Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 4, (April 2015), pp. 665–674
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Sum of fuzzy ideals of Γ-near-rings

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Received 21 August 2014; Accepted 14 October 2014

Abstract. In the present paper we introduce the concept on sum of fuzzy ideals of a Γ-near-ring and the sum of anti fuzzy ideals of a Γ-near-ring.

2010 AMS Classification: 16Y30, 03E72

Keywords: Γ-near-ring, Sum of fuzzy ideals, Direct sum of fuzzy ideals.

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1. Introduction

In 1965, Zadeh [10] has initiated the notion of fuzzy set. Then many researchers were applying it in various branches of mathematics (see [1, 2, 3, 6]). The algebraic system Γ-near-ring was introduced by Satyanarayana [8]. Later several mathematicians worked on this algebraic system. The notion of an anti fuzzy ideals of Γ-near-ring was studied by Srinivas, etc., [9]. Kim and Jun [4] has studied the concept of an anti fuzzy ideals in near-rings. The sum of the fuzzy ideals of a near-ring was studied by Narasimha swamy [7]. Now we are introducing the sum of fuzzy ideals of a Γ-near-ring and also the sum of anti fuzzy ideals of a Γ-near-ring. Also studied the concept of direct sum in both cases.

2. Preliminaries

A non-empty set N with two binary operations "+" and "·" is said to be a left near-ring (see [5]) if it satisfies the following three conditions;
(i) $(N, +)$ is a group (not necessarily abelian),
(ii) $(N, \cdot)$ is a semigroup,
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in N$.

We will use the word "near-ring" to mean "left near-ring". We denote xy instead of $x \cdot y$. Moreover, a near-ring N is said to be a zero-symmetric if $0 \cdot n = 0$ for all $n \in N$, where 0 is the additive identity in N.

Definition 2.1. Let $(M, +)$ be a group (not necessarily abelian) and Γ be a non empty set. Then M is said to be a Γ-near-ring, if there exist a mapping $M \times \Gamma \times M \longrightarrow M$ (the image of (x, α, y) is $x\alpha y$) satisfying the following conditions;
(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.2. A Γ-near-ring M is said to be a zero symmetric Γ-near-ring if $0\alpha n = 0$ for every $n \in M, \alpha \in \Gamma$, where 0 is the additive identity in M.

Definition 2.3. Let M be a Γ-near-ring. A normal subgroup $(I, +)$ of $(M, +)$ is called
(i) a right ideal, if $(x + i)\alpha y - x\alpha y \in I$ for all $x, y \in M, \alpha \in \Gamma, i \in I$,
(ii) a left ideal, if $x\alpha i \in I$ for all $x \in M, \alpha \in \Gamma, i \in I$,
(iii) an ideal, if it is both a left ideal and a right ideal.

A fuzzy set μ on a non-empty A is a mapping $\mu : A \to [0, 1]$.

Definition 2.4. A fuzzy set μ of a Γ-near-ring M is called a fuzzy ideal of M if
(i) $\mu(x - y) \geq Min\{\mu(x), \mu(y)\}$,
(ii) $\mu(y + x - y) \geq \mu(x)$,
(iii) $\mu((x + i)\alpha y - x\alpha y) \geq \mu(i)$ (or equivalently, $\mu(z\alpha y - x\alpha y) \geq \mu(z - x)$),
(iv) $\mu(x\alpha y) \geq \mu(y)$ for all $x, y, z, i \in M$ and $\alpha \in \Gamma$.

If μ satisfies (i), (ii) and (iii) then μ is called a fuzzy right ideal of M. If μ satisfies (i), (ii) and (iv) then μ is called a fuzzy left ideal of M.

Definition 2.5 ([9])**.** A fuzzy set μ of a Γ-near-ring M is called an anti fuzzy ideal of M, if
(i) $\mu(x - y) \leq Max\{\mu(x), \mu(y)\}$,
(ii) $\mu(y + x - y) \leq \mu(x)$,
(iii) $\mu((x + i)\alpha y - x\alpha y) \leq \mu(i)$ (or equivalently, $\mu(z\alpha y - x\alpha y) \leq \mu(z - x)$),
(iv) $\mu(x\alpha y) \leq \mu(y)$ for all $x, y, z, i \in M$ and $\alpha \in \Gamma$.

If μ satisfies (i), (ii) and (iii) then μ is called an anti fuzzy right ideal of a Γ-near-ring M. If μ satisfies (i), (ii) and (iv), then μ is called an anti fuzzy left ideal of a Γ-near-ring M.

