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A study of fuzzy soft AG-groupoids

Madad Khan, Faisal Ilyas, Muhammad Gulistan, Saima Anis

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Abstract. In this paper we characterize regular and intra-regular AG-groupoids using generalized fuzzy soft left and soft bi-ideals.

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Corresponding Author: Madad Khan (madadmath@yahoo.com)

1. Introduction

It is common knowledge that common models with their limited boundaries of truth and falsehood are not sufficient to detect the reality so there is a need to discover other systems which are able to address the daily life problems. In every branch of science problems arise which abound with uncertainties and impaction. Some of these problems are related to human life, some others are subjective while others are objective and classical methods are not sufficient to solve such problems because they can not handle various ambiguities involved. To overcome this problem, Zadeh [32] introduced the concept of a fuzzy set which provides a useful mathematical tool for describing the behavior of systems that are either too complex or are ill-defined to admit precise mathematical analysis by classical methods. The literature in fuzzy set theory is rapidly expanding and application of this concept can be seen in a variety of disciplines such as artificial intelligence, computer science, control engineering, expert systems, operating research, management science, and robotics.

Fuzzy sets are also closely related to other soft computing models such as rough sets [25], random sets [12] and soft sets [10], [11], [20]. Fuzzy groups have been first considered by Rosenfeld [22] and fuzzy semigroups by Kuroki [16]. Yaqoob et al. [24] introduced interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ideals of ternary semigroups. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [14]. Faisal et al. [5] defined $(\in, \in \vee q_k)$-fuzzy Γ-ideals of Γ-semigroups.

The concept of soft set was given by Molodtsov [20] in 1999, which is a completely new approach for modeling and uncertainty. Aktas and Cagman [2] introduce the basic properties of soft sets. Jun [13] applied soft sets to the theory of BCK/BCI-algebras, and introduced the concept of soft BCK/BCI-algebras. Feng et al. [9] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Maji at el. [17] presented the definition of fuzzy soft set. Aygunoglu and Aygun [3] introduce the concept of fuzzy soft group and in the meantime, discuss some properties and structural characteristics of fuzzy soft group. Maji et al. [18, 19] presented the notion of the intuitionistic fuzzy soft set theory which is based on a combination of the intuitionistic fuzzy set. Yaqoob et al. [23] defined intuitionistic fuzzy soft groups by using (t, s)-norm. Akram and Yaqoob [1] introduced the idea of intuitionistic fuzzy soft ordered ternary semigroups.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. If an AG-groupoid contains left identity then this left identity is unique. Mostly an AG-groupoid with left identity works like a commutative semigroup. Moreover every commutative AG-groupoid becomes a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e, moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

In this paper we have introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in a new non-associative algebraic structure, that is, in an AG-groupoid and developed some new results. We have defined a regular AG-groupoid and characterized it by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

2. AG-groupoids

A groupoid $(S, .)$ is called an AG-groupoid (LA-semigroup in some articles [21], if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [15], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all a, b, c, $d \in S$. If an AG-groupoid contains left identity, then the law $a(bc) = b(ac)$ holds for all $a, b, c \in S$. In [26] it is given that $L[a] = a \cup Sa$ is a principal left ideal of S generated by a.Moreover it is easy to see that $B[a] = a \cup a^2 \cup (aS)a$ is a principal bi-ideal generated by a. Faisal et al. [6, 8, 7] and Yaqoob et al. [29, 30, 28] studied several results on fuzzy AG-groupoids.

A fuzzy subset f of a given set S is a mapping from S into usual closed interval of real numbers $[0, 1]$ that is $f : S \longrightarrow [0,\ 1]$. For any two fuzzy subsets f and g of S, $f \leq g$ means that, $f(x) \leq g(x)$ for all x in S. Let f and g be any fuzzy subsets of an AG-groupoid S. Then, the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \sup\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b,\ c \in S, \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1. A fuzzy subset f of S is of the form

$$f(y) = \begin{cases} t(\neq 0) \ if \ y = x, \\ 0 \text{ otherwise,} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t, where $t \in (0, 1]$.

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $C \subseteq D$, we define $\chi^{\delta}_{\gamma C}$ be the fuzzy subset of X by $\chi^{\delta}_{\gamma C}(x) \geq \delta$ for all $x \in C$ and $\chi^{\delta}_{\gamma C}(x) \leq \gamma$ otherwise. Clearly, $\chi^{\delta}_{\gamma C}$ is the characteristic function of C if $\gamma = 0$ and $\delta = 1$. Moreover $\chi^{\delta}_{\gamma X}(x) = 1$, for all $x \in X$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that
(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Definition 2.2. A relation on $\mathcal{F}(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", defined as follows:

For any f, $g \in \mathcal{F}(X)$, $f \subseteq \vee q_{(\gamma,\delta)} g$ this implies $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee_{q\delta} g$ for all $x \in X$, and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ,δ)-equal, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$, for all $x \in X$.

Lemma 2.3. *Let f and g are fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$ *for all $x \in X$.*

Proof. It is same as in [31]. □

Lemma 2.4. *Let f, g and $h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

Proof. It is same as in [31]. □

It is shown in [31] that "$=_{(\gamma,\delta)}$" is an equivalence relation on $\mathcal{F}(X)$. It is also notified that $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 2.5. *Let A, B be any non empty subsets of an AG-groupoid S with left identity. Then we have*
(1) $A \subseteq B$ *if and only if* $\chi^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma B}$, *where* $r \in (\gamma, 1]$ *and* $\gamma, \delta \in [0, 1]$.
(2) $\chi^{\delta}_{\gamma A} \cap \chi^{\delta}_{\gamma B} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma (A \cap B)}$.
(3) $\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma (AB)}$.

Proof. It is same in [31]. □

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Soft Ideals of AG-groupoids

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U.

Definition 3.1 ([17]). A pair $\langle G, A \rangle$ is called a fuzzy soft set over U, where $A \subseteq E$ and G is a mapping given by $G : A \longrightarrow \mathcal{F}(U)$, where $\mathcal{F}(U)$ is the set of all fuzzy subsets of U.

In general, for every $\varepsilon \in A$, $G(\varepsilon)$ is a fuzzy set of U and it is called fuzzy value set of parameter ε.

