Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 4, (April 2015), pp. 593–600
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

On submetacompactness in fuzzy topological spaces

T. Baiju, Sunil Jacob John

Received 24 March 2014; Accepted 18 August 2014

Abstract. In this paper the concept of Submetacompactness in L-topological spaces is introduced by means of θ-sequence of α-Q covers. This fuzzy submetacompactness is a natural generalization of Lowen fuzzy compactness. Further some characterizations of fuzzy submetacompactness in the weakly induced L-topological spaces are also obtained.

2010 AMS Classification: 54D20, 54A40

Keywords: L- Topology, Fuzzy submetacompactness, θ-sequence, Point finite family, σ-discrete family.

Corresponding Author: T. Baiju (baijutmaths@gmail.com)

1. Introduction

The concept of a fuzzy set introduced by Zadeh [14] provides a natural frame work for the generalization of many concepts in general topology, known as fuzzy topological spaces. Compactness and its various generalizations is one of the important concepts in general topology. The class of submetacompact spaces was introduced by Worrel and Wicke [13] in 1965 and the term "submetacompact" was suggested by H. Junnila in 1978 [6]. The theory of submetacompact spaces in general topology provides an approach to a large portion of covering theory (see [3], [6]).

In [9] Fu-Gui Shi and Cheng-You Zheng introduced the concept of α-locally finite family to characterize fuzzy compactness and using this they have defined paracompactness in L-topological spaces in [10], which is a natural generalization of the Lowen fuzzy compactness. The authors have introduced point finite families and done some work in metacompactness in L-topological spaces and obtain a characterization for the same in [5]. In this paper we define θ-sequence of α-Q-covers and submetacompactness in L-topological spaces. Besides getting characterizations for subparacompactness in the weakly induced L-topological spaces that involves the concept of well monotone and directed α-Q-covers, it is also seen that submetacompactness is hereditary with respect to closed subsets. Further the invariance of these properties under perfect maps is also proved.

Let L be a complete lattice. Its universal bounds are denoted by $\bot$ and $\top$. We presume that L is consistent. i.e., $\bot$ is distinct from $\top$. Thus $\bot \leq \alpha \leq \top$ for all $\alpha \in L$. We note $\vee\phi = \bot$ and $\vee\phi = \top$. The two point lattice $\{\bot, \top\}$ is denoted by 2. A unary operation $'$ on L is a quasi-complementation. It is an involution (ie., $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering. (ie., $\alpha \leq \beta$ implies $\beta' \leq \alpha'$). In $(L,')$ the DeMorgan laws hold: $(\vee A)' = \wedge\{\alpha' : \alpha \in A\}$ and $(\vee A)' = \vee\{\alpha' : \alpha \in A\}$ for every $A \subset L$. Moreover, in particular, $\bot' = \top$ and $\top' = \bot$.

A molecule or co-prime element in a lattice L is a join irreducible element in L and the set of all non zero co-prime elements of L is denoted by $M(L)$. A complete lattice L is completely distributive if it satisfies either of the logically equivalent CD1 or CD2 below:

$$\text{CD1: } \wedge_{i\in I}\left(\vee_{j\in J_i} a_{i,j}\right) = \vee_{\phi\in\prod_{i\neq I} J_i}\left(\wedge_{i\in I} a_{i,\phi(i)}\right)$$
$$\text{CD2: } \vee_{i\in I}\left(\wedge_{j\in J_i} a_{i,j}\right) = \wedge_{\phi\in\prod_{i\neq I} J_i}\left(\vee_{i\in I} a_{i,\phi(i)}\right)$$

for all $\{\{a_{ij} : j \in J_i\} : i \in I\} \subset P(L)\backslash\{\phi\}$, $I \neq \phi$.

If L is a complete lattice, then for a set X, L^X is the complete lattice of all maps from X into L, called L-sets or L-subsets of X. Under point-wise ordering, $a \leq b$ in L^X if and only if $a(x) \leq b(x)$ in L for all $x \in X$. If $A \subset X$, $1_A \in 2^X \subset L^X$ is the characteristic function of A. The constant member of L^X with value α is denoted by α itself. Usually we will not distinguish between a crisp set and its characteristic function. Wang [11] proved that a complete lattice is completely distributive if and only if for each $\alpha \in L$, there exists $B \subseteq L$ such that (i) $a = \vee A$ and (ii) if $A \subseteq L$ and $a \leq \vee B$, then for each $b \in B$, there exists $c \in A$ such that $b \leq c$. B is called the minimal set of a and $\beta(a)$ denote the union of all minimal sets of a. Again $\beta^*(a) = \beta(a) \cap M(L)$. Clearly $\beta(a)$ and $\beta^*(a)$ are minimal sets of a.