Example 2.6 ([9])**.** Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. Define a binary operation "+" on M and a mapping $M \times \Gamma \times M \to M$ by the following tables;

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	a	a

Clearly $(M, +)$ is a group and (i) $x\gamma(y + z) = x\gamma y + x\gamma z$, for every $x, y, z \in M$, $\gamma \in \Gamma$, (ii) $(x\gamma y)\omega z = x\gamma(y\omega z)$ for every $x, y, z \in M$ and $\gamma, \omega \in \Gamma$. Thus M is a $\Gamma-$near-ring. Define a fuzzy set $\mu : M \to [0, 1]$ by $\mu(0) < \mu(a) = \mu(b) = \mu(c)$. The routine calculation shows that, μ is an anti fuzzy ideal of M.

3. Sum and direct sum of fuzzy ideals

Definition 3.1. Let μ and ν be two fuzzy ideals of a zero symmetric Γ-near-ring M. Then the sum $\mu+\nu$ is a fuzzy subset of M defined by

$$(\mu+\nu)(x) = \begin{cases} Sup(min(\mu(y),\nu(z))) & : x = y+z \\ 0 & : otherwise. \end{cases}$$

Theorem 3.2. *If μ and ν are two fuzzy ideals of a zero symmetric Γ-near-ring M, then $\mu+\nu$ is also a fuzzy ideal of M.*

Proof. Let $x, y, u \in M$ and $\alpha \in \Gamma$.
(i) Put $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, x_2, y_1, y_2 \in M$. Then

$$\begin{aligned} x - y &= x_1 + x_2 - (y_1 + y_2) \\ &= x_1 - y_1 + y_1 + x_2 - (y_1 + y_2). \\ (\mu+\nu)(x-y) &= (\mu+\nu)(x_1 - y_1 + y_1 + x_2 - y_1 - y_2) \\ &= \bigvee(\mu(x_1 - y_1) \wedge \nu(y_1 + x_2 - y_1 - y_2)) \\ &\geq \bigvee[(\mu(x_1) \wedge \mu(y_1)) \wedge (\nu(y_1 + x_2 - y_1) \wedge \nu(y_2))] \\ &\geq \bigvee[(\mu(x_1) \wedge \mu(y_1)) \wedge (\nu(x_2) \wedge \nu(y_2))] \\ &\geq (\bigvee(\mu(x_1) \wedge \nu(x_2))) \wedge (\bigvee(\mu(y_1) \wedge \nu(y_2))) \\ &= (\mu+\nu)(x) \wedge (\mu+\nu)(y). \end{aligned}$$

(ii) Put $x = x_1 + x_2$ where $x_1, x_2 \in M$. Then

$$\begin{aligned} y + x - y &= y + x_1 + x_2 - y = y + x_1 - y + y + x_2 - y. \\ (\mu+\nu)(y + x - y) &= (\mu+\nu)(y + x_1 - y + y + x_2 - y) \\ &= \bigvee[\mu(y + x_1 - y) \wedge \nu(y + x_2 - y)] \\ &\geq \bigvee[\mu(x_1) \wedge \nu(x_2)] \\ &= (\mu+\nu)(x). \end{aligned}$$

(iii) Let $u - x = t_1 + t_2$; $t_1, t_2 \in M$. Which implies $u = t_1 + t_2 + x$. Then

$$\begin{aligned} u\alpha y - x\alpha y &= (t_1 + t_2 + x)\alpha y - x\alpha y \\ &= (t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y + (t_2 + x)\alpha y - x\alpha y. \\ (\mu+\nu)(u\alpha y - x\alpha y) &= (\mu+\nu)((t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y + (t_2 + x)\alpha y - x\alpha y) \\ &= \bigvee[\mu((t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y) \wedge \nu((t_2 + x)\alpha y - x\alpha y)] \\ &\geq \bigvee[\mu(t_1) \wedge \nu(t_2)] \\ &= (\mu+\nu)(u - x). \end{aligned}$$

(iv) Put $y = y_1 + y_2$; $y_1, y_2 \in M$. Then

$$\begin{aligned} (\mu+\nu)(x\alpha y) &= (\mu+\nu)(x\alpha y_1 + x\alpha y_2) \\ &= \bigvee[\mu(x\alpha y_1) \wedge \nu(x\alpha y_2)] \end{aligned}$$

$$\begin{aligned} &\geq \bigvee [\mu(y_1) \wedge \nu(y_2)] \\ &= (\mu + \nu)(y). \end{aligned}$$