Definition 3.2. The extended intersection of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is a fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and defined as

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B. \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \tilde{\cap} \langle G, B\rangle$.

Definition 3.3 ([27])**.** Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that $\langle F, A\rangle$ is a fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle \subset \langle G, B\rangle$ if

(i) $A \subseteq B$,

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$.

$\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be fuzzy soft equal and write $\langle F, A\rangle = \langle G, B\rangle$ if $\langle F, A\rangle \subset \langle G, B\rangle$ and $\langle G, B\rangle \subset \langle F, A\rangle$.

Definition 3.4. Let $V \subseteq U$. A fuzzy soft set $\langle F, A\rangle$ over V is said to be a relative whole $(\gamma;\delta)$-fuzzy soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V; A)$, if $F(\varepsilon) = \mathcal{X}_{\gamma}^{\delta}$ for all $\varepsilon \in A$.

Definition 3.5. The product of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over an AG-groupoid S is a fuzzy soft set over S, denoted by $\langle F \circ G, C\rangle$, where $C = A \cup B$ and

$$(F \circ G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \circ G(\varepsilon) & \text{if } \varepsilon \in A \cap B. \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F \circ G, C\rangle = \langle F, A\rangle \odot \langle G, B\rangle$.

Definition 3.6. A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called fuzzy soft left (right) ideal over S if

$$\Sigma\,\langle S, E\rangle \odot \langle F, A\rangle \subset \langle F, A\rangle(\langle F, A\rangle \odot \Sigma\,\langle S, E\rangle \subset \langle F, A\rangle).$$

and fuzzy soft bi-ideal over S if

$$\langle F, A\rangle \odot \langle F, A\rangle \subset \langle F, A\rangle$$

and

$$[\langle F, A\rangle \odot \Sigma\,\langle S, A\rangle] \odot \langle F, A\rangle \subset \langle F, A\rangle.$$

Definition 3.7. $\langle F, A\rangle$ is an (γ, δ)-fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle \subset_{(\gamma,\delta)} \langle G, B\rangle$ if

(i) $A \subseteq B$,

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.

Definition 3.8. A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S if

$$\Sigma(S, E) \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle.$$

and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal over S if

(i) $\langle F, A\rangle \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle$, and

(ii) $[\langle F, A\rangle \odot \Sigma(S, E)] \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle$.

Example 3.9. Let $S = \{1, 2, 3\}$ and the binary operation " $\cdot$ " defines on S as follows:

$\cdot$	1	2	3
1	2	2	3
2	3	3	3
3	3	3	3

Then $(S, \cdot)$ is an AG-groupoid.

Example 3.10. Let $E = \{0.35, 0.4\}$ and define a fuzzy soft set $\langle F, A\rangle$ over S as follows:

$$F(\varepsilon)(x) = \begin{cases} 2\varepsilon \text{ if } x \in \{1, 2\}, \\ \frac{2}{5} \quad \text{otherwise.} \end{cases}$$

Then $\langle F, A\rangle$ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$- fuzzy soft left ideal of S.

Again let $E = \{0.7, 0.8\}$ and define a fuzzy soft set $\langle G, A\rangle$ over S as follows:

$$G(\varepsilon)(x) = \begin{cases} \varepsilon \text{ if } x \in \{1, 2\}, \\ \frac{2}{5} \quad \text{otherwise.} \end{cases}$$

Then $\langle F, A\rangle$ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.4})$- fuzzy soft bi-ideal of S.

Definition 3.11. An element a of an AG-groupoid S is called regular if there exist $x \in S$ such that $a = (ax)a$ and S is called regular, if every element of S is regular.

Corollary 3.12. *Let $(S, ., \leq)$ be an ordered semigroup and $P \subseteq S$. Then P is a left ideal(resp., right ideal, bi-ideal, quasi-ideal) of S if and only if $\Sigma(P, A)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi ideal) over S for any $A \subseteq E$.*

Theorem 3.13. *Let S be an AG-groupoid with left identity e. Then the following conditions are equivalent:*

(i) S is regular.

$(ii)$$I \cap L \subseteq IL$, for any left ideal L and ideal I of S.

$(iii)$$\langle F, I\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, I\rangle \odot \langle G, L\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle F, I\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle G, L\rangle$ over S.

(iv) $\langle F, A\rangle \subset_{(\gamma,\delta)} (\langle F, A\rangle \odot \Sigma(S, E)) \odot \langle F, A\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle F, A\rangle$.

Proof. $(i) \implies (iv)$ Let S be a regular and $\langle F, A\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal of S. Now we have to show that $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} (F(\varepsilon) \circ \chi^\delta_{\gamma S}) \circ F(\varepsilon)$. Let a be any element of S then there exist an elements x in S such that $a = (ax)a$. Then we have

$$\begin{aligned}
& \max\left\{((F(\varepsilon)\circ\chi^{\delta}_{\gamma S})\circ F(\varepsilon))(a),\gamma\right\} \\
= & \max\left\{\bigvee_{a=bc}\left\{(F(\varepsilon)\circ\chi^{\delta}_{\gamma S})(b)\wedge F(\varepsilon)(c)\right\},\gamma\right\} \\
\geq & \max\{(F(\varepsilon)\circ\chi^{\delta}_{\gamma S})(ax)\wedge F(\varepsilon)(a),\gamma\} \\
= & \max\{\bigvee_{(ax)=pq}(F(\varepsilon)(p)\wedge\chi^{\delta}_{\gamma S}(q))\wedge F(\varepsilon)(a),\gamma\} \\
\geq & \max\{(F(\varepsilon)(a)\wedge\chi^{\delta}_{\gamma S}(x)\wedge F(\varepsilon)(a),\gamma\} \\
= & \max\{(F(\varepsilon)(a)\wedge 1\wedge F(\varepsilon)(a),\gamma\} \\
= & \min\left\{\max\{F(\varepsilon)(a),\gamma\},1,\max\{F(\varepsilon)(a),\gamma\}\right\} \\
\geq & \min\{\min\{(F(\varepsilon)(a),\delta\},1,\min\{F(\varepsilon)(a),\delta\}\} \\
= & \min\left\{[F(\varepsilon)(a)\wedge 1\wedge F(\varepsilon)](a),\delta\right\} \\
= & \min\left\{F(\varepsilon)(a),\delta\right\}.
\end{aligned}$$

So, $\min\{F(\varepsilon)(a),\delta\}\leq\max\left\{((F(\varepsilon)\circ\chi^{\delta}_{\gamma S})\circ F(\varepsilon))(a),\gamma\right\}$. This implies that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ\chi^{\delta}_{\gamma S})\circ F(\varepsilon)$. Therefore, $H(\varepsilon)\subset\vee q_{(\gamma,\delta)}((F\circ\chi^{\delta}_{\gamma S})\circ F)(\varepsilon)$. Thus $\langle F,A\rangle\subset_{(\gamma,\delta)}(\langle F,A\rangle\odot\Sigma(S,E))\odot\langle F,A\rangle$.

