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A new closure operator and some kinds of fuzzy super continuous functions on smooth fuzzy topological spaces

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Abstract. We introduce the concepts R^r_τ-closure, R^r_τ-interior in a smooth fuzzy topological space, and we study several notions of continuities such as fuzzy super r_1-continuity, fuzzy super $[r,q]_1$-continuity, fuzzy super r_2-continuity, fuzzy super $[r,q]_2$-continuity, fuzzy super r_3-continuity, fuzzy super r_4-continuity, which are defined by using these closure operators. The properties of these functions are also obtained.

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1. Introduction

Ramadan [22] defined fuzzy topology on a fuzzy set in Šostak's sense [29] under the name of "*smooth fuzzy topological spaces*". Many works on smooth fuzzy topological spaces are going on. For example, we refer to [1, 2, 5, 6, 27, 28]. The fuzzy proper function and its continuity on Chang fuzzy topological spaces are introduced by Chakraborty and Ahsanullah [4]. Chaudhuri and Das [7] proved the equivalent conditions for continuity of fuzzy proper function in the context of Chang fuzzy topology. Fath Allah and Mahmoud [8] introduced the fuzzy graph, strongly fuzzy graph of a proper fuzzy proper function on Chang fuzzy topological space. The notions of smooth fuzzy continuity and weakly smooth fuzzy continuity of a fuzzy proper function on smooth fuzzy topological spaces and their properties are discussed in [22]. Roopkumar and Kalaivani [23] obtained the relations between continuity of fuzzy proper function on a fuzzy set and the continuity of fuzzy proper function at every fuzzy point belonging to the fuzzy set in the context of smooth fuzzy topological spaces. They also defined the projection maps as fuzzy proper functions and proved their properties in [23].

Mahmoud et.al. [19] introduced fuzzy semicontinuity of fuzzy proper function, fuzzy separation axioms and examined the validity of some characterization of these concepts. They also introduced fuzzy semi connected and fuzzy semi compact spaces and some of their properties are discussed. In [9], fuzzy γ-continuity of fuzzy proper function fuzzy γ-retracts in Chang fuzzy topology on fuzzy sets are introduced and some of their properties are established. In [24], (α,β)-weakly smooth fuzzy continuous proper function is introduced and its properties are derived. Further, in the same article, it is established that the product of connected sets is not connected for several notions of connectedness in a smooth fuzzy topological space and connectedness of images of smooth connected fuzzy sets under (α,β)-weakly smooth fuzzy continuous functions are also investigated.

Recently, there are plenty of research works on generalized/weaker forms of open sets such as fuzzy r-preopen, fuzzy r-semiopen sets, fuzzy semiopen sets, different notions of interior and closure operators, and weaker forms of continuous functions such as and weaker forms of fuzzy continuity such as fuzzy r-semicontinuity, fuzzy super continuity, fuzzy δ-continuity, fuzzy almost continuity maps, α-I-continuous functions, fuzzy γ-continuity, etc., For example, we refer to [3, 10, 11, 14, 15, 17, 18, 20, 21, 25, 26, 30, 31] .

In this paper, we introduce R^r_τ-closure, R^r_τ-interior and obtain their properties in a smooth fuzzy topological spaces using which we introduce various types of continuity of fuzzy proper functions. We also establish the relations among these different types of continuous proper functions, by proving lot of results and providing sufficient number of counterexamples wherever required.

2. Preliminaries

Let X, S be non-empty sets. We denote by I, I_0, I^X, 0_X, μ and ν, respectively the unit interval $[0,1]$, the interval $(0,1]$, the set of all fuzzy subsets of X, the zero function on X, a fixed fuzzy subset of X and a fixed fuzzy subset of S. For $X=\{x_1,x_2,\ldots,x_n\}$ and $\lambda_i\in I$, $i\in\{1,2,\ldots,n\}$, we denote the fuzzy subset μ of X which maps x_i to λ_i for every $i=1,2,\ldots,n$ by $\mu^{[\lambda_1,\lambda_2,\ldots,\lambda_n]}_{[x_1,x_2,\ldots,x_n]}$. A fuzzy point [15] in X is defined by $P^\lambda_x(t)=\begin{cases}\lambda & \text{if } t=x\\ 0 & \text{if } t\neq x\end{cases}$, where $0<\lambda\leq 1$. By $P^\lambda_x\in\mu$, we mean that $\lambda\leq\mu(x)$.

Definition 2.1 ([22]). Let $\Im_\mu=\{U\in I^X : U\leq\mu\}$. A smooth fuzzy topology on a fuzzy set $\mu\in I^X$ is a map $\tau:\Im_\mu\to I$, satisfying the following axioms:

(1) $\tau(0_X)=\tau(\mu)=1$,
(2) $\tau(A_1\wedge A_2)\geq\tau(A_1)\wedge\tau(A_2),\forall A_1,A_2\in\Im_\mu$,
(3) $\tau(\bigvee_{i\in\Gamma}A_i)\geq\bigwedge_{i\in\Gamma}\tau(A_i)$ for every family $(A_i)_{i\in\Gamma}\subseteq\Im_\mu$.

The pair (μ,τ) is called a smooth fuzzy topological space.

A fuzzy subset $U\leq\mu$ is called fuzzy open if $\tau(U)>0$ and is called fuzzy closed if $\tau(\mu-U)>0$.

Definition 2.2 ([4])**.** Let $U, V \in \Im_\mu$. We say that U and V are quasi-coincident referred to μ (written as $UqV[\mu]$) if there exists $x \in X$ such that $U(x)+V(x) > \mu(x)$. If U is not quasi-coincident with V, then we write, $U\bar{q}V[\mu]$.

A fuzzy set $U \in \Im_\mu$ is called a q-neighborhood of a fuzzy point P_x^λ in μ if $P_x^\lambda qU[\mu]$ and $\tau(U) > 0$.

Definition 2.3 ([4])**.** Let $\mu \in I^X$ and $\nu \in I^S$. A non-zero fuzzy subset F of $X \times S$ is said to be a fuzzy proper function from μ to ν if

(1) $F(x,s) \leq \min\{\mu(x), \nu(s)\}$, $\forall (x,s) \in X \times S$,
(2) for each $x \in X$ with $\mu(x) > 0$, there exists a unique $s_0 \in S$ such that $F(x,s_0) = \mu(x)$ and $F(x,s) = 0$ if $s \neq s_0$.

Definition 2.4 ([4])**.** Let F be a fuzzy proper function from μ to ν. If $U \in \Im_\mu$ and $V \in \Im_\nu$, then $F(U) : S \to I$ and $F^{-1}(V) : X \to I$ are defined by

$$\begin{aligned}(F(U))(s) &= \sup\{F(x,s) \wedge U(x) : x \in X\}, \forall s \in S,\\ (F^{-1}(V))(x) &= \sup\{F(x,s) \wedge V(s) : s \in S\}, \forall x \in X.\end{aligned}$$

The inverse image of a fuzzy subset V under a fuzzy proper function F can be easily obtained as $(F^{-1}(V))(x) = \mu(x) \wedge V(s)$, where $s \in S$ is the unique element such that $F(x,s) = \mu(x)$.

