda_i$ and *intersection* $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x) \ / \ x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x) \ / \ x \in X\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define the interior and closure of λ as $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([3])**.** *Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.*

Definition 2.4 ([9])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy dense* if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.5 ([10])**.** Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy dense set* if $cl_{T_1}(cl_{T_2}(\lambda)) = cl_{T_2}(cl_{T_1}(\lambda)) = 1$.

Definition 2.6 ([9])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy nowhere dense* if there exists no non - zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.7 ([12])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called *pairwise fuzzy nowhere dense* if $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$.

Definition 2.8 ([12])**.** A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy open set* if $\lambda \in T_i$ $(i = 1, 2)$ and a *pairwise fuzzy closed set* if $1 - \lambda \in T_i$ $(i = 1, 2)$.

Definition 2.9 ([12])**.** Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy first category set* if $\lambda = \bigvee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). A fuzzy set which is not of pairwise fuzzy first category, is called a *pairwise fuzzy second category set* in (X, T_1, T_2).

Definition 2.10 ([12])**.** If λ is a pairwise fuzzy first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a *pairwise fuzzy residual set* in (X, T_1, T_2).

Definition 2.11 ([11])**.** A fuzzy topological space (X, T) is called a *fuzzy Baire space* if $int(\bigvee_{k=1}^{\infty}(\lambda_k)) = 0$, where (λ_k)'s are fuzzy nowhere dense sets in (X, T).

Lemma 2.12 ([3])**.** *For a family $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee cl(\lambda_\alpha) \leq cl(\vee \lambda_\alpha)$. In case $\mathscr{A}$ is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee \lambda_\alpha)$. Also $\vee int(\lambda_\alpha) \leq int(\vee \lambda_\alpha)$.*

3. Pairwise fuzzy Baire spaces

Definition 3.1 ([12]). A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy Baire space* if $int_{T_i}(\bigvee_{k=1}^{\infty}(\lambda_k)) = 0$, $(i = 1, 2)$ where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

Theorem 3.2 ([12]). *Let (X, T_1, T_2) be a fuzzy bitopological space. Then the following are equivalent:*

(i). *(X, T_1, T_2) is a pairwise fuzzy Baire space.*
(ii). *$int_{T_i}(\lambda) = 0$, (i=1,2) for every pairwise fuzzy first category set λ in (X, T_1, T_2).*
(iii). *$cl_{T_i}(\mu) = 1$, (i=1,2) for every pairwise fuzzy residual set μ in (X, T_1, T_2).*

Proposition 3.3. *If $int_{T_i}(\lambda) = 0$, $(i = 1, 2)$ for a fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2), then $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$.*

Proof. Let $int_{T_i}(\lambda) = 0, (i = 1, 2)$ in (X, T_1, T_2). Then $int_{T_1}int_{T_2}(\lambda) = int_{T_1}(0) = 0$. Also, $int_{T_2}int_{T_1}(\lambda) = int_{T_2}(0) = 0$. Hence $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$. □

Proposition 3.4. *If $cl_{T_i}(\lambda) = 1$, $(i = 1, 2)$ for a fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2), then λ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proof. Let $cl_{T_i}(\lambda) = 1, (i = 1, 2)$ in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_1}(1) = 1$. Also, $cl_{T_2}cl_{T_1}(\lambda) = cl_{T_2}(1) = 1$. Hence we have $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = 1$. Therefore λ is a pairwise fuzzy dense set in (X, T_1, T_2). □

Proposition 3.5. *If (X, T_1, T_2) is a pairwise fuzzy Baire space, then,*

(i). *$int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, for a pairwise fuzzy first category set λ in (X, T_1, T_2).*
(ii). *$cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$, for a pairwise fuzzy residual set λ in (X, T_1, T_2).*

Proof. Proof follows from theorem 3.2, propositions 3.3 and 3.4. □

Proposition 3.6. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space, then no pairwise fuzzy closed set other than 1, is a pairwise fuzzy residual set in (X, T_1, T_2).*

Proof. Let $\lambda(\neq 1)$, be a pairwise fuzzy closed set in a pairwise fuzzy Baire space (X, T_1, T_2). Then $cl_{T_i}(\lambda) = \lambda, (i = 1, 2)$. Suppose λ is a pairwise fuzzy residual set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Baire space, by theorem 3.2, $cl_{T_i}(\lambda) = 1$. This implies that $cl_{T_i}(\lambda) \neq \lambda$, a contradiction to λ being a pairwise fuzzy closed set. Hence no pairwise fuzzy closed set other than 1, is a pairwise fuzzy residual set in (X, T_1, T_2). □