For $\alpha \in L$ and $A \in L^X$, we use the following notations.

$$A_{[\alpha]} = \{x \in X : A(x) \geq \alpha\}$$
$$A^{[\alpha]} = \{x \in X : A(x) \leq \alpha\}$$
$$A^{(\alpha)} = \{x \in X : A(x) \not\geq \alpha\}$$
$$A_{(\alpha)} = \{x \in X : A(x) \not\leq \alpha\}$$

Clearly L^X has a quasi complementation $'$ defined point-wisely $\alpha'(x) = \alpha(x)'$ for all $\alpha \in L$ and $x \in X$. Thus the DeMorgan laws are inherited by $(L^X,')$.

Let $(L,')$ be a complete lattice equipped with an order reversing involution and X be any non empty set. A subfamily $\tau \subset L^X$ which is closed under the formation of sups and finite infs (both formed in L^X) is called an L-topology on X and its members are called open L-sets. The pair (X, τ) is called an L-topological space (L-ts). The category of all L-topological spaces, together with L-continuous mappings and the composition and identities of set is denoted by L-**Top**. Quasi complements of open L-sets are called closed L-sets.

We know that the set of all non zero co-prime elements in a completely distributive lattice is $\vee$-generating. Moreover for a continuous lattice L and a topological space (X, T), $T = i_L\omega_L(T)$ is not true in general. By proposition 3.5 in Kubiak [7] we know that one sufficient condition for $T = i_L\omega_L(T)$ is that L is completely distributive.

In [12] Wang extended the Lowen functor ω for completely distributive lattices as follows: For a topological space (X, T), $(X, \omega(T))$ is called the induced space of (X, T) where $\omega(T) = \{A \in L^X : \forall \alpha \in M(L), A^{(\alpha')} \in T\}$. In 1992 Kubiak also extended the Lowen functor ω_L for a complete lattice L. In fact when L is completely distributive, $\omega_L = \omega$.

An L-topological space (X, τ) is called weakly induced space if $\forall\ \alpha \in M(L)$, $\forall A \in \tau$ it is true that $A^{(\alpha')} \in [\tau]$ where $[\tau]$ is the set of all crisp open sets in τ.

Based on these facts, in this paper we use a complete, completely distributive lattice L in L^X. For a standardized basic fixed-basis terminology, we follow Hohle and Rodabaugh [4].

2. Preliminaries and basic definitions

Definition 2.1 ([8])**.** Let (X, τ) be an L-ts. A fuzzy point x_α is quasi coincident with $D \in L^X$ (and write $x_\alpha \prec D$) if $x_\alpha \not\leq D'$. Also D quasi coincides with E at x (DqE at x) if $D(x) \not\leq E'(x)$. We say D quasi coincident with E and write DqE if DqE at x for some $x \in X$. Further $D\neg qE$ means D not quasi coincides with E. We say $U \in \tau$ is quasi coincident nbd of $x_\alpha(Q - nbd)$ if $x_\alpha qU$. The family of all Q-nbds of x_α is denoted by $Q_\tau(x_\alpha)$ or $Q(x_\alpha)$.

Definition 2.2 ([8])**.** Let (X, τ) be an L-ts, $A \in L^X$. $\Phi \subset L^X$ is called a Q-cover of A if for every $x \in \mathrm{Supp}\,(A)$, there exist $U \in \Phi$ such that $x_{A(x)} \prec U$. Φ is a Q-cover of (X, τ) if Φ is a Q-cover of $\top$. If $\alpha \in M(L)$, then $C \in \tau$ is an α-Q-nbd of A if $C \in Q(x_\alpha)$ for every $x_\alpha \leq A$. Φ is called an α-Q-cover of A, if for each $x_\alpha \leq A$, there exists $U \in \Phi$ such that $x_\alpha \prec U$. Φ is called an open α-Q-cover of A if $\Phi \subset \tau$ and Φ is an α-Q-cover of A. $\Phi_0 \subset L^X$ is called a sub α-Q-cover of A if $\Phi 0 \subset \Phi$ and Φ_0 is also an α-Q-cover of A.