Definition 2.5 ([8])**.** A fuzzy proper function $F : \mu \to \nu$ is said to be injective (or one-to-one) if $F(x_1,s) > 0$ and $F(x_2,s) > 0$, for some $x_1, x_2 \in X$ and $s \in S$, then $x_1 = x_2$.

Theorem 2.6 ([13])**.** *Let (μ,τ) be a smooth fuzzy topological space. For $r \in I_0, A \in \Im_\mu$, if $C_\tau : \Im_\mu \times I_0 \to \Im_\mu$ is defined by $C_\tau(A,r) = \bigwedge\{K \in \Im_\mu : A \leq K, \tau(\mu - A) \geq r\}$, then,*

(1) $C_\tau(0_X, r) = 0_X$,
(2) $A \leq C_\tau(A,r)$,
(3) $C_\tau(A,r) \vee C_\tau(B,r) = C_\tau(A \vee B, r)$,
(4) $C_\tau(A,r) \leq C_\tau(B,s)$ *if* $r \leq s$,
(5) $C_\tau(C_\tau(A,r),r) = C_\tau(A,r)$,

where $A, B \in \Im_\mu$ and $r, s \in I_0$.

Theorem 2.7 ([13])**.** *Let (μ,τ) be a smooth fuzzy topological space. For $r \in I_0, A \in \Im_\mu$, if $I_\tau : \Im_\mu \times I_0 \to \Im_\mu$ is defined by $I_\tau(A,r) = \bigvee\{S \in \Im_\mu : S \leq A, \tau(S) \geq r\}$, then*

(1) $I_\tau(\mu - A, r) = \mu - C_\tau(A,r)$,
(2) *If* $I_\tau(C_\tau(A,r),r) = A$, *then* $C_\tau(I_\tau(\mu - A, r), r) = \mu - A$,
(3) $I_\tau(\mu, r) = \mu$,
(4) $I_\tau(A,r) \leq A$,
(5) $I_\tau(A,r) \wedge I_\tau(B,r) = I_\tau(A \wedge B, r)$,
(6) $I_\tau(A,r) \geq I_\tau(A,q)$, *if* $r \leq q$,
(7) $I_\tau(I_\tau(A,r),r) = I_\tau(A,r)$,

where $A, B \in \Im_\mu$ and $r, s \in I_0$.

Definition 2.8 ([16])**.** Let (μ,τ) be a smooth fuzzy topological space and let $A \in \Im_\mu$, $r \in I_0$. Then, A is called a

(1) Q^r_τ-neighborhood of P^λ_x if $P^\lambda_x qA[\mu]$ with $\tau(\mu) \geq r$.
(2) R^r_τ-neighborhood of P^λ_x if $P^\lambda_x qA[\mu]$ with $A = I_\tau(C_\tau(A, r), r)$.

Definition 2.9 ([16])**.** Let (μ, τ) be a smooth fuzzy topological space. Then the δ-closure operator is a function $D_\tau : \Im_\mu \times I_0 \to I$ defined as follows.

$$D_\tau(A, r) = \bigvee\{P^\lambda_x \in \mu : UqA[\mu],\ forevery\ R^r_\tau - neighborhood\ U\ of\ P^\lambda_x\}$$

Result 2.10 ([16])**.** Let (μ, τ) be a smooth fuzzy topological space and let $A \in \Im_\mu$, $r \in I_0$. Then,

(1) $C_\tau(A, r) = \bigvee\{P^\lambda_x : UqA[\mu],\ forevery\ Q^r_\tau - neighborhood\ U\ of\ P^\lambda_x\}$.
(2) $D_\tau(A, r) = \bigwedge\{K \in \Im_\mu : A \leq K, K = C_\tau(I_\tau(K, r), r)\}$.

Definition 2.11 ([16])**.** Let (μ, τ) and (ν, σ) be smooth fuzzy topological spaces and $F : \mu \to \nu$ be a fuzzy proper function. Then, F is called fuzzy super continuous or FSC if for every Q^r_σ-neighborhood V of $F(P^\lambda_x)$, there exists a Q^r_τ-neighborhood U of P^λ_x such that $F(I_\tau(C_\tau(U, r), r)) \leq V$.

Theorem 2.12 ([12])**.** *Let $F : \mu \to \nu$ be a fuzzy proper function such that $\nu = F(\mu)$. If F is one-to-one, then $F^{-1}(\nu - V) = \mu - F^{-1}(V)$, $\forall V \in \Im_\nu$.*

3. Smooth fuzzy R^r_τ-closure operator

Definition 3.1. Let (μ, τ) be a smooth fuzzy topological space. For $A \in \Im_\mu$ and $r \in I_0$, smooth fuzzy R^r_τ-closure $\mathbb{D}_\tau(A, r)$ of A is defined by

$$\mathbb{D}_\tau(A, r) = \bigvee\{P^\lambda_x \in \mu : C_\tau(U, r)qA[\mu], \forall R^r_\tau - neighborhood\ U\ of\ P^\lambda_x\}.$$

Theorem 3.2. *Let (μ, τ) be a smooth fuzzy topological space. For $A \in \Im_\mu$ and $r \in I_0$, $\mathbb{D}_\tau(A, r) = \bigwedge\{K \in \Im_\mu : A \leq I_\tau(K, r), K = C_\tau(I_\tau(K, r), r)\}$.*

Proof. If $P^\lambda_x \notin \bigwedge\{K \in \Im_\mu : A \leq I_\tau(K, r), K = C_\tau(I_\tau(K, r), r)\}$, then $P^\lambda_x \notin K$, for some $K \in \Im_\mu$ such that $A \leq I_\tau(K, r)$, $K = C_\tau(I_\tau(K, r), r)$. Therefore,

$$(\mu - K)(x) > \mu(x) - \lambda,\ \mu - A \geq \mu - I_\tau(K, r) \text{ and } \mu - K = \mu - C_\tau(I_\tau(K, r), r)$$

and hence

$$P^\lambda_x q(\mu - K)[\mu],\ \mu - K = I_\tau(C_\tau(\mu - K, r), r) \text{ and } A\bar{q}C_\tau(\mu - K, r)[\mu].$$

Since $(\mu - K)$ is an R^r_τ-neighborhood of P^λ_x such that $C_\tau(\mu - K, r)\bar{q}A[\mu]$, we get $P^\lambda_x \notin \mathbb{D}_\tau(A, r)$.