Hence $\mu + \nu$ is a fuzzy ideal of M. □

Example 3.3. From example 2.6, M is a zero symmetric Γ-near-ring. Now define two fuzzy sets $\mu : M \to [0, 1]$ and $\nu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.8 & : x = 0 \\ 0.5 & : otherwise \end{cases}$$

and

$$\nu(x) = \begin{cases} 1 & : x = 0 \\ 0.2 & : otherwise. \end{cases}$$

The routine calculation shows that, μ and ν are fuzzy ideals of M. Now for any $y, z \in M$,

$$(\mu + \nu)(0) = \bigvee_{0=y+z} \{min(\mu(y), \nu(z))\}$$
$$= \bigvee \{min(\mu(0), \nu(0)), min(\mu(a), \nu(a)), min(\mu(b), \nu(b)), min(\mu(c), \nu(c))\}$$
$$= \bigvee \{0.8, 0.2, 0.2, 0.2\} = 0.8,$$

$$(\mu + \nu)(a) = \bigvee_{a=y+z} \{min(\mu(y), \nu(z))\}$$
$$= \bigvee \{min(\mu(0), \nu(a)), min(\mu(a), \nu(0)), min(\mu(b), \nu(c)), min(\mu(c), \nu(b))\}$$
$$= \bigvee \{0.2, 0.5, 0.2, 0.2\} = 0.5,$$

$$(\mu + \nu)(b) = \bigvee_{b=y+z} \{min(\mu(y), \nu(z))\}$$
$$= \bigvee \{min(\mu(0), \nu(b)), min(\mu(a), \nu(c)), min(\mu(b), \nu(0)), min(\mu(c), \nu(a))\}$$
$$= \bigvee \{0.2, 0.2, 0.5, 0.2\} = 0.5,$$

$$(\mu + \nu)(c) = \bigvee_{c=y+z} \{min(\mu(y), \nu(z))\}$$
$$= \bigvee \{min(\mu(0), \nu(c)), min(\mu(a), \nu(b)), min(\mu(b), \nu(a)), min(\mu(c), \nu(0))\}$$
$$= \bigvee \{0.2, 0.2, 0.2, 0.5\} = 0.5.$$

Therefore

$$(\mu + \nu)(x) = \begin{cases} 0.8 & : x = 0 \\ 0.5 & : otherwise. \end{cases}$$

The routine calculation shows that, $\mu + \nu$ is a fuzzy ideal of M.

Now we extend the above theorem 3.2 to the sum of finite number of fuzzy ideals of a zero symmetric Γ-near-ring M.

Definition 3.4. Let M be a zero symmetric Γ-near-ring and let $\mu_1, \mu_2, ..., \mu_n$ be the fuzzy ideals of a Γ-near-ring M. For any $x \in M$, put
$S(x) = \{\mu_1(x_1) \wedge \mu_2(x_2) \wedge ... \wedge \mu_n(x_n) : x = x_1 + x_2 + ... + x_n, x_i \in M, i = 1 \text{ to } n\}$. Define $(\mu_1 + \mu_2 + ... + \mu_n)(x) = SupS(x) = Sup\{\mu_1(x_1) \wedge \mu_2(x_2) \wedge ... \wedge \mu_n(x_n) : x = x_1 + x_2 + ... + x_n\}$.

Remark 3.5 ([7]). Let $x = x_1+x_2+...+x_n$. Consider a transposition of the indices $(1, k), k > 1$. Then $x = x_1+x_2+...+x_{k-1}+x_k+x_{k+1}+...+x_n = y+x_k-y+x_1+x_2+...+x_{k-1}+x_{k+1}+...+x_n($ where $y = x_1+x_2+...+x_{k-1}) = (y+x_k-y)+z-z+x_1+z+x_{k+1}+...+x_n($ where $z = x_2+...+x_{k-1}) = x'_k+z+x'_1+x_{k+1}+...+x_n($ where $x'_k = y + x_k - y, x'_1 = -z + x_1 + z) = x'_k + x_2 + ... + x_{k-1} + x'_1 + x_{k+1} + ... + x_n$. Thus $\mu_1(x'_k) \wedge \mu_2(x_2) \wedge ... \wedge \mu_{k-1}(x_{k-1}) \wedge \mu_k(x'_1) \wedge \mu_{k+1}(x_{k+1}) \wedge .... \wedge \mu_n(x_n) = \mu_1(x_k) \wedge \mu_2(x_2) \wedge ... \wedge \mu_{k-1}(x_{k-1}) \wedge \mu_K(x_1) \wedge \mu_{k+1}(x_{k+1}) \wedge .... \wedge \mu_n(x_n) \in S(x)$. This is true for every transposition (i, j) of the indices. Since every permutation is a product of transpositions, then for any permutation $\begin{pmatrix} 1 & 2 & . & . & . & n \\ i_1 & i_2 & . & . & . & i_n \end{pmatrix}$ we have $\mu_1(x_{i_1}) \wedge \mu_2(x_{i_2}) \wedge ... \wedge \mu_n(x_{i_n})$ belongs to $S(x)$ for $x = x_1 + x_2 + ... + x_n$. Hence $\mu_1 + \mu_2 + ... + \mu_n = \mu_{i_1} + \mu_{i_2} + ... + \mu_{i_n}$.