$(iv)\Longrightarrow(iii)$ Assume that (iv) holds. Let any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft ideal $\langle F,I\rangle$ and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft left ideal $\langle G,L\rangle$ over S. Then we have

$$\begin{aligned}
&((\langle F,I\rangle\tilde{\cap}\langle G,L\rangle)\odot\Sigma(S,E))\tilde{\cap}(\Sigma(S,E)\odot(\langle F,I\rangle)\tilde{\cap}\langle G,L\rangle)) \\
&=\{(\langle F,I\rangle\odot\Sigma(S,E))\tilde{\cap}(\langle G,L\rangle\odot\Sigma(S,E))\}\tilde{\cap}\{(\Sigma(S,E)\odot\langle F,I\rangle)\tilde{\cap}(\Sigma(S,E)\odot\langle G,L\rangle)\} \\
&\subset_{(\gamma,\delta)}(\langle F,I\rangle\tilde{\cap}\langle G,L\rangle)
\end{aligned}$$

But by (iv), we have

$$\begin{aligned}
\langle F,I\rangle\tilde{\cap}\langle G,L\rangle & \subset_{(\gamma,\delta)}\{(\langle F,I\rangle\tilde{\cap}\langle G,L\rangle)\odot\Sigma(S,E)\}\odot(\langle F,I\rangle\tilde{\cap}\langle G,L\rangle) \\
& \subset_{(\gamma,\delta)}(\langle F,I\rangle\odot\Sigma(S,E))\odot(\langle F,I\rangle\tilde{\cap}\langle G,L\rangle) \\
& \subset_{(\gamma,\delta)}\langle F,I\rangle\odot(\langle F,I\rangle\tilde{\cap}\langle G,L\rangle) \\
& \subset_{(\gamma,\delta)}\langle F,I\rangle\odot\langle G,L\rangle.
\end{aligned}$$

$(iii)\Longrightarrow(ii)$ Let I be an ideal and L be the left ideal of S, then $\Sigma(I,E)$ and $\Sigma(L,E)$ are $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft ideal and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft left ideal of S respectively. Assume that (iii) holds, we have

$$\Sigma(I,E)\tilde{\cap}\Sigma(L,E)\subset_{(\gamma,\delta)}\Sigma(I,E)\odot\Sigma(L,E).$$

So,

$$\chi^{\delta}_{\gamma(I\cap L)}=_{(\gamma,\delta)}\chi^{\delta}_{\gamma I}\cap\chi^{\delta}_{\gamma L}\subseteq\vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I}\odot\chi^{\delta}_{\gamma L})=_{(\gamma,\delta)}\chi^{\delta}_{\gamma IL}.$$

Thus $I\cap L\subseteq IL$.

$(ii)\Longrightarrow(i)$ Let I be an ideal and L be a left ideal of S. Assume that (ii) holds, So

$$\begin{aligned}
(a\cup aS)\cap(Sa) & \subseteq (a\cup aS)(Sa)=a(Sa)\cup(aS)(Sa) \\
& = Sa^2\cup(aS)a\subseteq(aS)a.
\end{aligned}$$

Hence S is regular. □

Theorem 3.14. *Let S be an AG-groupoid with left identity e. Then the following statements are equivalent:*
(i) S is a regular
(ii) $L_1 \cap I \cap L_2 \subseteq (L_1 I)L_2$, for any left ideals L_1, L_2 and ideal I of S.
(iii) $\langle F, A\rangle \tilde{\cap} \langle G, I\rangle \tilde{\cap} \langle F, A\rangle \subset_{(\gamma,\delta)} (\langle F, A\rangle \odot \langle G, I\rangle) \odot \langle F, A\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideals $\langle F, A\rangle$, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle G, I\rangle$ over S.

Proof. $(i) \Longrightarrow (iii)$ Let S be a regular and $\langle F, A\rangle$ and $\langle F, A\rangle$ are any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideals and $\langle F, I\rangle$ be a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S.
Then for any $\varepsilon \in I \cap L$, There are following cases.
Case1: $\varepsilon \in I - A$. Then $(F \circ G) = G(\varepsilon)$.
Case2: $\varepsilon \in A - I$. Then $(F \circ G) = F(\varepsilon)$.
Case3: $\varepsilon \in I \cap A$. Then $(F \circ G) = F(\varepsilon) \circ G(\varepsilon)$.
Since for any $a \in S$ there exist $x \in S$ such that $a = (ax)a = [((ax)(a))x]a = [(xa)(ax)]a$. First we have to show that

$$F(\varepsilon) \cap G(\varepsilon) \cap F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon)) \circ F(\varepsilon).$$