Conversely, suppose that $P^\lambda_x \notin \mathbb{D}_\tau(A, r)$. Then, there is an R^r_τ-neighborhood U of P^λ_x such that $C_\tau(U, r)\bar{q}A[\mu]$. Therefore,

$$U(x) + \lambda > \mu(x),\ U = I_\tau(C_\tau(U, r), r),\ C_\tau(U, r)(k) + A(k) \leq \mu(k),$$

for every $k \in X$. Hence, it follows that

$$(\mu - U)(x) < \lambda,\ \mu - U = \mu - I_\tau(C_\tau(U, r), r),\ A \leq \mu - C_\tau(U, r).$$

Therefore, $P^\lambda_x \notin \mathbb{D}_\tau(A, r)$. This completes the proof of the theorem. □

Theorem 3.3. *Let (μ, τ) be a smooth fuzzy topological space. For $A, B \in \Im_\mu$ and $r, q \in I_0$, R^r_τ-closure operator satisfies the following properties:*

(1) $\mathbb{D}_\tau(0_X, r) = 0_X$,

(2) $A \leq \mathbb{D}_\tau(A, r)$,
(3) $A \leq B \Rightarrow \mathbb{D}_\tau(A, r) \leq \mathbb{D}_\tau(B, r)$,
(4) $\mathbb{D}_\tau(A, r) \leq \mathbb{D}_\tau(B, q)$ *if* $r \leq q$,
(5) $\mathbb{D}_\tau(A, r) \vee \mathbb{D}_\tau(B, r) = \mathbb{D}_\tau(A \vee B, r)$,
(6) $\mathbb{D}_\tau(A \wedge B, r) \leq \mathbb{D}_\tau(A, r) \wedge \mathbb{D}_\tau(B, r)$.

Proof.

(1) From Theorems 2.6 and 2.7, we have

$$I_\tau(0_X, r) = 0_X \text{ and } C_\tau(I_\tau(0_X, r), r) = C_\tau(0_X, r) = 0_X.$$

Hence, $\mathbb{D}_\tau(0_X, r) = \bigwedge\{U \in \Im_\mu : I_\tau(U, r) \geq 0_X, I_\tau(C_\tau(U, r), r) = U\} = 0_X$.

(2) Since $U \geq I_\tau(U, r), \forall U \in \Im_\mu$,

$$\begin{aligned} \mathbb{D}_\tau(A, r) &\geq U, \forall U \in \Im_\mu \text{ with } I_\tau(U, r) \geq A, C_\tau(I_\tau(U, r), r) = U \\ &\geq I_\tau(U, r), \forall U \in \Im_\mu \text{ with } I_\tau(U, r) \geq A, C_\tau(I_\tau(U, r), r) = U \\ &\geq A. \end{aligned}$$

(3) Since $A \leq B$, we have $\{U \in \Im_\mu : I_\tau(U, r) \geq A, C_\tau(I_\tau(U, r), r) = U\} \supseteq \{U \in \Im_\mu : I_\tau(U, r) \geq B, C_\tau(I_\tau(U, r), r) = U\}$. Therefore,

$$\begin{aligned} \mathbb{D}_\tau(B, r) &= \bigwedge\{U \in \Im_\mu : I_\tau(U, r) \geq B, C_\tau(I_\tau(U, r), r) = U\} \\ &\geq \bigwedge\{U \in \Im_\mu : I_\tau(U, r) \geq A, C_\tau(I_\tau(U, r), r) = U\} = \mathbb{D}_\tau(A, r). \end{aligned}$$

(4) Using Theorems 2.6 and 2.7, we get $I_\tau(A, r) \geq I_\tau(A, q)$ and $C_\tau(I_\tau(A, r), r) \geq C_\tau(I_\tau(A, q), q)$ if $r \leq q$. Therefore,

$$\begin{aligned} \mathbb{D}_\tau(A, q) &= \bigwedge\{U \in \Im_\mu : I_\tau(U, q) \geq A, C_\tau(I_\tau(U, q), q) = U\} \\ &\geq \bigwedge\{U : I_\tau(U, r) \geq A, C_\tau(I_\tau(U, r), r) = U\} = \mathbb{D}_\tau(A, r). \end{aligned}$$

(5) From (3), it is clear that $\mathbb{D}_\tau(A, r) \vee \mathbb{D}_\tau(B, r) \leq \mathbb{D}_\tau(A \vee B, r)$. Let $P_x^\lambda \in \mathbb{D}_\tau(A \vee B, r)$. Then, $C_\tau(U, r)q(A \vee B)[\mu]$, for every R_τ^r-neighborhood U of P_x^λ. Therefore, $C_\tau(U, r)qA[\mu]$ or $C_\tau(U, r)qB[\mu]$, for every R_τ^r-neighborhood U of P_x^λ. Thus, we get $P_x^\lambda \in \mathbb{D}_\tau(A, r) \vee \mathbb{D}_\tau(B, r)$ and hence $\mathbb{D}_\tau(A \vee B, r) = \mathbb{D}_\tau(A, r) \vee \mathbb{D}_\tau(B, r)$

(6) By (3), we have $\mathbb{D}_\tau(A \wedge B, r) \leq \mathbb{D}_\tau(A, r)$ and $\mathbb{D}_\tau(A \wedge B, r) \leq \mathbb{D}_\tau(B, r)$. Thus, we get $\mathbb{D}_\tau(A \wedge B, r) \leq \mathbb{D}_\tau(A, r) \wedge \mathbb{D}_\tau(B, r)$.

□

The following example shows that the equality does not hold in (6) of the previous theorem.

Counterexample 3.4. Let $X = \{x, y\}$, $\mu_{[x,y]}^{[0.9,0.8]} \in I^X$, $U1_{[x,y]}^{[0.4,0.4]} \in \Im_\mu$.

Define $\tau : \Im_\mu \to I$ by $\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise}. \end{cases}$

Let $A^{[0.45,0]}_{[x,y]}$ and $B^{[0.2,0.45]}_{[x,y]}$. Since U_1 and μ are the R^r_τ-neighborhoods of $P^{0.45}_y$, from the following inequalities,

$$\begin{aligned} C_\tau(U_1,r)(y)+B(y) &= 0.4+0.45=0.85>0.8=\mu(y)\\ C_\tau(U_1,r)(x)+A(x) &= 0.5+0.45=0.95>0.9=\mu(x), \end{aligned}$$

we conclude that $BqC_\tau(U_1,r)[\mu]$ and $AqC_\tau(U_1,r)[\mu]$. Therefore,

$$P^{0.45}_y \in \mathbb{D}_\tau(A,r)\wedge\mathbb{D}_\tau(B,r).$$

But, $C_\tau(U_1,r)(x)+(A\wedge B)(x)=0.7<0.9=\mu(x)$ and $C_\tau(U_1,r)(y)+(A\wedge B)(y)=0.4<0.8=\mu(y)$ imply that $P^{0.45}_y\notin\mathbb{D}_\tau(A\wedge B)$.

The following example shows that $\mathbb{D}_\tau(\mathbb{D}_\tau(A,r),r)\neq\mathbb{D}_\tau(A,r)$.

Counterexample 3.5. Let $X=\{x,y\}$, $\mu^{[0.8,0.7]}_{[x,y]}$, $U_1{}^{[0.4,0.3]}_{[x,y]}$ and $A^{[0.4,0.2]}_{[x,y]}$.

Define $\tau:\Im_\mu\to I$ by $\tau(U)=\begin{cases}1, & U=0_X \text{ or } \mu,\\ 0.6, & U=U_1,\\ 0, & \text{otherwise.}\end{cases}$ If $C_\tau(I_\tau(U,r),r)=U$, then $U=0_X$ or μ or $(\mu-U_1)^{[0.4,0.4]}_{[x,y]}$. We observe that

$$I_\tau(\mu-U_1)=U_1\geq A \text{ and } C_\tau(I_\tau(\mu-U_1,r),r)=C_\tau(U_1,r)=\mu-U_1.$$

Therefore, $\mathbb{D}_\tau(A,r)=\mu-U_1$. Since $I_\tau(\mu-U_1)=U_1\ngeq\mu-U_1$, we get that $\mathbb{D}_\tau(\mathbb{D}_\tau(A,r),r)=\mathbb{D}_\tau(\mu-U_1,r)=\mu$ and hence $\mathbb{D}_\tau(\mathbb{D}_\tau(A,r),r)\neq\mathbb{D}_\tau(A,r)$.