Theorem 3.6. *Let M be a zero symmetric Γ-near-ring. If $\mu_1, \mu_2, ..., \mu_n$ are the fuzzy ideals of M, then $\mu_1 + \mu_2 + ... + \mu_n$ is also a fuzzy ideal of M.*

Proof. Put $\mu = \mu_1 + \mu_2 + ... + \mu_n$.

(i) Let $x = x_1 + x_2 + ... + x_n$, $y = y_1 + y_2 + ... + y_n$;$x_i, y_i \in M, i = 1, 2, ..., n$.Then $x - y = x_1 + x_2 + ... + x_n - y_1 - y_2 - ... - y_n$. This can be expressed as $x - y = x'_1 - y'_1 + x'_2 - y'_2 + ... + x'_n - y'_n$, where x'_i is a conjugate of x_i and y'_i is a conjugate of y_i. Therefore $x-y = (x'_1-y'_1)+(x'_2-y'_2)+...+(x'_n-y'_n)$. Which implies $\mu(x-y) = \mu((x'_1-y'_1)+(x'_2-y'_2)+...+(x'_n-y'_n)) = \bigvee[\mu_1(x'_1-y'_1)\wedge\mu_2(x'_2-y'_2)\wedge...\wedge\mu_n(x'_n-y'_n)] \geq \bigvee[\mu_1(x'_1) \wedge \mu_1(y'_1) \wedge \mu_2(x'_2) \wedge \mu_2(y'_2) \wedge ... \wedge \mu_n(x'_n) \wedge \mu_n(y'_n)] = [\bigvee(\mu_1(x'_1) \wedge \mu_2(x'_2) \wedge ... \wedge \mu_n(x'_n))] \wedge [\bigvee(\mu_1(y'_1) \wedge \mu_2(y'_2) \wedge ... \wedge \mu_n(y'_n))] = SupS(x) \wedge SupS(y) = \mu(x) \wedge \mu(y)$.

(ii) Let $x, y \in M$ and $x = x_1 + x_2 + ... + x_n; x_i \in M, i = 1, 2, ..., n$. Then $y+x-y = y+x_1+x_2+...+x_n-y = y+x_1-y+y+x_2-y+y+x_3-y+...+y+x_n-y$. This implies that $\mu(y + x - y) = \mu(y + x_1 - y + y + x_2 - y + ... + y + x_n - y) = \bigvee[\mu_1(y+x_1-y)\wedge\mu_2(y+x_2-y)\wedge...\wedge\mu_n(y+x_n-y)] \geq \bigvee[\mu_1(x_1)\wedge\mu_2(x_2)\wedge...\wedge\mu_n(x_n)]. = SupS(x) = \mu(x)$.

(iii) Let $x, y, u \in M$ and $\alpha \in \Gamma$. And let $u - x = t_1 + t_2 + ... + t_n; t_i \in M, i = 1, 2, ..., n$. Which implies $u = t_1 + t_2 + ... + t_n + x$. And so $u\alpha y - x\alpha y = (t_1 + t_2 + ... + t_n + x)\alpha y - x\alpha y = (t_1+t_2+...+t_n+x)\alpha y - (t_2+t_3+...+t_n+x)\alpha y + (t_2+t_3+...+t_n+x)\alpha y - (t_3+t_4+...+t_n+x)\alpha y + (t_3+t_4+...+t_n+x)\alpha y - ... + (t_n+x)\alpha y - x\alpha y$. Now $\mu(u\alpha y - x\alpha y) = \mu\{(t_1 + t_2 + ... + t_n + x)\alpha y - (t_2 + t_3 + ... + t_n + x)\alpha y + (t_2 + t_3 + ... + t_n+x)\alpha y - (t_3+t_4+...+t_n+x)\alpha y + (t_3+t_4+...+t_n+x)\alpha y - ... + (t_n+x)\alpha y - x\alpha y\} = \bigvee\{\mu_1((t_1+t_2+...+t_n+x)\alpha y - (t_2+t_3+...+t_n+x)\alpha y) \wedge \mu_2((t_2+t_3+...+t_n+x)\alpha y - (t_3+t_4+...+t_n+x)\alpha y) \wedge ... \wedge \mu_n((t_n+x)\alpha y - x\alpha y)\} \geq \bigvee(\mu_1(t_1) \wedge \mu_2(t_2) \wedge ... \wedge \mu_n(t_n)) = SupS(u - x) = \mu(u - x)$.