Theorem 3.7 ([13]). *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space, then no non-zero pairwise fuzzy open set is a pairwise fuzzy first category set in (X, T_1, T_2).*

Proposition 3.8. *If λ is a non-zero pairwise fuzzy open set in a pairwise fuzzy Baire space (X, T_1, T_2), then $\lambda \neq \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).*

Proof. Let λ be a non-zero pairwise fuzzy open set in a pairwise fuzzy Baire space (X, T_1, T_2). Then, by theorem 3.7, λ is not a pairwise fuzzy first category set in (X, T_1, T_2). Hence, $\lambda \neq \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). □

4. Inter-relations between pairwise fuzzy strongly irresolvable spaces, pairwise fuzzy submaximal spaces and pairwise fuzzy Baire spaces

Definition 4.1 ([13])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy strongly irresolvable space* if $cl_{T_1}int_{T_2}(\lambda) = 1 = cl_{T_2}int_{T_1}(\lambda)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).

Proposition 4.2. *Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space . If λ is a pairwise fuzzy dense set in (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy nowhere dense set .*

Proof. Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, $cl_{T_1}int_{T_2}(\lambda) = 1 = cl_{T_2}int_{T_1}(\lambda)$. This implies that $1 - cl_{T_1}int_{T_2}(\lambda) = 0 = 1 - cl_{T_2}int_{T_1}(\lambda)$. Therefore $int_{T_1}cl_{T_2}(1 - \lambda) = 0 = int_{T_2}cl_{T_1}(1 - \lambda)$ and hence $1 - \lambda$ is a pairwise fuzzy nowhere dense set. □

Theorem 4.3 ([12])**.** *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2) then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proposition 4.4. *Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space. Then λ is a pairwise fuzzy dense set in (X, T_1, T_2) if and only if $1 - \lambda$ is a pairwise fuzzy nowhere dense set.*

Proof. Proof follows from proposition 4.2 and theorem 4.3. □

Theorem 4.5 ([13])**.** *If $cl_{T_i}(\wedge_{k=1}^{\infty}(\lambda_k)) = 1$ $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense sets in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proposition 4.6. *If (λ_k)'s are pairwise fuzzy dense sets in a pairwise fuzzy strongly irresolvable and pairwise fuzzy Baire space (X, T_1, T_2), then $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$, $(i = 1, 2)$.*

Proof. Let (μ_k)'s $(k = 1 \text{ to } \infty)$ be pairwise fuzzy nowhere dense sets in a pairwise fuzzy strongly irresolvable and pairwise fuzzy Baire space (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Baire space, $int_{T_i}\big(\vee_{k=1}^{\infty}(\mu_k)\big) = 0$. Then $1 - int_{T_i}\big(\vee_{k=1}^{\infty} (\mu_k)\big) = 1$. This implies that $cl_{T_i}\big(\wedge_{k=1}^{\infty}(1 - \mu_k)\big) = 1 \longrightarrow (1)$. Since (μ_k)'s are pairwise fuzzy nowhere dense sets in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), by proposition 4.4, $(1 - \mu_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Let $\lambda_k = 1 - \mu_k$. Then we have from (1), $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$, $(i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2). □

Proposition 4.7. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, then (X, T_1, T_2) is a pairwise fuzzy Baire space if and only if $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$, $(i = 1, 2)$.*

Proof. Proof follows from proposition 4.6 and theorem 4.5. ☐

Proposition 4.8. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$ is a pairwise fuzzy dense set in (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2).*

Proof. Let $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$ be a pairwise fuzzy dense set in (X, T_1, T_2). Then $cl_{T_1} cl_{T_2}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1 = cl_{T_2} cl_{T_1}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big)$. But $cl_{T_1}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq \wedge_{k=1}^{\infty}\big(cl_{T_1}(\lambda_k)\big)$ and hence $cl_{T_2} cl_{T_1}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq cl_{T_2}[\wedge_{k=1}^{\infty}\big(cl_{T_1}(\lambda_k)\big)]$. Thus $1 \leq cl_{T_2}[\wedge_{k=1}^{\infty}\big(cl_{T_1}(\lambda_k)\big)] \leq \wedge_{k=1}^{\infty}[cl_{T_2}\big(cl_{T_1}(\lambda_k)\big)]$. Then $\wedge_{k=1}^{\infty}[cl_{T_2}\big(cl_{T_1}(\lambda_k)\big)] = 1$. This implies that, $cl_{T_2}\big(cl_{T_1}(\lambda_k)\big) = 1$. Similarly, we can show that $cl_{T_1}\big(cl_{T_2}(\lambda_k)\big) = 1$. Thus (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, by proposition 4.2, $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets. Therefore, we have $1 - \lambda = \vee_{k=1}^{\infty}(1 - \lambda_k)$, where $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets. Hence $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2). ☐

Proposition 4.9. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$ is a pairwise fuzzy dense set in (X, T_1, T_2), then λ is a pairwise fuzzy residual set in (X, T_1, T_2).*

Proof. Let $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, by proposition 4.8, $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2). Therefore λ is a pairwise fuzzy residual set in (X, T_1, T_2). ☐

Definition 4.10 ([13])**.** A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy submaximal space* if each pairwise fuzzy dense set in (X, T_1, T_2), is a pairwise fuzzy open set in (X, T_1, T_2). That is., if λ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2), then $\lambda \in T_i$ $(i = 1, 2)$.