Lemma 3.6. *Let (μ,τ) be smooth fuzzy topological space and let $r\in I_0$. If $U,A\in\Im_\mu$ are such that $C_\tau(U,r)\leq\mu-A$, $I_\tau(C_\tau(U,r),r)=U$, then $\mu-U\geq\mathbb{D}_\tau(A,r)$.*

Proof. If $C_\tau(U,r)\leq\mu-A$ and $I_\tau(C_\tau(U,r),r)=U$, then $\mu-C_\tau(U,r)\geq A$ and $\mu-I_\tau(C_\tau(U,r),r)=\mu-U$. Applying Theorem 2.7(1), we get that $I_\tau(\mu-U,r)\geq A$ and $C_\tau(I_\tau(\mu-U,r),r)=\mu-U$. Therefore, $\mu-U\geq\mathbb{D}_\tau(A,r)$. □

Definition 3.7. Let (μ,τ) be a smooth fuzzy topological space. For $A\in\Im_\mu$ and $r\in I_0$, the smooth fuzzy R^r_τ-interior $\mathbb{I}_\tau(A,r)$ of A is defined by

$$\mathbb{I}_\tau(A,r)=\bigvee\{K\in\Im_\mu : A\geq C_\tau(K,r),\ K=I_\tau(C_\tau(K,r),r)\}.$$

Theorem 3.8. *Let (μ,τ) be a smooth fuzzy topological space. For $A,B\in\Im_\mu$ and $r,q\in I_0$,*

(1) $\mathbb{I}_\tau(\mu,r)=\mu$,
(2) $\mathbb{I}_\tau(A,r)\leq A$,
(3) $A\leq B\to\mathbb{I}_\tau(A,r)\leq\mathbb{I}_\tau(B,r)$,
(4) $\mathbb{I}_\tau(A,r)\geq\mathbb{I}_\tau(A,q)$, *if* $r\leq q$,
(5) $\mathbb{I}_\tau(\mu-A,r)=\mu-C_\tau(A,r)$,
(6) $\mathbb{I}_\tau(A,r)\wedge\mathbb{I}_\tau(B,r)=\mathbb{I}_\tau(A\wedge B,r)$,
(7) $\mathbb{I}_\tau(A\vee B)\geq\mathbb{I}_\tau(A,r)\vee\mathbb{I}_\tau(B,r)$,
(8) *If* $\mathbb{I}_\tau(\mathbb{D}_\tau(A,r),r)=A$, *then* $\mathbb{D}_\tau,(\mathbb{I}_\tau(\mu-A,r),r)=\mu-A$.

Proof. (1) By Theorem 2.6 and Theorem 2.7 we have,

$$C_\tau(\mu,r)=\mu \text{ and } I_\tau(C_\tau(\mu,r),r)=I_\tau(\mu,r)=\mu.$$

Therefore, $\mathbb{I}_\tau(\mu,r)=\bigvee\{T\in\Im_\mu : C_\tau(T,r)\leq\mu, I_\tau(C_\tau(T,r),r)=T\}=\mu$.

(2) $\mathbb{I}_\tau(A,r) = \bigvee\{T \in \mathfrak{I}_\mu : C_\tau(T,r) \le A, I_\tau(C_\tau(T,r),r) = T\} \le A.$

(3) Since $A \le B$, we have $C_\tau(T,r) \le A \Rightarrow C_\tau(T,r) \le B$. Therefore,

$$\begin{aligned}\mathbb{I}_\tau(B,r) &= \bigvee\{T \in \mathfrak{I}_\mu : C_\tau(T,r) \le B, I_\tau(C_\tau(T,r),r) = T\}\\ &\ge \bigvee\{T \in \mathfrak{I}_\mu : C_\tau(T,r) \le A, I_\tau(C_\tau(T,r),r) = T\} = \mathbb{I}_\tau(A,r).\end{aligned}$$

(4) Applying Theorem 2.6, we have $C_\tau(T,r) \le C_\tau(T,q)$, if $r \le q$. Therefore,

$$\begin{aligned}\mathbb{I}_\tau(A,r) &= \bigvee\{T \in \mathfrak{I}_\mu : C_\tau(T,r) \le A, I_\tau(C_\tau(T,r),r) = T\}\\ &\ge \bigvee\{T \in \mathfrak{I}_\mu : C_\tau(T,q) \le A, I_\tau(C_\tau(T,q),q) = T\} = \mathbb{I}_\tau(A,q).\end{aligned}$$

(5) Using Lemma 3.6, we obtain
$\mu - \mathbb{D}_\tau(A,r)$

$$\begin{aligned}&= \mu - \bigwedge\{K \in \mathfrak{I}_\mu : I_\tau(K,r) \ge A, C_\tau(I_\tau(K,r),r) = K\}\\ &= \bigvee\{\mu - K : \mu - I_\tau(K,r) \le \mu - A, \mu - C_\tau(I_\tau(K,r),r) = \mu - K\}\\ &= \bigvee\{\mu - K : C_\tau(\mu - K,r) \le \mu - A, I_\tau(C_\tau(\mu - K,r),r) = \mu - K\}\\ &= \bigvee\{U \in \mathfrak{I}_\mu : C_\tau(U,r) \le \mu - A, I_\tau(C_\tau(U,r),r) = U\} = \mathbb{I}_\tau(\mu - A,r).\end{aligned}$$

(6) Using (5) and Theorem 3.3 (5), we obtain

$$\begin{aligned}\mathbb{I}_\tau(A \wedge B,r) &= \mu - \mathbb{D}_\tau(\mu - (A \wedge B),r)\\ &= \mu - \mathbb{D}_\tau((\mu - A) \vee (\mu - B),r)\\ &= \mu - [\mathbb{D}_\tau(\mu - A,r) \vee \mathbb{D}_\tau(\mu - B,r)]\\ &= [\mu - \mathbb{D}_\tau(\mu - A,r)] \wedge [\mu - \mathbb{D}_\tau(\mu - B,r)] = \mathbb{I}_\tau(A,r) \wedge \mathbb{I}_\tau(B,r).\end{aligned}$$

(7) In view of (3), we have $\mathbb{I}_\tau(A \vee B,r) \ge \mathbb{I}_\tau(A,r)$ and $\mathbb{I}_\tau(A \vee B,r) \ge \mathbb{I}_\tau(B,r)$. Thus, we obtain $\mathbb{I}_\tau(A \vee B,r) \ge \mathbb{I}_\tau(A,r) \vee \mathbb{I}_\tau(B,r)$.