(iv) Let $x, y \in M$ and $\alpha \in \Gamma$. Put $y = y_1+y_2+...+y_n; y_i \in M, i = 1, 2, ..., n$. Then $\mu(x\alpha y) = \mu(x\alpha(y_1 + y_2 + ... + y_n)) = \mu(x\alpha y_1 + x\alpha y_2 + ... + x\alpha y_n) = \bigvee(\mu_1(x\alpha y_1) \wedge \mu_2(x\alpha y_2) \wedge ... \wedge \mu_n(x\alpha y_n)) \geq \bigvee(\mu_1(y_1) \wedge \mu_2(y_2) \wedge ... \wedge \mu_n(y_n)) = SupS(y) = \mu(y)$. Hence μ is a fuzzy ideal of M. □

Definition 3.7. Let M be a zero symmetric Γ-near-ring and $\mu_1, \mu_2, ..., \mu_n$ be the fuzzy ideals of M. Then the sum $\mu = \mu_1 + \mu_2 + ... + \mu_n$ is said to be direct, if $(\mu_1 + \mu_2 + ... + \mu_{i-1} + \mu_{i+1} + ... + \mu_n) \wedge \mu_i = 0$.

Theorem 3.8. *Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ be the direct sum of Γ-near-rings $M_1, M_2, ..., M_n$ with left or right identity $e = (e_1, e_2, ..., e_n)$ and μ be a fuzzy ideal of M. Then there exists fuzzy ideals $\mu_1, \mu_2, ..., \mu_n$ of M such that $\mu = \mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n$.*

Proof. Let $x_i = (0, 0, ..., 0, x_i, 0, ..., 0)$ and $e_i = (0, 0, ..., 0, e_i, 0, ..., 0), \alpha \in \Gamma$. Then for $x = (x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$, we have $\mu(x) = \mu(x_1 + x_2 + ... + x_n) \geq \mu(x_1) \wedge \mu(x_2) \wedge ... \wedge \mu(x_n)$. But $\mu(x_i) = \mu(e_i \alpha x) \geq \mu(x)$, for $i = 1, 2, ..., n$. That is $\mu(x_1) \wedge \mu(x_2) \wedge ... \wedge \mu(x_n) \geq \mu(x)$ Thus $\mu(x) = \mu(x_1) \wedge \mu(x_2) \wedge ... \wedge \mu(x_n)$. Define μ_i on M by

$$\mu_i(x) = \begin{cases} \mu(x) & : x \in M_i \\ 0 & : otherwise. \end{cases}$$

Hence $\mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n = \mu$. □

4. Sum, Direct sum of anti fuzzy ideals

Definition 4.1. Let μ and ν be two anti fuzzy ideals of a zero symmetric Γ-near-ring M. Then the sum $\mu + \nu$ is a fuzzy set of M defined by

$$(\mu + \nu)(x) = \begin{cases} Inf(max(\mu(y), \nu(z))) & : x = y + z \\ 0 & : otherwise. \end{cases}$$

Theorem 4.2. *If μ and ν are two anti fuzzy ideals of a zero symmetric Γ-near-ring M, then $\mu + \nu$ is also an anti fuzzy ideal of M.*

Proof. Let $x, y, u \in M$ and $\alpha \in \Gamma$.
(i) Put $x = x_1 + x_2,\ y = y_1 + y_2;\ x_1, x_2, y_1, y_2 \in M$. Then

$$\begin{aligned} x - y &= x_1 + x_2 - (y_1 + y_2) \\ &= x_1 - y_1 + y_1 + x_2 - (y_1 + y_2). \\ (\mu + \nu)(x - y) &= (\mu + \nu)(x_1 - y_1 + y_1 + x_2 - y_1 - y_2) \\ &= \bigwedge(\mu(x_1 - y_1) \vee \nu(y_1 + x_2 - y_1 - y_2)) \\ &\leq \bigwedge[(\mu(x_1) \vee \mu(y_1)) \vee (\nu(y_1 + x_2 - y_1) \vee \nu(y_2))] \\ &\leq \bigwedge[(\mu(x_1) \vee \mu(y_1)) \vee (\nu(x_2) \vee \nu(y_2))] \\ &= [\bigwedge(\mu(x_1) \vee \nu(x_2))] \vee [\bigwedge(\mu(y_1) \vee \nu(y_2))] \\ &= (\mu + \nu)(x) \vee (\mu + \nu)(y). \end{aligned}$$

(ii) Put $x = x_1 + x_2;\ x_1, x_2 \in M$. Then

$$\begin{aligned} y + x - y &= y + x_1 + x_2 - y = y + x_1 - y + y + x_2 - y. \\ (\mu + \nu)(y + x - y) &= (\mu + \nu)(y + x_1 - y + y + x_2 - y) \\ &= \bigwedge[\mu(y + x_1 - y) \vee \nu(y + x_2 - y)] \\ &\leq \bigwedge[\mu(x_1) \vee \nu(x_2)] \\ &= (\mu + \nu)(x). \end{aligned}$$