Definition 2.3 ([8])**.** Let (X, τ) be an L-ts, $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, $x_\lambda \in M(L^X)$. $\mathbf{A}$ is called locally finite at x_λ, if there exist $U \in Q(x_\lambda)$ and a finite subset T_0 of T such that $t \in T \backslash T_0 \Rightarrow A_t \neg qU$. And $\mathbf{A}$ is called $*$-locally finite at x_λ if there exist $U \in Q(x_\lambda)$ and a finite subset T_0 of T such that $t \in T_0 \Rightarrow \chi_{At(0)} \neg qU$. $\mathbf{A}$ is called locally finite ($*$-locally finite) for short, if $\mathbf{A}$ is locally finite ($*$-locally finite) at every molecule $x_\lambda \in M(L^X)$.

$\mathbf{A}$ is called discrete at x_λ if there exist $U \in Q(x_\lambda)$ and a singleton $T_0 = \{t_0\} \subset T$ such that $t \in T_0 \Rightarrow A_t \neg qU$. And $\mathbf{A}$ is called $*$-discrete at x_λ if there exist $U \in Q(x_\lambda)$ and a singleton $T_0 = \{t_0\} \subset T$ such that $t \in T\ T_0 \Rightarrow \chi_{At(0)} \neg qU$. $\mathbf{A}$ is called discrete ($*$-discrete) for short, if $\mathbf{A}$ is discrete ($*$-discrete) at every molecule $x_\lambda \in M(L^X)$.

The previous notions "locally finite family" and "discrete family" are defined for L-ts. They can be also defined for L-subsets:

Definition 2.4 ([8])**.** Let (X, τ) be an L-ts. $A \in L^X$, $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, $x_\lambda \in M(L^X)$. $\mathbf{A}$ is called locally finite (discrete) in A, if $\mathbf{A}$ is locally finite (discrete) at every molecule $x_\lambda \in M(\downarrow A)$.

It is easy to find that the above definition coincides with Definition 2.3 provided $A = \top$.

Definition 2.5 ([5]). Let (X, τ) be an L-ts. $A = \{A_t : t \in T\} \subset L^X$, $x_\lambda \in M(L^X)$. $\mathbf{A}$ is called point finite at x_λ if $x_\lambda \prec A_t$ for at most finitely many $t \in T$. And $\mathbf{A}$ is $*$-point finite at x_λ if there exists at most finitely many $t \in T$ such that $x_\lambda \prec \chi A_{t(0)}$, $\mathbf{A}$ is called point finite (resp. $*$-point finite) for short, if $\mathbf{A}$ is point finite (resp. $*$-point finite) at every molecule x_λ of L^X.

Definition 2.6. A sequence $\{\mathbf{G}_n\}$ of α-Q covers of $\top$ is said to be a θ-sequence ($*$-θ-sequence) of α-Q covers if for each $x_\alpha \in M(L^X)$, there is some $k \in N$ such that the family $\mathbf{G}_k$ is point finite ($*$-point finite) at x_α.

Definition 2.7 ([8]). Let (X, τ) be an L-ts. (X, τ) is called weakly α-induced if $U_{(\alpha)} \in [\tau]$ for every $U \in \tau$.

Definition 2.8 ([8]). Let (X, τ) be an L-ts. Then by $[\tau]$ we denote the family of support sets of all crisp subsets in τ. $(X, [\tau])$ is a topology and it is the background space. (X, τ) is weakly induced if $U \in \tau$ is a lower semi continuous function from the background space $(X, [\tau])$ to L.

Definition 2.9 ([8]). Let (X, τ) be an L-ts. Then the following conditions are equivalent.
(i) (X, τ) is weakly induced.
(ii) (X, τ) is weakly γ-induced for every $\gamma \in \mathrm{pr}(L)$.
(iii) (X, τ) is weakly α-induced for every $\alpha \in L$.

Definition 2.10 ([2]). Let (X, τ) be an L-ts. $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, $B \in L^X$.

$\mathbf{A}$ is called σ-discrete in B if $\mathbf{A}$ is countable union of sub families which are discrete in B. $\mathbf{A}$ is called σ-discrete for short, if $\mathbf{A}$ is σ-discrete in $\top$.

$\mathbf{A}$ is called σ^*-discrete in B if $\mathbf{A}$ is countable union of sub families which are $*$-discrete in B. $\mathbf{A}$ is called σ^*-discrete for short, if $\mathbf{A}$ is σ^*-discrete in $\top$.

Definition 2.11 ([8]). Let (X, τ) be an L-ts. $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$ is a closure preserving collection if for every subfamily $\mathbf{A}_0$ of $\mathbf{A}$, $\mathrm{cl}\,[\vee \mathbf{A}_0] = \vee[\mathrm{cl}\,\mathbf{A}_0]$.