So

$$\begin{aligned}
& \max\{(F(\varepsilon) \circ G(\varepsilon)) \circ F(\varepsilon)(a), \gamma\} \\
= \; & \max\left\{\bigvee_{a=uv} \{(F(\varepsilon) \circ G(\varepsilon))(u) \wedge F(\varepsilon)(v)\}, \gamma\right\} \\
\geq \; & \max\{(F(\varepsilon) \circ G(\varepsilon))((xa)(ax)) \wedge F(\varepsilon)(ea), \gamma\} \\
= \; & \max\{\bigvee_{xa(ax)=rs} (F(\varepsilon)(r) \wedge G(\varepsilon)(s)) \wedge F(\varepsilon)(ea), \gamma\} \\
\geq \; & \max\{(F(\varepsilon)(xa) \wedge G(\varepsilon)(ax) \wedge F(\varepsilon)(ea), \gamma\} \\
= \; & \min\{\max\{F(\varepsilon)(xa), \gamma\}, \max\{G(\varepsilon)(ax), \gamma\}, \max\{F(\varepsilon)(ea), \gamma\}\} \\
\geq \; & \min\{\min\{F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(a), \delta\}, \min\{F(\varepsilon)(a), \delta\}\} \\
= \; & \min\{F(\varepsilon)(a) \wedge G(\varepsilon)(a) \wedge F(\varepsilon)(a), \delta\} \\
= \; & \min\{(F(\varepsilon) \wedge G(\varepsilon) \wedge F(\varepsilon))(a), \delta\} \\
= \; & \min\{(F(\varepsilon) \cap G(\varepsilon) \cap F(\varepsilon))(a), \delta\}.
\end{aligned}$$

So, $\min\{(F(\varepsilon) \cap G(\varepsilon) \cap F(\varepsilon))(a), \delta\} \leq \max\{(F(\varepsilon) \circ G(\varepsilon)) \circ F(\varepsilon)(a), \gamma\}$ .This implies that

$$F(\varepsilon) \cap G(\varepsilon) \cap F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon)) \circ F(\varepsilon),$$

that is, $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}((F \circ g) \circ F)(\varepsilon)$. Therefore

$$\langle F, A\rangle \tilde{\cap} \langle G, I\rangle \tilde{\cap} \langle F, A\rangle \subset_{(\gamma,\delta)} (\langle F, A\rangle \odot \langle G, I\rangle) \odot \langle F, A\rangle.$$

$(iii) \Longrightarrow (ii)$ Assume that (iii) holds. Let L_1 and L_2 are left ideals and I be an ideal of S, then by $\Sigma(L_1, E)$, $\Sigma(L_2, E)$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideals and $\Sigma(I, E)$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S. Now by (iii), we have

$$\Sigma(L_1, E) \tilde{\cap} \Sigma(I, E) \tilde{\cap} \Sigma(L_2, E) \subset_{(\gamma,\delta)(} \Sigma(L_1, E) \odot \Sigma(I, E)) \odot \Sigma(L_2, E).$$

So we have

$$\begin{aligned}\chi^{\delta}_{\gamma(L_1\cap I\cap L_2)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L_1}\cap\chi^{\delta}_{\gamma I}\cap\chi^{\delta}_{\gamma L_2}\\ &\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L_1}\odot\chi^{\delta}_{\gamma I})\odot\chi^{\delta}_{\gamma L_2}\\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L_1 I}\odot\chi^{\delta}_{\gamma L_2}\\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(L_1 I)L_2}.\end{aligned}$$

This implies that $L_1\cap I\cap L_2\subseteq(L_1I)L_2$.

$(ii)\implies(i)$ Assume that (ii) holds. Let L_1 and L_2 are left ideals and I be an ideal of S. So by (ii),

$$\begin{aligned}(Sa)\cap(Sa\cup aS)\cap(Sa) &\subseteq [(Sa)(Sa\cup aS)](Sa)\\ &= [((Sa)(Sa))\cup((Sa)(aS))](Sa)\\ &= [Sa^2\cup(Sa)(aS)](Sa)\\ &= (Sa^2)(Sa)\cup[(Sa)(aS)](Sa)\\ &\subseteq (Sa^2)(S)\cup((Sa)(S))((aS)a)\\ &\subseteq (Sa^2)S\cup S((aS)(a))\\ &= (Sa^2)S\cup(aS)(Sa)\\ &\subseteq (aS)a\cup(aS)a=(aS)a.\end{aligned}$$

Thus S is regular. □

Theorem 3.15. *Let S be an AG-groupoid with left identity e. Then the following conditions are equivalent:*

(i) S is regular.

(ii) $I\cap L\subseteq I(IL)$, for any ideal I and left ideal L of S.

(iii) $\langle F,I\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}\langle F,I\rangle\odot(\langle F,I\rangle\odot\langle G,L\rangle)$, for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,I\rangle$ and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal $\langle G,L\rangle$ over S.

Proof. $(i)\implies(iii)$ Let S be a regular and $\langle F,I\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and $\langle G,L\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S. Now we have to show that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon))$. Then for any $\varepsilon\in I\cap L$, we consider the following cases.

Case1**:** $\varepsilon\in I-L$. Then $(F\circ G)=F(\varepsilon)$.

Case2: $\varepsilon\in L-I$. Then $(F\circ G)=G(\varepsilon)$.

Case3: $\varepsilon\in I\cap L$. Then $(F\circ G)(\varepsilon)=F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon))$.

Let a be an element of S then there exist an element x in S such that $a = (ax)a = (ax)\{(ax)a\}$. Then we have

$$\begin{aligned}
& \max\{(F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon)))(a),\gamma\} \\
= & \max\left\{\bigvee_{a=pq}\{F(\varepsilon)(p)\wedge(F(\varepsilon)\circ G(\varepsilon))(q)\},\gamma\right\} \\
\geq & \max\{F(\varepsilon)\,(ax)\wedge(F(\varepsilon)\circ G(\varepsilon))((ax)a),\gamma\} \\
= & \max\{F(\varepsilon)\,(ax)\wedge\bigvee_{(ax)a=uv}F(\varepsilon)(u)\wedge G(\varepsilon)(v),\gamma\} \\
\geq & \max\{F(\varepsilon)\,(ax)\wedge(F(\varepsilon)\,(ax)\wedge G(\varepsilon)(ea)),\gamma\} \\
= & \max\{F(\varepsilon)\,(ax)\wedge F(\varepsilon)\,(ax)\wedge G(\varepsilon)(ea),\gamma\} \\
= & \min\{\max\{F(\varepsilon)(ax),\gamma\},\max\{F(\varepsilon)(ax),\gamma\},\max\{G(\varepsilon)(ea),\gamma\}\} \\
\geq & \min\{\min\{F(\varepsilon)(a),\delta\},\min\{F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(a),\delta\}\} \\
= & \min\{F(\varepsilon)(a)\wedge F(\varepsilon)(a)\wedge G(\varepsilon)(a),\delta\} \\
= & \min\{F(\varepsilon)(a)\wedge G(\varepsilon)(a),\delta\} \\
= & \min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}.
\end{aligned}$$