(iii) Let $u - x = t_1 + t_2;\ t_1, t_2 \in M$. Which implies $u = t_1 + t_2 + x$. Then

$$\begin{aligned}
u\alpha y - x\alpha y &= (t_1 + t_2 + x)\alpha y - x\alpha y \\
&= (t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y + (t_2 + x)\alpha y - x\alpha y. \\
(\mu+\nu)(u\alpha y - x\alpha y) &= (\mu+\nu)((t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y + (t_2 + x)\alpha y - x\alpha y) \\
&= \bigwedge[\mu((t_1 + t_2 + x)\alpha y - (t_2 + x)\alpha y) \vee \nu((t_2 + x)\alpha y - x\alpha y)] \\
&\leq \bigwedge\{\mu(t_1) \vee \nu(t_2)\} \\
&= (\mu+\nu)(u - x).
\end{aligned}$$

(iv) Put $y = y_1 + y_2$. Then

$$\begin{aligned}
(\mu+\nu)(x\alpha y) &= (\mu+\nu)(x\alpha y_1 + x\alpha y_2) \\
&= \bigwedge[\mu(x\alpha y_1) \vee \nu(x\alpha y_2)] \\
&\leq \bigwedge[\mu(y_1) \vee \nu(y_2)] \\
&= (\mu+\nu)(y).
\end{aligned}$$

Hence $\mu + \nu$ is an anti fuzzy ideal of M. □

Example 4.3. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be a non-empty set. Define a binary operation " + " on M and a mapping $M \times \Gamma \times M \to M$ by the following tables;

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	a	a

Clearly, $(M, +)$ is a group and
(i) $x\gamma(y + z) = x\gamma y + x\gamma z$, for every $x, y, z \in M$, $\gamma \in \Gamma$,
(ii) $(x\gamma y)\omega z = x\gamma(y\omega z)$ for every $x, y, z \in M$ and $\gamma, \omega \in \Gamma$.
And also $0\alpha n = 0$ for all $\alpha \in \Gamma, n \in M$; 0 is the additive identity in M. Thus M is a zero symmetric $\Gamma-$near-ring. Define two fuzzy sets μ and ν on M as follows;
$\mu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.5 & : x = 0 \\ 0.8 & : otherwise \end{cases}$$

and $\nu : M \to [0, 1]$ by

$$\nu(x) = \begin{cases} 0.2 & : x = 0 \\ 0.9 & : otherwise. \end{cases}$$

Then the routine calculation shows that μ and ν are anti fuzzy ideals of M. Now

$$\begin{aligned}
(\mu+\nu)(0) &= \bigwedge_{0=y+z} \{max(\mu(y), \nu(z))\} \\
&= \bigwedge\{max(\mu(0), \nu(0)), max(\mu(a), \nu(a)), max(\mu(b), \nu(b)), max(\mu(c), \nu(c))\} \\
&= \bigwedge\{0.5, 0.9, 0.9, 0.9\} = 0.5,
\end{aligned}$$

$$(\mu+\nu)(a) = \bigwedge_{a=y+z}\{max(\mu(y),\nu(z))\}$$
$$= \bigwedge\{max(\mu(0),\nu(a)), max(\mu(a),\nu(0)), max(\mu(b),\nu(c)), max(\mu(c),\nu(b))\}$$
$$= \bigwedge\{0.9, 0.8, 0.9, 0.9\} = 0.8,$$
$$(\mu+\nu)(b) = \bigwedge_{b=y+z}\{max(\mu(y),\nu(z))\}$$
$$= \bigwedge\{max(\mu(0),\nu(b)), max(\mu(a),\nu(c)), max(\mu(b),\nu(0)), max(\mu(c),\nu(a))\}$$
$$= \bigwedge\{0.9, 0.9, 0.8, 0.9\} = 0.8,$$
$$(\mu+\nu)(c) = \bigwedge_{c=y+z}\{min(\mu(y),\nu(z))\}$$
$$= \bigwedge\{max(\mu(0),\nu(c)), max(\mu(a),\nu(b)), max(\mu(b),\nu(a)), max(\mu(c),\nu(0))\}$$
$$= \bigwedge\{0.9, 0.9, 0.9, 0.8\} = 0.8.$$

Therefore,

$$(\mu+\nu)(x) = \begin{cases} 0.5 & : x = 0 \\ 0.8 & : otherwise. \end{cases}$$

We can easily verify that, $\mu+\nu$ is an anti fuzzy ideal of M.

Now we extend the theorem 4.2 to the sum of finite number of anti fuzzy ideals of a zero symmetric Γ-near-ring M.