(8) If $\mathbb{I}_\tau(\mathbb{D}_\tau((A,r),r) = A$, then we get

$$\begin{aligned}\mathbb{D}_\tau(\mathbb{I}_\tau((\mu - A,r),r) &= \mathbb{D}_\tau(\mu - \mathbb{D}_\tau(A,r),r)\\ &= \mu - \mathbb{I}_\tau(\mu - (\mu - \mathbb{D}_\tau(A,r)),r), \quad \text{(using (5))}\\ &= \mu - \mathbb{I}_\tau(\mathbb{D}_\tau((A,r),r) = \mu - A.\end{aligned}$$

Hence, the theorem follows. □

The following example shows that the equality does not hold in (7) of the previous theorem.

Counterexample 3.9. Let $X = \{x,y\}$, $\mu^{[0.8,0.7]}_{[x,y]} \in I^X$, $U_1{}^{[0.4,0.3]}_{[x,y]} \in \mathfrak{I}_\mu$.

Define $\tau : \mathfrak{I}_\mu \to I$ by $\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu,\\ 0.6, & U = U_1,\\ 0, & \text{otherwise.}\end{cases}$ Let $A^{[0.4,0.2]}_{[x,y]}$ and $B^{[0.2,0.5]}_{[x,y]}$ and $r = 0.5$. We first observe that $(A \vee B)^{[0.4,0.5]}_{[x,y]}$, $C_\tau(U_1,r) = (\mu - U_1)^{[0.4,0.4]}_{[x,y]}$ and if $I_\tau(C_\tau(U,r),r) = U$, then $U = 0_X$ or $U = U_1$. Since $C_\tau(U_1,r) \nleq A$ and $C_\tau(U_1,r) \nleq B$, we get that $\mathbb{I}_\tau(A,r) = 0_X$ and $\mathbb{I}_\tau(B,r) = 0_X$, which implies that

$\mathbb{I}_\tau(A,r) \vee \mathbb{I}_\tau(B,r) = 0_X$. Since $C_\tau(U_1,r) \le A \vee B$, we have $\mathbb{I}_\tau(A \vee B, r) = U_1 \ne \mathbb{I}_\tau(A,r) \vee \mathbb{I}_\tau(B,r)$.

The following example shows that $\mathbb{I}_\tau(\mathbb{I}_\tau(A,r),r)$ need not be equal to $\mathbb{I}_\tau(A,r)$

Counterexample 3.10. Let $X = \{x,y\}$, $\mu_{[x,y]}^{[0.8,0.7]}$, $U_1{}_{[x,y]}^{[0.4,0.3]}$, $A_{[x,y]}^{[0.4,0.5]}$.

If $\tau : \Im_\mu \to I$ is defined by $\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise,} \end{cases}$ then (μ,τ) is a smooth fuzzy topological space. If $I_\tau(C_\tau(U,r),r) = U$, then $U = 0_X$ or $U = \mu$ or $U = U_1$. Clearly, we have

$$C_\tau(U_1,r) = (\mu - U_1)_{[x,y]}^{[0.4,0.4]} \le A \text{ and } I_\tau(C_\tau(U_1,r),r) = I_\tau(\mu - U_1, r) = U_1.$$

Therefore, $\mathbb{I}_\tau(A,r) = U_1$. Since the only fuzzy r-closed sets in (μ,τ) are 0_X, $\mu - U_1$ and μ, and $\mu - U_1 \nleq U_1$, we get $\mathbb{I}_\tau(\mathbb{I}_\tau(A,r),r) = \mathbb{I}_\tau(U_1,r) = 0_X \ne U_1 = \mathbb{I}_\tau(A,r)$.

4. Some kinds of fuzzy super continuous functions

Definition 4.1. Let (μ,τ) and (ν,σ) be smooth fuzzy topological spaces and $F : \mu \to \nu$ be a fuzzy proper function. We say that F is

(1) fuzzy super r_1-continuous or FS-r_1-C if $F(\mathbb{D}_\tau(A,r)) \le C_\sigma(F(A),r)$, $\forall A \in \Im_\mu$, $\forall r \in I_0$.
(2) fuzzy super $[r,q]_1$-continuous or FS-$[r,q]_1$-C if $F(\mathbb{D}_\tau(A,r)) \le C_\sigma(F(A),q)$, $\forall A \in \Im_\mu$ and $r,q \in I_0$.
(3) fuzzy super r_2-continuous or FS-r_2-C if $\mathbb{D}_\tau(F^{-1}(V),r) \le F^{-1}(C_\sigma(V,r))$, $\forall V \in \Im_\nu$, $\forall r \in I_0$.
(4) fuzzy super $[r,q]_2$- continuous or FS-$[r,q]_2$-C if $\mathbb{D}_\tau(F^{-1}(V),r) \le F^{-1}(C_\sigma(V,q))$, $\forall V \in \Im_\nu$ and $r,q \in I_0$.
(5) fuzzy super r_3-continuous or FS-r_3-C if $\mathbb{D}_\tau(F^{-1}(V),r) = F^{-1}(V)$, $\forall V \in \Im_\nu$ with $V = C_\sigma(V,r)$.
(6) fuzzy super r_4-continuous or FS-r_4-C if $\mathbb{D}_\tau(\mu - F^{-1}(V),r) = \mu - F^{-1}(V)$ $\forall V \in \Im_\nu$ with $V = I_\sigma(V,r)$.

Theorem 4.2. *Let $F : (\mu,\tau) \to (\nu,\sigma)$ be a one-to-one fuzzy proper function with $\nu = F(\mu)$. If F is fuzzy super continuous, then F is fuzzy super r_1-continuous.*

Proof. Suppose that there exist $A \in \Im_\mu$ and $r \in I_0$ such that $F(\mathbb{D}_\tau(A,r))(s) > C_\sigma(F(A),r)(s)$, for some $s \in S$. Observing that $F(\mathbb{D}_\tau(A,r))(s) > 0$, we can find $x \in X$ such that $F(x,s) = \mu(x)$. Since F is one-to-one and $F(\mu) = \nu$, we have $F(C)(s) = C(x)$, $\forall C \in \Im_\mu$. In particular,

$$\mathbb{D}_\tau(A,r))(x) = F(\mathbb{D}_\tau(A,r))(s) > C_\sigma(F(A),r)(s).$$

Now, we choose a real number η such that $\mathbb{D}_\tau(A,r)(x) > \eta > C_\sigma(F(A),r)(s)$, which implies that $P_s^\eta \notin C_\sigma(F(A),r)$. Therefore, there exists a Q_τ^r-neighborhood V of $F(P_x^\eta)$ such that $V\ \bar{q}F(A)[\nu]$ and hence $F(A) \le \nu - V$. Since F is fuzzy super continuous, there exists a Q_τ^r-neighborhood U of P_x^η such that $F(I_\tau(C_\tau(U,r),r)) \le$

V, which implies that $F(A) \leq \nu - F(I_\tau(C_\tau(U,r),r))$. Using Theorem 2.12, we obtain

$$\begin{aligned} A \leq F^{-1}(F(A)) &\leq F^{-1}(\nu - F(I_\tau(C_\tau(U,r),r))) \\ &= \mu - F^{-1}(F(I_\tau(C_\tau(U,r),r))) \\ &\leq \mu - I_\tau(C_\tau(U,r),r). \end{aligned}$$

Since $I_\tau(U,r) = U$ and $U \leq I_\tau(C_\tau(U,r),r)$, we have

$$\begin{aligned} I_\tau(C_\tau(U,r),r) \leq I_\tau(C_\tau(I_\tau(C_\tau(U,r),r),r),r) &\leq I_\tau(C_\tau(C_\tau(U,r),r),r) \\ &= I_\tau(C_\tau(U,r),r), \end{aligned}$$

and hence $I_\tau(C_\tau(U,r),r)$ is an R^r_τ-neighborhood of P^η_x and $A + I_\tau(C_\tau(U,r),r) \leq \mu$. Therefore, $P^\eta_x \notin \mathbb{D}_\tau(A,r)$, which is a contradiction to $\mathbb{D}_\tau(A,r)(x) > \eta$. Thus, F is fuzzy super r_1-continuous. □

The statement of the above theorem is not true when F is not one-to-one or $F(\mu) \neq \nu$. The following examples justify our statement.