Proposition 4.11. *If (X, T_1, T_2) is a pairwise fuzzy submaximal space and λ is a pairwise fuzzy first category set, then $1 - \lambda$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy first category set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). By theorem 4.3, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, $(1 - \lambda_k)$'s are pairwise fuzzy open sets in (X, T_1, T_2). Also $1 - \lambda = 1 - \big(\vee_{k=1}^{\infty}(\lambda_k)\big) = \wedge_{k=1}^{\infty}(1 - \lambda_k)$, where $(1 - \lambda_k)$'s are pairwise fuzzy open sets in (X, T_1, T_2). Therefore $1 - \lambda$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). ☐

Proposition 4.12. *If (X, T_1, T_2) is a pairwise fuzzy submaximal space, then every pairwise fuzzy first category set is a pairwise fuzzy F_σ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy first category set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 4.11, $1 - \lambda$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2) and hence λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). ☐

Theorem 4.13 ([13])**.** *If (X, T_1, T_2) is a pairwise fuzzy Baire space, then every pairwise fuzzy residual set is a pairwise fuzzy dense set in (X, T_1, T_2).*

Definition 4.14 ([14]). A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy P-space* if every non-zero pairwise fuzzy G_δ-set in (X, T_1, T_2), is a pairwise fuzzy open set in (X, T_1, T_2). That is, if (X, T_1, T_2) is a pairwise fuzzy P-space if $\lambda \in T_i$, $(i = 1, 2)$ for $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy open sets in (X, T_1, T_2).

Proposition 4.15. *If (X, T_1, T_2) is a pairwise fuzzy submaximal and pairwise fuzzy P-space, then every pairwise fuzzy residual set is a pairwise fuzzy open set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy residual set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 4.12, λ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy P-space, λ is a pairwise fuzzy open set in (X, T_1, T_2). Hence every pairwise fuzzy residual set is a pairwise fuzzy open set in a pairwise fuzzy submaximal and pairwise fuzzy P-space. □

Definition 4.16 ([13]). A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy nodec space* if every non-zero pairwise fuzzy nowhere dense set in (X, T_1, T_2), is a pairwise fuzzy closed set in (X, T_1, T_2). That is., if λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda \in T_i$ $(i = 1, 2)$.

Proposition 4.17. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space, then (X, T_1, T_2) is a pairwise fuzzy submaximal space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space. Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, by proposition 4.2, $1 - \lambda$ is a pairwise fuzzy nowhere dense set. Since (X, T_1, T_2) is a pairwise fuzzy nodec space, $1 - \lambda$ is a pairwise fuzzy closed set. Then λ is a pairwise fuzzy open set in (X, T_1, T_2). Hence every pairwise fuzzy dense set is pairwise fuzzy open set in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy submaximal space. □

Proposition 4.18. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable and pairwise fuzzy second category space, then $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$ where (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2).*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy second category space. Let us assume that $\wedge_{k=1}^{\infty}(\lambda_k) = 0$. Since (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), by proposition 4.2, $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Now $1 - \wedge_{k=1}^{\infty}(\lambda_k) = 1$, implies that $\vee_{k=1}^{\infty}(1 - \lambda_k) = 1$ and $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Hence (X, T_1, T_2) is a pairwise fuzzy first category space which is a contradiction. Therefore $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$ where (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2). □

Proposition 4.19. *If λ is a pairwise fuzzy first category set in a pairwise fuzzy submaximal and pairwise fuzzy P-space (X, T_1, T_2), then λ is not a pairwise fuzzy dense set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy first category set in a pairwise fuzzy submaximal and pairwise fuzzy P-space (X, T_1, T_2). Hence $1 - \lambda$ is a pairwise fuzzy residual set in (X, T_1, T_2). Then, by proposition 4.15, $1 - \lambda$ is a pairwise fuzzy open set in

(X, T_1, T_2), so λ is a pairwise fuzzy closed set. Hence $cl_{T_1}(\lambda) = cl_{T_2}(\lambda) = \lambda$ implies that $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = \lambda \neq 1$ and therefore λ is not a pairwise fuzzy dense set in (X, T_1, T_2). □

Remark 4.20. If $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$ do not imply $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$ in a fuzzy bitopological space (X, T_1, T_2).