Definition 4.4. Let M be a zero symmetric Γ-near-ring and let $\mu_1, \mu_2, ..., \mu_n$ be the anti fuzzy ideals of a Γ-near-ring M. For any $x \in M$ put $I(x) = \{\mu_1(x_1) \vee \mu_2(x_2) \vee ...\mu_n(x_n) : x = x_1+x_2+...+x_n, x_i \in M, i = 1 \text{ to } n\}$. Define $(\mu_1+\mu_2+...+\mu_n)(x) = Inf I(x) = Inf\{\mu_1(x_1) \vee \mu_2(x_2) \vee ... \vee \mu_n(x_n) : x = x_1 + x_2 + ... + x_n\}$.

Remark 4.5. Let $x = x_1 + x_2 + ... + x_n$. Consider a transposition of the indices $(1, k), k > 1$. Then $x = x_1+x_2+...+x_{k-1}+x_k+x_{k+1}+...+x_n = y+x_k-y+x_1+x_2+...+x_{k-1}+x_{k+1}+...+x_n$(where $y = x_1+x_2+...+x_{k-1}$) $= (y+x_k-y)+z-z+x_1+z+x_{k+1}+...+x_n$(where $z = x_2+...+x_{k-1}$) $= x'_k+z+x'_1+x_{k+1}+...+x_n$(where $x'_k = y + x_k - y, x'_1 = -z + x_1 + z$) $= x'_k + x_2 + ... + x_{k-1} + x'_1 + x_{k+1} + ... + x_n$. Thus $\mu_1(x'_k) \vee \mu_2(x_2) \vee ... \vee \mu_{k-1}(x_{k-1}) \vee \mu_k(x'_1) \vee \mu_{k+1}(x_{k+1}) \vee .... \vee \mu_n(x_n) = \mu_1(x_k) \vee \mu_2(x_2) \vee ... \vee \mu_{k-1}(x_{k-1}) \vee \mu_K(x_1) \vee \mu_{k+1}(x_{k+1}) \vee .... \vee \mu_n(x_n) \in I(x)$. This is true for every transposition (i, j) of the indices. As every permutation is a product of transpositions, then for any permutation $\begin{pmatrix} 1 & 2 & . & . & . & n \\ i_1 & i_2 & . & . & . & i_n \end{pmatrix}$ we have $\mu_1(x_{i_1}) \vee \mu_2(x_{i_2}) \vee ... \vee \mu_n(x_{i_n})$ belongs to $I(x)$ for any $x = x_1 + x_2 + ... + x_n$. Hence $\mu_1 + \mu_2 + ... + \mu_n = \mu_{i_1} + \mu_{i_2} + ... + \mu_{i_n}$.

Theorem 4.6. *Let M be a zero symmetric Γ-near-ring. If $\mu_1, \mu_2, ..., \mu_n$ are the anti fuzzy ideals of M, then $\mu_1 + \mu_2 + ... + \mu_n$ is an anti fuzzy ideal of M.*

Proof. Put $\mu = \mu_1 + \mu_2 + ... + \mu_n$.

(i) Let $x = x_1 + x_2 + ... + x_n$, $y = y_1 + y_2 + ... + y_n$;$x_i, y_i \in M, i = 1, 2, ..., n$.Then $x - y = x_1 + x_2 + ... + x_n - y_1 - y_2 - ... - y_n$. This can be expressed as $x - y =$

$x_1' - y_1' + x_2' - y_2' + ... + x_n' - y_n'$, where x_i' is a conjugate of x_i and y_i' is a conjugate of y_i. Therefore $x - y = (x_1' - y_1') + (x_2' - y_2') + ... + (x_n' - y_n')$. Which implies $\mu(x-y) = \mu((x_1'-y_1')+(x_2'-y_2')+...+(x_n'-y_n')) = \bigwedge[\mu_1(x_1'-y_1') \vee \mu_2(x_2'-y_2') \vee ... \vee \mu_n(x_n'-y_n')] \leq \bigwedge[\mu_1(x_1') \vee \mu_1(y_1') \vee \mu_2(x_2') \vee \mu_2(y_2') \vee ... \vee \mu_n(x_n') \vee \mu_n(y_n')] = [\bigwedge(\mu_1(x_1') \vee \mu_2(x_2') \vee ... \vee \mu_n(x_n'))] \vee [\bigwedge(\mu_1(y_1') \vee \mu_2(y_2') \vee ... \vee \mu_n(y_n'))] = InfI(x) \vee InfI(y) = \mu(x) \vee \mu(y)$.

(ii) Let $x, y \in M$ and $x = x_1 + x_2 + ... + x_n; x_i \in M, i = 1, 2, ..., n$. Then $y+x-y = y+x_1+x_2+...+x_n-y = y+x_1-y+y+x_2-y+y+x_3-y+...+y+x_n-y$. Which implies $\mu(y + x - y) = \mu(y + x_1 - y + y + x_2 - y + ... + y + x_n - y) = \bigwedge[\mu_1(y+x_1-y) \vee \mu_2(y+x_2-y) \vee ... \vee \mu_n(y+x_n-y)] \leq \bigwedge[\mu_1(x_1) \vee \mu_2(x_2) \vee ... \vee \mu_n(x_n)]. = InfI(x) = \mu(x)$.