Counterexample 4.3. Let $X = \{x,y\}$, $S = \{s,t\}$ and $\mu^{[0.6,0.5]}_{[x,y]}$, $\nu^{[0.6,0]}_{[s,t]}$ be fuzzy subsets of X and S respectively. Define the fuzzy subsets ${U_1}^{[0.3,0.2]}_{[x,y]} \in \Im_\mu$ and ${V_1}^{[0.3,0]}_{[s,t]} \in \Im_\nu$. If $\tau : \Im_\mu \to I$ and $\sigma : \Im_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$$F(x,s) = 0.6, F(x,t) = 0, F(y,s) = 0.5, F(y,t) = 0.$$

Clearly, F is not one-to-one and $F(\mu)^{[0.6,0]}_{[s,t]} = \nu$. If V_1 is a Q^r_σ-neighborhood of $F(P^\eta_l)$, for an arbitrary $P^\eta_l \in \mu$, then U_1 is a Q^r_τ-neighborhood of P^η_l such that $F(I_\tau(C_\tau(U_1,r),r)) \leq V_1$. Indeed, $C_\tau(U_1,r) = (\mu - U_1)$ implies that $I_\tau(C_\tau(U_1,r),r) = I_\tau(\mu - U_1, r) = U_1$, and hence $F(I_\tau(C_\tau(U_1,r),r)) = F(U_1)^{[0.3,0]}_{[s,t]} = V_1$. For ν, we choose μ as the Q^r_τ-neighborhood P^η_l such that $F(I_\tau(C_\tau(\mu,r),r)) = \nu$. Hence F is fuzzy super continuous.

Next, we claim that $F(\mathbb{D}_\tau(A,r)) \nleq C_\sigma(F(A),r)$, for $A^{[0,0.3]}_{[x,y]} \in \Im_\mu$ and $r = 0.5$. If $U \in \Im_\mu$ with $U = I_\tau(C_\tau(U,r),r)$, then $U = 0_X$ or $U = \mu$ or $U = U_1$. Since $P^{0.35}_y q U_1[\mu]$ and $P^{0.35}_y q \mu[\mu]$, we have that U_1 and μ are the R^r_τ-neighborhoods of $P^{0.35}_y$. Since,

$$C_\tau(U_1,r)(y) + A(y) = (\mu - U_1)(y) + A(y) = 0.3 + 0.3 = 0.6 > 0.5 = \mu(y),$$

it follows that $C_\tau(U_1,r) q A[\mu]$. Therefore,

$$P^{0.35}_y \in \mathbb{D}_\tau(A,r) \text{ and } F(P^{0.35}_y) \in F(\mathbb{D}_\tau(A,r)).$$

Since, $V_1(s) + 0.35 = 0.3 + 0.35 = 0.65 > 0.6 = \nu(s)$, V_1 is a Q^r_τ-neighborhood of $P^{0.35}_s = F(P^{0.35}_y)$. However, from $F(A)^{[0.3,0]}_{[s,t]} \bar{q} V_1[\nu]$, we conclude that $P^{0.35}_s \notin C_\sigma(F(A),r)$. Therefore, F is not super r_1-continuous.

Counterexample 4.4. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu^{[0.7,0.6]}_{[x,y]} \in I^X$, $\nu^{[0.9,0.9]}_{[s,t]} \in I^X$, $U1^{[0.2,0.1]}_{[x,y]} \in \Im_\mu$ and $V1^{[0.4,0.4]}_{[s,t]} \in \Im_\nu$. We define $\tau : \Im_\mu \to I$ and $\sigma : \Im_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Define a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ by

$$F(x, s) = 0.7, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.6.$$

Obviously, F is one-to-one and $F(\mu)^{[0.7,0.6]}_{[s,t]} \neq \nu$. As in the previous counterexample, we can verify that F is fuzzy super continuous by showing that if V_1 is a Q^r_σ-neighborhood of $F(P^\eta_l)$, then U_1 is a required Q^r_τ-neighborhood of P^η_l such that $F(I_\tau(C_\tau(U_1, r), r)) \leq V_1$, and for μ, we choose ν as a required neighborhood of P^η_l.

Theorem 4.5. *Let $F : \mu \to \nu$ be a fuzzy proper function, where (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Then, $(a) \Rightarrow (b) \Rightarrow (c)$, where*

- (a) *F is fuzzy super r_1-continuous*
- (b) *F is fuzzy super r_2-continuous*
- (c) *F is fuzzy super r_3-continuous*

Proof. Let $V \in \Im_\nu$ be arbitrary.

$(a) \Rightarrow (b)$**:** From (a), we have

$$F(\mathbb{D}_\tau(F^{-1}(V), r)) \leq C_\sigma(F(F^{-1}(V)), r) \leq C_\sigma(V, r).$$

Therefore $\mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(F(\mathbb{D}_\tau(F^{-1}(V), r))) \leq F^{-1}(C_\sigma(V, r))$.

$(b) \Rightarrow (c)$**:** If $V = C_\sigma(V, r)$, then applying (b), we get $\mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(V)$. Using Theorem 3.3 (2), we get $\mathbb{D}_\tau(F^{-1}(V), r) \geq F^{-1}(V)$.

Hence the theorem follows. □

Theorem 4.6. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function with $\nu = F(\mu)$. If F is fuzzy super r_3-continuous, then F is fuzzy super r_4-continuous.*

Proof. If $V \in \Im_\nu$ is such that $V = I_\sigma(V, r)$, then $\nu - V = \nu - I_\sigma(V, r) = C_\sigma(\nu - V, r)$. Using hypothesis, we get $\mathbb{D}_\tau(F^{-1}(\nu - V), r) = F^{-1}(\nu - V)$. Since F is one-to-one and $\nu = F(\mu)$, by Theorem 2.12, we have $F^{-1}(\nu - V) = \mu - F^{-1}(V)$. Therefore, $\mathbb{D}_\tau(\mu - F^{-1}(V), r) = \mu - F^{-1}(V)$. □

The statement of the above theorem is not true when F is not one-to-one or $F(\mu) \neq \nu$. The following examples justify our statement.