Proposition 4.21. *If $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$ for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$ in (X, T_1, T_2).*

Proof. Let $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$ for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2). Then $1 - int_{T_1}int_{T_2}(\lambda) = 1$ and $1 - int_{T_2}int_{T_1}(\lambda) = 1$ imply that $cl_{T_1}cl_{T_2}(1 - \lambda) = 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) = 1$. That is., $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set $1 - \lambda$ in (X, T_1, T_2), we have $cl_{T_1}int_{T_2}(1 - \lambda) = 1$ and $cl_{T_2}int_{T_1}(1 - \lambda) = 1$. Then we have, $int_{T_1}cl_{T_2}(\lambda) = 0$ and $int_{T_2}cl_{T_1}(\lambda) = 0$. Hence $int_{T_1}(\lambda) \leq int_{T_1}cl_{T_2}(\lambda) = 0$ and $int_{T_2}(\lambda) \leq int_{T_2}cl_{T_1}(\lambda) = 0$ implies that $int_{T_1}(\lambda) \leq 0$ and $int_{T_2}(\lambda) \leq 0$. That is., $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$. □

Proposition 4.22. *If $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$ for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2).*

Proof. Let $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$ for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2). Hence $1 - cl_{T_1}cl_{T_2}(\lambda) = 0$ and $1 - cl_{T_2}cl_{T_1}(\lambda) = 0$. This implies that $int_{T_1}int_{T_2}(1 - \lambda) = 0$ and $int_{T_2}int_{T_1}(1 - \lambda) = 0$. Now, by proposition 4.21, $int_{T_1}(1 - \lambda) = 0$ and $int_{T_2}(1 - \lambda) = 0$ and hence $1 - cl_{T_1}(\lambda) = 0$ and $1 - cl_{T_2}(\lambda) = 0$. Therefore $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2). □

Proposition 4.23. *If (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, then*

(i). *$int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, for a fuzzy set λ in (X, T_1, T_2) implies that $int_{T_i}(\lambda) = 1$, $(i = 1, 2)$.*

(ii). *$cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$, for a fuzzy set λ in (X, T_1, T_2) implies that $cl_{T_i}(\lambda) = 1$, $(i = 1, 2)$.*

Proof. Proof follows from propositions 4.21 and 4.22 □

Proposition 4.24. *If (X, T_1, T_2) is a pairwise fuzzy submaximal space, then every pairwise fuzzy residual set is a pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy residual set in (X, T_1, T_2). Then $1 - \lambda$ is a pairwise fuzzy first category set (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 4.12, $1 - \lambda$ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Therefore λ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). □

Theorem 4.25 ([13]). *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space, then (X, T_1, T_2) is not a pairwise Baire space.*

Proposition 4.26. *If every pairwise fuzzy G_δ-set is fuzzy pairwise dense in a pairwise fuzzy submaximal and pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then (X, T_1, T_2) pairwise fuzzy Baire space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy submaximal space in which every pairwise fuzzy G_δ set is pairwise fuzzy dense in (X, T_1, T_2). Let λ be a pairwise fuzzy residual set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 4.24, λ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). By hypothesis, λ is a pairwise fuzzy dense set in (X, T_1, T_2). That is, $cl_{T_1} cl_{T_2}(\lambda) = 1 = cl_{T_2} cl_{T_1}(\lambda)$. Then, by proposition 4.22, $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2). Hence, by theorem 3.2, (X, T_1, T_2) is a pairwise fuzzy Baire space. □

Theorem 4.27 ([13])**.** *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy nodec space, then (X, T_1, T_2) is not a pairwise fuzzy Baire space.*

Proposition 4.28. *If every pairwise fuzzy G_δ-set is a pairwise fuzzy dense set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proof. Let λ be a pairwise fuzzy G_δ-set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space (X, T_1, T_2) such that $cl_{T_1} cl_{T_2}(\lambda) = 1 = cl_{T_2} cl_{T_1}(\lambda)$. By proposition 4.17, (X, T_1, T_2) is a pairwise fuzzy submaximal space. Again, by proposition 4.26, (X, T_1, T_2) is a pairwise fuzzy Baire space. □

5. Conclusions

In this paper several characterizations of pairwise fuzzy Baire spaces are studied. The conditions underwhich a fuzzy bitopological space becomes a pairwise fuzzy Baire space are investigated. For this work, pairwise fuzzy strongly irresolvable space, pairwise fuzzy submaximal space, pairwise fuzzy P-space are considered.

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