(iii) Let $x, y, u \in M$ and $\alpha \in \Gamma$. And let $u - x = t_1 + t_2 + ... + t_n; t_i \in M, i = 1, 2, ..., n$. Which implies $u = t_1 + t_2 + ... + t_n + x$. And so $u\alpha y - x\alpha y = (t_1 + t_2 + ... + t_n + x)\alpha y - x\alpha y = (t_1 + t_2 + ... + t_n + x)\alpha y - (t_2 + t_3 + ... + t_n + x)\alpha y + (t_2 + t_3 + ... + t_n + x)\alpha y - (t_3 + t_4 + ... + t_n + x)\alpha y + (t_3 + t_4 + ... + t_n + x)\alpha y - ... + (t_n + x)\alpha y - x\alpha y$. Thus $\mu(u\alpha y - x\alpha y) = \mu\{(t_1 + t_2 + ... + t_n + x)\alpha y - (t_2 + t_3 + ... + t_n + x)\alpha y + (t_2 + t_3 + ... + t_n + x)\alpha y - (t_3 + t_4 + ... + t_n + x)\alpha y + (t_3 + t_4 + ... + t_n + x)\alpha y - ... + (t_n + x)\alpha y - x\alpha y\} = \bigwedge\{\mu_1((t_1 + t_2 + ... + t_n + x)\alpha y - (t_2 + t_3 + ... + t_n + x)\alpha y) \vee \mu_2((t_2 + t_3 + ... + t_n + x)\alpha y - (t_3 + t_4 + ... + t_n + x)\alpha y) \vee ... \vee \mu_n((t_n + x)\alpha y - x\alpha y)\} \leq \bigwedge(\mu_1(t_1) \vee \mu_2(t_2) \vee ... \vee \mu_n(t_n)) = InfI(u - x) = \mu(u - x)$.

(iv) Let $x, y \in M$ and $\alpha \in \Gamma$. Put $y = y_1 + y_2 + ... + y_n; y_i \in M, i = 1, 2, ..., n$. Then $\mu(x\alpha y) = \mu(x\alpha(y_1 + y_2 + ... + y_n)) = \mu(x\alpha y_1 + x\alpha y_2 + ... + x\alpha y_n) = \bigwedge(\mu_1(x\alpha y_1) \vee \mu_2(x\alpha y_2) \vee ... \vee \mu_n(x\alpha y_n)) \leq \bigwedge(\mu_1(y_1) \vee \mu_2(y_2) \vee ... \vee \mu_n(y_n)) = InfI(y) = \mu(y)$. Hence $\mu_1 + \mu_2 + ... + \mu_n$ is an anti fuzzy ideal of M. □

Definition 4.7. Let M be a zero symmetric Γ-near-ring and $\mu_1, \mu_2, ..., \mu_n$ be the anti fuzzy ideals of M. Then a sum $\mu = \mu_1 + \mu_2 + ... + \mu_n$ is said to be direct, if $(\mu_1 + \mu_2 + ... + \mu_{i-1} + \mu_{i+1} + ... + \mu_n) \vee \mu_i = 0$.

Theorem 4.8. *Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ be the direct sum of Γ-near-rings $M_1, M_2, ..., M_n$ with left or right identity $e = (e_1, e_2, ..., e_n)$ and μ be an anti fuzzy ideal of M. Then there exists anti fuzzy ideals $\mu_1, \mu_2, ..., \mu_n$ of M such that $\mu = \mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n$.*

Proof. Let $x_i = (0, 0, ..., 0, x_i, 0, ..., 0)$ and $e_i = (0, 0, ..., 0, e_i, 0, ..., 0), \alpha \in \Gamma$.
Then for $x = (x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$, we have $\mu(x) = \mu(x_1 + x_2 + ... + x_n) \leq \mu(x_1) \vee \mu(x_2) \vee ... \vee \mu(x_n)$. But $\mu(x_i) = \mu(e_i\alpha x) \leq \mu(x)$, for $i = 1, 2, ..., n$. That is $\mu(x_1) \vee \mu(x_2) \vee ... \vee \mu(x_n) \leq \mu(x)$. Thus $\mu(x) = \mu(x_1) \vee \mu(x_2) \vee ... \vee \mu(x_n)$. Define μ_i on M by

$$\mu_i(x) = \begin{cases} \mu(x) & : x \in M_i \\ 0 & : otherwise. \end{cases}$$

Hence $\mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n = \mu$. □

Acknowledgements. The authors wish to thank the anonymous reviewers for their comments and valuable suggestions.

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