Counterexample 4.7. Let $X = \{x, y\}$ and $S = \{s, t\}$. If $\mu^{[0.8,0.9]}_{[x,y]}$, $\nu^{[0.9,0]}_{[s,t]}$, $U1^{[0.4,0.4]}_{[x,y]}$, $U2^{[0.4,0.5]}_{[x,y]}$ and $V1^{[0.5,0]}_{[s,t]}$, then $U_1, U_2 \in \mathscr{I}_\mu$ and $V_1 \in \mathscr{I}_\nu$. We define smooth fuzzy topologies τ on μ and σ on ν, respectively, by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1 \text{ or } U_2, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let the fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0.9, F(y, t) = 0.$$

Then, F is not one-to-one and $F(\mu)^{[0.9,0]}_{[s,t]} = \nu$. We fix $r = 0.5$. If $C_\sigma(V,r) = V$, then $V = 0_S$ or $V = \nu$ or $V = (\nu - V_1)^{[0.4,0]}_{[s,t]}$. Obviously, we have

$$\mathbb{D}_\tau(F^{-1}(0_S), r) = F^{-1}(0_S),\ \mathbb{D}_\tau(F^{-1}(\nu), r) = F^{-1}(\nu).$$

For an arbitrary $U \in \mathfrak{I}_\mu$, $C_\tau(I_\tau(U,r), r)$ is any one of the following four sets 0_X, μ, U_1, and U_2. Using $I_\tau(U_1, r) = U_1 \geq F^{-1}(\nu - V_1)^{[0.4,0.4]}_{[x,y]}$, we obtain

$$\mathbb{D}_\tau(F^{-1}(\nu - V_1), r) = U_1 = F^{-1}(\nu - V_1).$$

Therefore, F is fuzzy super r_3-continuous. However,

$$\mathbb{D}_\tau((\mu - F^{-1}(V_1))^{[0.3,0.4]}_{[x,y]}, r) = U_1 \wedge U_2 \wedge \mu = U_1 \neq \mu - F^{-1}(V_1)$$

and hence F is not fuzzy super r_4-continuous.

Counterexample 4.8. Let $X = \{x, y\}$, $S = \{s, t\}$. Define $\mu^{[0.6,0.6]}_{[x,y]} \in I^X$, $\nu^{[0.9,0.8]}_{[s,t]} \in I^S$, $U_1{}^{[0.3,0.3]}_{[x,y]} \in \mathfrak{I}_\mu$ and $V_1{}^{[0.6,0.5]}_{[s,t]} \in \mathfrak{I}_\nu$. If $\tau : \mathfrak{I}_\mu \to I$ and $\sigma : \mathfrak{I}_\nu \to I$ are, respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.6, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.6.$$

Clearly, F is one-to-one and $F(\mu)^{[0.6,0.6]}_{[s,t]} \neq \nu$. We fix $r = 0.5$. If $V \in \mathscr{I}_\nu$ is such that $V = C_\sigma(V, r)$, then $V = 0_S$ or $V = \nu$ or $V = \nu - V_1$. Since $\mathbb{D}_\tau(F^{-1}(V), r) = F^{-1}(V)$, for each $V \in \left\{0_S, \nu, F^{-1}(\nu - V_1)^{[0.3,0.3]}_{[x,y,z]}\right\}$, we conclude that F is fuzzy super r_3-continuous. But $\mathbb{D}_\tau\left((\mu - F^{-1}(V_1))^{[0,0.1]}_{[x,y]}, r\right) = U_1 \neq \mu - F^{-1}(V_1)$ implies that F is not fuzzy super r_4-continuous.

Theorem 4.9. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function. If F is fuzzy super r_4-continuous, then F is fuzzy super continuous.*

Proof. If W is a Q^r_σ-neighborhood of $F(P^\lambda_x)$, then $\sigma(W) \geq r$ and $F(P^\lambda_x) q W[\nu]$ and $\mu - F^{-1}(W) = \mathbb{D}_\tau(\mu - F^{-1}(W))$, by hypothesis. If $s \in S$ is such that $F(x, s) = \mu(x)$, then

$$\lambda + F^{-1}(W)(x) = \lambda + (\mu(x) \wedge W(s)) = (\lambda + \mu(x)) \wedge (\lambda + W(s)) > \mu(x) \wedge \nu(s) = \mu(x).$$

Therefore, $P^\lambda_x \notin \mu - F^{-1}(W) = \mathbb{D}_\tau(\mu - F^{-1}(W), r)$. Then, there exists an R^r_τ-neighborhood U of P^λ_x such that $C_\tau(U, r)\bar{q}(\mu - F^{-1}(W))[\mu]$, which implies that $C_\tau(U, r) + (\mu - F^{-1}(W)) \leq \mu$ and hence $C_\tau(U, r) \leq F^{-1}(W)$. Thus, $U \leq F^{-1}(W)$ and $F(U) \leq F(F^{-1}(W)) \leq W$. Since U is an R^r_τ-neighborhood of P^λ_x, we have $P^\lambda_x q U[\mu]$ and $I_\tau(C_\tau(U, r), r) = U$. Thus, $F(I_\tau(C_\tau(U, r), r)) = F(U) \leq W$. Hence the theorem follows. $\square$

Theorem 4.10. *Let $r, q \in I_0$ be such that $r < q$. If $F : (\mu, \tau) \to (\nu, \sigma)$ is fuzzy super r_1-continuous, then F is fuzzy super $[r, q]_1$-continuous.*

Proof. Let $A \in \Im_\mu$. Using the assumption and using Theorem 2.6[4], we have $F(\mathbb{D}_\tau(A,r)) \leq C_\sigma(F(A),r) \leq C_\sigma(F(A),q)$. Hence, F is fuzzy super $[r,q]_1$-continuous. □

The statement of the above theorem is not true when $r > q$.

Counterexample 4.11. Let $X = \{x,y\}$, $S = \{s,t\}$, $\mu^{[0.6,0.6]}_{[x,y]} \in I^X$, $\nu^{[0.6,0.6]}_{[s,t]} \in I^S$ and $V1^{[0.3,0.3]}_{[s,t]} \in \Im_\nu$. If $\tau : \Im_\mu \to I$ and $\sigma : \Im_\nu \to I$ are, respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$$F(x,s) = 0.6, F(x,t) = 0, F(y,s) = 0, F(y,t) = 0.6.$$

We fix $r = 0.7$ and $q = 0.5$. If $A = 0_X$, then $F(\mathbb{D}_\tau(0_X,r)) \leq C_\sigma(F(0_X),r)$. If $A \neq 0_X$, then $C_\sigma(F(A),r) = \nu \geq F(\mathbb{D}_\tau(A,r))$. Hence, F is fuzzy super r_1-continuous. Let $A^{[0,0.3]}_{[x,y]} \in \Im_\mu$ and $P^{0.35}_y \in \mu$. Since the only R^r_τ-neighborhood of $p^{0.35}_y$ is μ, we have $p^{0.35}_y \in \mathbb{D}(A,r)$ and hence $F(p^{0.35}_y) \in F(\mathbb{D}(A,r))$. But V_1 is a Q^q_σ-neighborhood of $p^{0.35}_t = F(P^{0.35}_y)$ such that $V\bar{q}F(A)[\nu]$. Hence, it follows that $p^{0.35}_t \notin C_\sigma(F(A),q)$. Thus, F is not fuzzy super $[r,q]_1$-continuous.