So, $\min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\} \leq \max\{(F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon)))(a),\gamma\}$. This implies that

$$F(\varepsilon)\cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon)).$$

Thus we have

$$H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F\circ(F\circ G))(\varepsilon).$$

Therefore,

$$\langle F,I\rangle\tilde{\cap}\langle G,L\rangle \subset_{(\gamma,\delta)} \langle F,I\rangle\odot(\langle F,I\rangle\odot\langle G,L\rangle).$$

$(iii) \Longrightarrow (ii)$ Assume that the condition (ii) holds. Let L be left ideal and I be an ideal of S, then $\Sigma(L,E)$ and $\Sigma(I,E)$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal of S respectively. Now by the assumption, we have

$$\Sigma(I,E)\tilde{\cap}\Sigma(L,E) \subset_{(\gamma,\delta)} \Sigma(I,E)\odot(\Sigma(I,E)\odot\Sigma(L,E)).$$

Hence

$$\begin{aligned}
\chi^{\delta}_{\gamma(I\cap L)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma I}\cap\chi^{\delta}_{\gamma L} \\
&\subseteq \vee q_{(\gamma,\delta)}\chi^{\delta}_{\gamma I}\odot(\chi^{\delta}_{\gamma I}\odot\chi^{\delta}_{\gamma L}) \\
&= \vee q_{(\gamma,\delta)}\,\chi^{\delta}_{\gamma I}\odot\chi^{\delta}_{\gamma IL} \\
&= \vee q_{(\gamma,\delta)}\,\chi^{\delta}_{\gamma I(IL).}
\end{aligned}$$

Thus $I\cap L \subseteq I(IL)$.

$(ii) \Longrightarrow (i)$ Let I be an ideal and L be a left ideal of S. So by the assumption (ii), we have

$$\begin{aligned}
(a \cup aS) \cap (Sa) &\subseteq (a \cup aS)[(a \cup aS)(Sa)] \\
&= (a \cup aS)[a(Sa) \cup (aS)(Sa)] \\
&= (a \cup aS)[Sa^2 \cup (aS)(Sa)] \\
&\subseteq (a \cup aS)[(aS)a] \\
&= a((aS)a) \cup (aS)((aS)a) \\
&\subseteq (aS)(aa) \cup (aS)(aS)S \\
&\subseteq (aS)(Sa) \cup (aS)(SS)a \\
&= (aS)(Sa) \cup (aS)(Sa) \\
&\subseteq (aS)a.
\end{aligned}$$

Thus S is regular. $\square$

Theorem 3.16. *Let S be an AG-groupoid with left identity e. Then the following conditions are equivalent:*
(i) S is regular.
(ii) $L \subseteq (LS)L$, for any left ideal L of S.
(iii) $\langle F, L\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \Sigma(S, E)) \odot \langle F, L\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F, L\rangle$ over S.

Proof. $(i) \Longrightarrow (iii)$ Let S be a regular and $\langle F, L\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S. We have to show that $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ \chi^\delta_{\gamma S}) \circ F(\varepsilon)$. For any element a of S then there exist an elements x in S such that $a = (ax)a$. We have

$$\begin{aligned}
&\max\left\{((F(\varepsilon) \circ \chi^\delta_{\gamma S}) \circ F(\varepsilon))(a), \gamma\right\} \\
= &\max\left\{\bigvee_{a=pq}\left\{(F(\varepsilon) \circ \chi^\delta_{\gamma S})(p) \wedge F(\varepsilon)(q)\right\}, \gamma\right\} \\
\geq &\max\{\left\{(F(\varepsilon) \circ \chi^\delta_{\gamma S})(ax) \wedge F(\varepsilon)(a)\right\}, \gamma\} \\
= &\max\left\{\bigvee_{(ax)=uv} F(\varepsilon)(u) \wedge \chi^\delta_{\gamma S}(v) \wedge F(\varepsilon)(a), \gamma\right\} \\
\geq &\max\{F(\varepsilon)(ea) \wedge \chi^\delta_{\gamma S}(x) \wedge F(\varepsilon)(ea), \gamma\} \\
= &\max\{F(\varepsilon)(ea) \wedge 1 \wedge F(\varepsilon)(ea), \gamma\} \\
= &\min\{\max\{F(\varepsilon)(ta), \gamma\}, 1, \max\{F(\varepsilon)(ea), \gamma\}\} \\
\geq &\min\{\min\{F(\varepsilon)(a), \delta\}, 1, \min\{F(\varepsilon)(a), \delta\}\} \\
= &\min\{F(\varepsilon)(a) \wedge F(\varepsilon)(a) \wedge 1, \delta\} \\
= &\min\{F(\varepsilon)(a), \delta\}
\end{aligned}$$

So, $\min\{F(\varepsilon)(a), \delta\} \leq \max\left\{((F(\varepsilon) \circ \chi^\delta_{\gamma S}) \circ F(\varepsilon))(a), \gamma\right\}$. This implies that,

$$F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ \chi^\delta_{\gamma S}) \circ F(\varepsilon).$$

Thus we have

$$H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}((F \circ \chi^\delta_{\gamma S}) \circ F)(\varepsilon).$$

Therefore,

$$\langle F, L\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \Sigma(S, E)) \odot \langle F, L\rangle.$$

$(iii) \Longrightarrow (ii)$ Assume that the given condition (iii) holds. Let L be a left ideal of S, then $\Sigma(L, E)$ be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal of S. Now by the assumption, we have

$$\Sigma(L, E) \subset_{(\gamma,\delta)} (\Sigma(L, E) \odot \Sigma(S, E)) \odot \Sigma(L, E).$$

So we have

$$\begin{aligned} \chi^{\delta}_{\gamma L} &= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L} \odot \chi^{\delta}_{\gamma S}) \odot \chi^{\delta}_{\gamma L} \\ &\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L} \odot \chi^{\delta}_{\gamma S}) \odot \chi^{\delta}_{\gamma L} \\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma LS} \odot \chi^{\delta}_{\gamma L} \\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(LS)L}. \end{aligned}$$

Thus $L \subseteq (LS)L$.