Theorem 4.12. *Let $r,q \in I_0$ be such that $q < r$. If $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy super $[r,q]_1$-continuous, then F is fuzzy super r_1-continuous and F is fuzzy super q_1-continuous.*

Proof. Let $A \in \Im_\mu$. By hypothesis and by Theorems 3.3[4], 2.6[4], we obtain $F(\mathbb{D}_\tau(A,q)) \leq F(\mathbb{D}_\tau(A,r)) \leq C_\sigma(F(A),q) \leq C_\sigma(F(A),r)$. Hence, F is fuzzy super r_1-continuous and F is fuzzy super q_1-continuous. □

The statement of the above theorem is not true when $q > r$.

Counterexample 4.13. Let $X = \{x,y\}$, $S = \{s,t\}$, $\mu^{[0.6,0.6]}_{[x,y]}$, $\nu^{[0.6,0.6]}_{[s,t]}$. Define a fuzzy subset $V_1 \in \Im_\nu$ by $V1^{[0.3,0.3]}_{[s,t]}$. If $\tau : \Im_\mu \to I$ and $\sigma : \Im_\nu \to I$ are defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

then obviously (μ,τ) and (ν,σ) are smooth fuzzy topological space. Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$$F(x,s) = 0.6, F(x,t) = 0, F(y,s) = 0, F(y,t) = 0.6.$$

We fix $r = 0.5$ and $q = 0.7$. If $A = 0_X$, then $F(\mathbb{D}_\tau(0_X,r)) \leq C_\sigma(F(0_X),q)$. If $A \neq 0_X$, then $C_\sigma(F(A),q) = \nu \geq F(\mathbb{D}_\tau(A,r))$. Hence, F is fuzzy super $[r,q]_1$-continuous. As in the previous counterexample, we can verify that F is not fuzzy super r_1-continuous, by showing that

$$F(P^{0.35}_y) \in F(\mathbb{D}(A,r)) \text{ and } F(P^{0.35}_y) = P^{0.35}_t \notin C_\sigma(F(A),r).$$

Theorem 4.14. *Let $r, q \in I_0$ be such that $r < q$. If $F : (\mu, \tau) \to (\nu, \sigma)$ is fuzzy super r_2-continuous, then F is fuzzy super $[r, q]_2$-continuous.*

Proof. Let $A \in \mathfrak{I}_\mu$. By assumption and by Theorem 2.6[4], we immediately get $\mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(C_\sigma(V, r)) \leq F^{-1}(C_\sigma(V, q))$. Hence, F is fuzzy super $[r, q]_1$-continuous. □

The statement of the above theorem is not true when $q < r$

Counterexample 4.15. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu_{[x,y]}^{[0.8,0.8]} \in I^X$, $\nu_{[s,t]}^{[0.8,0.8]} \in I^S$, $U1_{[x,y]}^{[0.4,0.4]} \in \mathfrak{I}_\mu$ and $V1_{[s,t]}^{[0.4,0.4]} \in \mathfrak{I}_\nu$. If $\tau : \mathfrak{I}_\mu \to I$ and $\sigma : \mathfrak{I}_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Define a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.8.$$

We fix $r = 0.8$ and $q = 0.4$. If $B = 0_S$, then $\mathbb{D}_\tau(F^{-1}(0_s), r) \leq F^{-1}(C_\sigma(0_S, r))$. If $B \neq 0_S$, then $F^{-1}(C_\sigma(B, r)) = \mu \geq \mathbb{D}_\tau(F^{-1}(B), r)$. Hence, F is fuzzy super r_2-continuous. Let $B_{[s,t]}^{[0,0.4]} \in \mathfrak{I}_\nu$ and $P_t^{0.45} \in \nu$. Clearly, we have

$$F^{-1}(P_t^{0.45}) = P_y^{0.45} \in \mathbb{D}_\tau(F^{-1}(B), r).$$

Since V_1 is a Q_σ^q-neighborhood of $P_t^{0.45}$ such that $V_1(t) + B(t) = 0.4 + 0.4 = 0.8 = \nu(t)$ and $V_1(s) + B(s) = 0.4 + 0 = 0.4 < 0.8 = \nu(s)$, we get that $P_t^{0.45} \notin C_\sigma(B, q)$ and hence $P_y^{0.45} \notin F^{-1}(C_\sigma(B, q))$. Thus, F is not fuzzy super $[r, q]_2$-continuous.

Theorem 4.16. *Let $r, q \in I_0$ be such that $q < r$. If $F : (\mu, \tau) \to (\nu, \sigma)$ is fuzzy super $[r, q]_2$-continuous, then F is fuzzy super r_2-continuous and F is fuzzy super q_2-continuous.*

Proof. Let $A \in \mathfrak{I}_\mu$. Using the assumption and using Theorems 3.3(4), 2.6(4), we get $\mathbb{D}_\tau(F^{-1}(V), q) \leq \mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(C_\sigma(V, q)) \leq F^{-1}(C_\sigma(V, r))$. Hence, F is fuzzy super r_2-continuous and F is fuzzy super q_2-continuous. □

The statement of the above theorem is not true when $q > r$

Counterexample 4.17. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu_{[x,y]}^{[0.8,0.8]} \in I^X$, $\nu_{[s,t]}^{[0.8,0.8]} \in I^S$, $V1_{[s,t]}^{[0.4,0.4]} \in \mathfrak{I}_\nu$. We define $\tau : \mathfrak{I}_\mu \to I$ and $\sigma : \mathfrak{I}_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases} \quad \textit{and} \quad \sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Define a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.8.$$

We fix $r = 0.4$ and $t = 0.8$. If $B = 0_S$, then $\mathbb{D}_\tau(F^{-1}(0_S), r) \leq F^{-1}(C_\sigma(0_S, t))$. If $B \neq 0_S$, then $F^{-1}(C_\sigma(B, q)) = \mu \geq \mathbb{D}_\tau(F^{-1}(B), r)$. Hence, F is fuzzy super $[r, q]_2$-continuous. Let $B_{[s,t]}^{[0,0.4]} \in \Im_\nu$ and $P_t^{0.45} \in \nu$. As in the previous counterexample, we have

$$F^{-1}(P_t^{0.45}) = P_y^{0.45} \in \mathbb{D}_\tau(F^{-1}(B), r) \text{ but } P_y^{0.45} \notin F^{-1}(C_\sigma(B, r)).$$

Thus, F is not fuzzy super r_2-continuous.

The results obtained in this section are summarized in the following implication diagram.

$$\begin{array}{ccccc}
 & & FS\text{-}[r,q]_1\text{-}C & & FS\text{-}[r,q]_2\text{-}C \\
 & & (q>r \Uparrow)\ (\Downarrow q<r) & & (q>r \Uparrow)\ (\Downarrow q<r) \\
FSC & \text{1-1, } \overset{F(\mu)=\nu}{\Longrightarrow} & FS\text{-}r_1\text{-}C & \Longrightarrow & FS\text{-}r_2\text{-}C \\
\Uparrow & & & & \Downarrow \\
FS\text{-}r_4\text{-}C & & \text{1-1, } \overset{F(\mu)=\nu}{\Longleftarrow} & & FS\text{-}r_3\text{-}C
\end{array}$$

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