$(ii) \Longrightarrow (i)$ Let L be a left ideal of S. Then by (ii),

$$\begin{aligned} Sa &\subseteq ((Sa)S)(Sa) = [(Sa)(SS)](Sa) \\ &= [(SS)(aS)](Sa) = [S(aS)](Sa) \\ &= [a(SS)](Sa) = (aS)(Sa) \subseteq (aS)a. \end{aligned}$$

Hence S is regular. □

Theorem 3.17. *Let S be an AG-groupoid with left identity e. Then the following conditions are equivalent:*

(i) S is regular.

(ii) $I \cap L \subseteq (SI)L$, for any ideal I and left ideal L of S.

(iii) $\langle F, I\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\Sigma(S, E) \odot \langle F, I\rangle) \odot \langle G, L\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle F, I\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle G, L\rangle$ over S.

Proof. $(i) \Longrightarrow (iii)$ Let S be a regular and $\langle F, I\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal and $\langle G, L\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal of S. Now we have to show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}\{(\chi^{\delta}_{\gamma S} \circ F(\varepsilon)) \circ G(\varepsilon)\}$. Then for any $\varepsilon \in I \cap L$, we consider the following cases.

Case1: $\varepsilon \in I - L$. Then $(F \circ G)(\varepsilon) = F(\varepsilon)$.

Case2: $\varepsilon \in L - I$. Then $(F \circ G)(\varepsilon) = G(\varepsilon)$.

Case3: $\varepsilon \in I \cap L$. Then $(F \circ G)(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$.

As S is regular, therefore for any a in S there exist x in S such that $a = (ax)a$

$$a = ((ax)a)x)a = ((xa)(ax))a = (a(ax))(xa).$$

Then we have

$$\begin{aligned}
& \max\left\{(\chi^{\delta}_{\gamma S}\circ F(\varepsilon))\circ G(\varepsilon),\gamma\right\}\\
= & \max\left\{\bigvee_{a=bc}\left\{(\chi^{\delta}_{\gamma S}\circ F(\varepsilon))(b)\wedge G(\varepsilon)(c)\right\},\gamma\right\}\\
\geq & \max\{(\chi^{\delta}_{\gamma S})\circ F(\varepsilon))(a(ax))\wedge G(\varepsilon)(xa),\gamma\}\\
= & \max\left\{\bigvee_{a(ax)=pq}\left\{(\chi^{\delta}_{\gamma S}(p)\wedge F(\varepsilon)(q))\wedge G(\varepsilon)(xa)\right\},\gamma\right\}\\
\geq & \max\{\chi^{\delta}_{\gamma S}(a)\wedge F(\varepsilon)(ax)\wedge G(\varepsilon)(xa),\gamma\}\\
= & \max\{1\wedge F(\varepsilon)(a)\wedge G(\varepsilon)(a),\gamma\}\\
= & \min\{1,\max\{F(\varepsilon)(a),\gamma\},\max\{G(\varepsilon)(a),\gamma\}\}.\\
\geq & \min\{1,\min\{F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(a),\delta\}\}\\
= & \min\{1\wedge F(\varepsilon)(a)\wedge G(\varepsilon)(a),\delta\}\\
= & \min\{(F(\varepsilon)\wedge G(\varepsilon))(a),\delta\}\\
= & \min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}.
\end{aligned}$$

Thus, $F(\varepsilon)\cap G(\varepsilon)\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma S}\circ F(\varepsilon))\circ G(\varepsilon)$. That is $H(\varepsilon)\subseteq \vee q_{(\gamma,\delta)}((\chi^{\delta}_{\gamma S}\circ F)\circ G)(\varepsilon)$. Therefore,

$$\langle F,I\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}\{\Sigma(S,E)\odot\langle F,I\rangle\}\odot\langle G,L\rangle.$$

$(iii)\implies(ii)$ Assume that the given condition (iii) holds. Let I be an ideal and L be the left ideal of S, then $\Sigma(I,E)$ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and $\Sigma(L,E)$ be a $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal of S. Now by the assumption, we have

$$\Sigma(I,E)\tilde{\cap}\Sigma(L,E)\subset_{(\gamma,\delta)}\{\Sigma(S,E)\odot\Sigma(I,E)\}\odot\Sigma(L,E).$$

Hence

$$\begin{aligned}
\chi^{\delta}_{\gamma I\cap L} &= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I}\tilde{\cap}\chi^{\delta}_{\gamma L})\\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma S}\odot\chi^{\delta}_{\gamma I})\odot\chi^{\delta}_{\gamma L}\\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma SI}\odot\chi^{\delta}_{\gamma L}\\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(SI)L}.
\end{aligned}$$

Thus $I\cap L\subseteq(SI)L$.

$(ii)\implies(i)$ Assume that (ii) holds. let I and L be ideal and left ideal of S. Then we have

$$\begin{aligned}
(a\cup aS)\cap(Sa) &\subseteq (S(a\cup aS))(Sa)\\
&= [Sa\cup S(aS)](Sa)\\
&= [Sa\cup a(SS)](Sa)\\
&= (Sa\cup aS)(Sa)\\
&= (Sa)(Sa)\cup(aS)(Sa)\\
&= (SS)(aa)\cup(aS)(Sa)\\
&= Sa^2\cup(aS)(Sa)\\
&\subseteq (aS)a.
\end{aligned}$$

Therefore S is regular. □

Definition 3.18. An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.

Lemma 3.19. *If (F, A) is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft two sided ideal of an intra-regular AG-groupoid S, then (F, A) is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft semiprime ideal of S.*

Proof. Let $F(\varepsilon)$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an intra-regular AG-groupoid. Then for any $a \in S$, there exists some $x, y \in S$ such that $a = (xa^2)y$. Now

$$\max\{F(\varepsilon)(a), \gamma\} = \max\{F(\varepsilon)(xa^2)y, \gamma\} \geq \min\{F(\varepsilon)(a^2), \delta\}.$$

Hence (F, A) is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft semiprime ideal of S. □

Theorem 3.20. *Let S be an AG- groupoid then the following conditions are equivalent:*

(i) S is intra-regular.

(ii) For every ideal A of S, $A \subseteq A^2$ and A is semiprime.

(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy soft ideal $\langle F, A\rangle$ of S, $\langle F, A\rangle \subseteq \vee q_{(\gamma,\delta)}(\langle F, A\rangle \odot \langle F, A\rangle)$, and $\langle F, A\rangle$ is fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$. Let S be an intra-regular AG-groupoid with left identity and $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal of S. Since S is intra-regular, therefore for any a in S, there exist x, y in S such that $a = (xa^2)y$. For any a in S there exist u and v in S such that $a = uv$, then

$$\begin{aligned}
\max\{(F(\varepsilon) \circ F(\varepsilon))(a), \gamma\} &= \max\left\{\bigvee_{a=uv}\{\{F(\varepsilon)(u) \wedge F(\varepsilon)(v)\}, \gamma\}\right\} \\
&\geq \max\{F(\varepsilon)(y(xa)) \wedge F(\varepsilon)(a)\}, \gamma\} \\
&= \max\{\min\{F(\varepsilon)(y(xa)), F(\varepsilon)(a)\}, \gamma\} \\
&= \min\{\max\{F(\varepsilon)(y(xa)), \gamma\}, \max\{F(\varepsilon)(a), \gamma\}\} \\
&\geq \min\{\min\{F(\varepsilon)(a), \delta\}, \min\{F(\varepsilon)(a), \delta\}\} \\
&= \min\{F(\varepsilon)(a) \wedge F(\varepsilon)(a), \delta\} \\
&= \min\{F(\varepsilon)(a), \delta\} \\
\min\{F(\varepsilon)(a), \delta\} &\leq \max\{(F(\varepsilon) \circ F(\varepsilon))(a), \gamma\}.
\end{aligned}$$

This implies that $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ F(\varepsilon))$, i.e $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F \circ F)(\varepsilon)$. Therefore $\langle F, A\rangle \subset_{(\gamma,\delta)} \{\langle F, A\rangle \odot \langle F, A\rangle\}$. Now we show that $\langle F, A\rangle$ is a fuzzy soft semiprime ideal of intra-regular AG-groupoid S, Since S is intra-regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Then

$$\begin{aligned}
\max\{F(\varepsilon)(a), \gamma\} &= \max\{F(\varepsilon)((xa^2)y), \gamma\} \\
&\geq \min\{F(\varepsilon)(a^2), \delta\}.
\end{aligned}$$

Hence $\langle F, A\rangle$ is fuzzy soft semiprime in S.

$(iii) \Rightarrow (ii)$.Assume that (iii) holds. Let A be any ideal of S. Then $\Sigma(A, E)$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S, So by assumption (iii), we get

$$\begin{aligned} \chi^\delta_{\gamma A} &= \chi^\delta_{\gamma A\cap A} =_{(\gamma,\delta)} \chi^\delta_{\gamma A} \cap \chi^\delta_{\gamma A} \\ &\subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma A} \odot \chi^\delta_{\gamma A} \\ &= {}_{(\gamma,\delta)} X^\delta_{\gamma A^2}. \end{aligned}$$

Hence, we get $A \subseteq A^2$. Now we show that A is semiprime. Let A be an ideal and $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal of S. Let $a^2 \in A$, then since $\chi^\delta_{\gamma A}$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an AG-groupoid S, So by (iii), we have $\max\{\chi^\delta_{\gamma A}(a), \gamma\} \geq \min\{\chi^\delta_{\gamma A}(a^2), \delta\} = \delta$, this implies that $\chi^\delta_{\gamma A}(a) \geq \delta$. Thus $a \in A$. Hence A is semiprime.

$(ii) \Rightarrow (i)$. Assume that every ideal is semiprime of S. Since Sa^2 is an ideal of an AG-groupoid S generated by a^2. Therefore

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is intra-regular. □

Lemma 3.21. *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft two sided ideal of an AG-groupoid S, is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft interior ideal of S.*

Proof. Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft two sided ideal of an AG-groupoid S, then for any $a, x, y \in S$,

$$\begin{aligned} \max\{F(\varepsilon)((xa)y), \gamma\} &\geq \max\{F(\varepsilon)(xa), \gamma\} \\ &\geq \min\{F(\varepsilon)(a), \delta\}. \end{aligned}$$

Hence $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft interior ideal of S. □

Theorem 3.22. *For an AG-groupoid S with left identity the following are equivalent:*

(i) S is intra-regular.

(ii) Every two sided ideal is semiprime.

(iii) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft two sided ideal $\langle F, A\rangle$ of S is fuzzy soft semiprime.

(iv) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft interior ideal $\langle F, A\rangle$ of S is fuzzy soft semiprime.

(v) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft generalized interior ideal $\langle F, A\rangle$ of S is soft semiprime.

Proof. $(i) \Rightarrow (v)$ Assume that (i) holds. Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft generalized interior ideal of an AG-groupoid S. Since S is intra-regular, then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$. We have

$$\begin{aligned} \max\{F(\varepsilon)(a), \gamma\} &= \max\{F(\varepsilon)((xa^2)y), \gamma\} \\ &\geq \min\{F(\varepsilon)(a^2), \delta\} \\ \min\{F(\varepsilon)(a^2), \delta\} &\leq \max\{F(\varepsilon)(a), \delta\}. \end{aligned}$$

Hence $\langle F, A\rangle$ is soft semiprime.

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$ it is obvious by Lemma 3.21.

$(iii) \Rightarrow (ii)$ Assume that (iii) holds. Let $\langle F, A\rangle$ be a fuzzy soft two sided ideal of an AG-groupoid S, then$\chi^{\delta}_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Let $a^2 \in A$, then since $\chi^{\delta}_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal therefore $\chi^{\delta}_{\gamma A}(a^2) \geq \delta$, thus by (iii), $\max\{\chi^{\delta}_{\gamma A}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma A}(a^2), \delta\} = \delta$, this implies that $\chi^{\delta}_{\gamma A}(a) \geq \delta$. Thus $a \in \langle F, A\rangle$. Hence A is soft semiprime.

$(ii) \Rightarrow (i)$ Assume that every two sided ideal is semiprime and since Sa^2 is a two sided ideal contains a^2. Thus

$$\begin{aligned} a &\in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S \\ &= ((SS)(aa))S = (Sa^2)S. \end{aligned}$$

Hence S is an intra-regular. □

Theorem 3.23. *Let S be an AG-groupoid with left identity, then the following conditions equivalent:*

(i) *S is intra-regular.*

(ii) *Every bi-ideal ideal of S is semiprime.*

(iii) *Every fuzzy soft bi-ideal of S is fuzzy soft semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle F, A\rangle$ of S is semiprime.*

(v) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft generalized bi-ideal $\langle F, A\rangle$ of S is fuzzy soft semiprime.*

Proof. $(i) \Rightarrow (v)$. Let S be an intra-regular. Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft generalized bi-ideal of intra-regular AG-groupoid S. Since S is intra-regular, so for all $b \in S$ there exists x, y in S such that $b = (xb^2)y$.

$$\begin{aligned} b &= (xb^2)y = (x(bb))y = (b(xb))y = (y(xb))b = \{y(x((xb^2)y)))\}b \\ &= \{x(y((xb^2)y))\}b = \{x((xb^2)y^2))\}b = \{(xb^2)(xy^2)\}b = \{x^2(b^2y^2)\}b \\ &= \{b^2(x^2y^2)\}b = \{b(x^2y^2)\}b^2 = \{((xb^2)y)(x^2y^2)\}b^2 \\ &= \{(y^2y)(x^2(xb^2))\}b^2 = \{(y^2x^2)(y(xb^2))\}b^2 = \{(y^2x^2)((y_1y_2)(xb^2))\}b^2 \\ &= \{(y^2x^2)((b^2y_2)(xy_1))\}b^2 = \{(y^2x^2)((b^2x)(y_2y_1))\}b^2 \\ &= \{(y^2x^2)(((y_2y_1)x)(bb))\}b^2 = \{(y^2x^2)(b^2(x(y_2y_1)))\}b^2 \\ &= \{(bb)(x((x^2y^2))(y_2y_1))\}b^2 \\ &= \{b^2\{(x(x^2y^2)(y_2y_1))\}\}b^2 = (b^2t)b^2, \text{ where } t = (x(x^2y^2)(y_2y_1)). \end{aligned}$$

we have

$$\begin{aligned} \max\{F(\varepsilon)(b), \gamma\} &= \max\{F(\varepsilon)(b^2t)b^2, \gamma\} \\ &\geq \max\{F(\varepsilon)(b^2), \gamma\} \\ &= \min\{F(\varepsilon)(b^2), \delta\}. \end{aligned}$$

Therefore $\max\{F(\varepsilon)(b), \gamma\} \geq \min\{F(\varepsilon)(b^2), \delta\}$. Hence $\langle F, A\rangle$ is fuzzy soft semiprime.

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$. Let $\langle F, B\rangle$ be a fuzzy soft bi-ideal of S, then $\chi^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal of an AG-groupoid S. Let $b^2 \in B$ then since $\chi^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal therefore $\chi^{\delta}_{\gamma B}(b^2) \geq \delta$, thus by (iv),

$$\max\{\chi^{\delta}_{\gamma B}(b), \gamma\} \geq \min\{\chi^{\delta}_{\gamma B}(b^2), \delta\} = \delta$$

this implies $\chi^{\delta}_{\gamma B}(b) \geq \delta$. Thus $b \in B$. Hence B is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal of S is semiprime and since Sb^2 is an ideal containing b. Thus

$$\begin{aligned} b &\in (Sb^2) \subseteq (SS)b^2 \subseteq (b^2S)S = ((bb)(SS))S \\ &= ((SS)(bb))S = (Sb^2)S. \end{aligned}$$

Hence S is an intra-regular. □

Theorem 3.24. *Let S be an AG-groupoid with left identity, then the following conditions are equivalent:*

(i) *S is intra-regular.*

(ii) *Every ideal of S is semiprime.*

(iii) *Every fuzzy soft quasi-ideal of S is fuzzy soft semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle F, A\rangle$ of S is fuzzy soft semiprime.*

Proof. $(i) \Rightarrow (iv)$. Let S be an intra-regular AG-groupoid with left identity and $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi ideal over S. As S is intra-regular, so for any $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now by left invertive law and medial law, then

$$\begin{aligned} a &= (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2((y_2y_1)x) \\ &= a^2t, \text{ where } t = (y_2y_1)x. \end{aligned}$$

we have

$$\begin{aligned} \max\{F(\varepsilon)(a), \gamma\} &= \max\{F(\varepsilon)(a^2t), \gamma\} \\ &\geq \min\{F(\varepsilon)(a^2), \delta\}. \end{aligned}$$

Therefore

$$\max\{F(\varepsilon)(a), \gamma\} \geq \min\{F(\varepsilon)(a^2), \delta\}.$$

Hence $\langle F, A\rangle$ is fuzzy soft semiprime.

$(iv) \Rightarrow (iii)$. Let $\langle F, Q\rangle$ be a fuzzy soft quasi ideal of an AG-groupoid S, and $\langle F, Q\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy soft quasi ideal then $\chi^{\delta}_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of S. Let $a^2 \in \langle F, Q\rangle$ then since $\chi^{\delta}_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal, then $\chi^{\delta}_{\gamma Q}(a^2) \geq \delta$. So by (iv),

$$\max\{\chi^{\delta}_{\gamma Q}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma Q}(a^2), \delta\} = \delta$$

this implies that $\chi^{\delta}_{\gamma Q}(a) \geq \delta$. Thus $a \in \langle F, Q\rangle$. Hence $\langle F, Q\rangle$ is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. As every ideal of S is semiprime and since Sa^2 is an ideal containing a^2. Thus

$$a \in (Sa^2) \subseteq (Sa)(Sa) = (SS)(aa) = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular. □

Conclusion. In this paper, we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semirpime ideals in AG-groupoids. We characterize regular and intra-regular AG-groupoids with left identity using the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

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Madad Khan (madadmath@yahoo.com)
Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan

Faisal Ilya (faisalilyas12@yahoo.com)
Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan

Muhammad Gulistan (gulistanm21@yahoo.com)
Department of Mathematics, Hazara University Mansehra, Pakistan

Saima Anis (sanis@uchicago.edu)
Department of Mathematics, University of Chicago, USA