Proposition 2.12 ([8]). *Let (X, τ) be an L-ts. $A \subset L^X$ is closure preserving. Then for every sub family $\mathbf{A}_0 = \{A_t : t \in T\} \subset \mathbf{A}$, $\vee_{t \in T} cl\, A_t$ is a closed subset.*

Theorem 2.13 ([8]). *Every locally finite family of subsets is closure preserving; particularly, every discrete family of subsets is closure preserving.*

Definition 2.14. Let (X, τ) be an L-ts. $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$ is a interior preserving collection if for every subfamily $\mathbf{A}_0$ of $\mathbf{A}$, $\mathrm{int}\,[\wedge A_0] = \wedge[\mathrm{int}\,\mathbf{A}_0]$.

Definition 2.15 ([1]). Let (X, τ) be an L-ts. $\mathbf{A}, \mathbf{B} \subset L^X$. $\mathbf{A}$ is called a refinement of $\mathbf{B}$ ($\mathbf{A} < \mathbf{B}$) if for every $A \in \mathbf{A}$, there exists $B \in \mathbf{B}$ such that $A \leq B$.

Definition 2.16 ([5]). A collection $\mathbf{U}$ of fuzzy subsets of an L-topological space (X, τ) is said to be well monotone if the subset relation '$<$' is a well order on $\mathbf{U}$.

Definition 2.17 ([5]). A collection $\mathbf{U}$ of fuzzy subsets of an L-topological space (X, τ) is said to be directed if $U, V \in \mathbf{U}$ implies there exists $W \in \mathbf{U}$ such that $U \vee V < W$.

3. Fuzzy sub-metacompactness

Definition 3.1 ([2])**.** Let (X,τ) be an L-ts, $A \in L^X$, $\alpha \in M(L)$. A is called α-subparacompact (α^*-subparacompact) if for every open α-Q-cover Φ of A, there exist a closed refinement Ψ of Φ which is σ-discrete (σ^*-discrete) in A and Ψ is also an α-Q-cover of A. A is subparacompact ($*$-subparacompact) if A is α-subparacompact (α^*-subparacompact) for every $\alpha \in M(L)$. And (X,τ) is subparacompact ($*$-subparacompact) if $\top$ is subparacompact ($*$-subparacompact).

Definition 3.2. Let (X,τ) be an L-ts, $A \in L^X$, $\alpha \in M(L)$. A is called α-submetacompact (α^*-submetacompact) if for every open α-Q-cover of A has a θ-sequence ($*$-θ-sequence) of α-Q-cover refinements. A is submetacompact ($*$-submetacompact) if A is α-submetacompact (α^*-submetacompact) for every $\alpha \in M(L)$. And (X,τ) is submetacompact ($*$-submetacompact) if $\top$ is submetacompact ($*$-submetacompact).

Remark 3.3. Clearly we have $*$-point finite $\Rightarrow$ point finite.

Proposition 3.4. *Let (X,τ) be an L-ts, $A \in L^X$, $\alpha \in M(L)$. Then*
(i) A is α^*-submetacompactness $\Rightarrow$ A is α-submetacompactness.
(ii) A is $*$-submetacompactness $\Rightarrow$ A is submetacompactness.

Proposition 3.5. *Every discrete ($*$-discrete) family is point finite ($*$-point finite)*

Proof of Proposition 3.5 follows immediately from the definitions.

Remark 3.6. From the Proposition 3.5 it follows that subparacompact ($*$-subparacompact) $\Rightarrow$ submetacompact ($*$-submetacompact).

Proposition 3.7. *A point finite closure preserving closed collection is always locally finite.*

Proof. Let $\{A_t : t \in T\}$ be a point finite closure preserving closed collection and let $x_\lambda \in M(L^X)$. Therefore $x_\lambda \prec A_t$, for $t \in T_0$ where T_0 is an at most finite subset of T.

Now take

$$\begin{aligned} V &= \text{cl } \{\vee A_t : t \notin T_0\} \\ &= \vee\{\text{cl } A_t : t \notin T_0\} \text{ since the collection is closure preserving.} \\ &= \vee\{A_t : t \notin T_0\} \text{ since each } A_t \text{ is closed.} \end{aligned}$$

Take $U = V' = (\vee\{A_t : t \notin T_0\})' = \wedge\{A'_t : t \notin T_0\}$

Now if $t \in T \backslash T_0$, $x_\lambda \neg q A_t$ implies $x_\lambda q A'_t$ for every $t \in T\backslash T_0$. Therefore it follows that $x_\lambda q(\vee\{A_t : t \notin T_0\})'$. That is $x_\lambda q U$, ie, $x_\lambda \not\leq U'$. Now since $x_\lambda \neg q A_t$ it follows that $x_\lambda \leq A'_t$.

Combining these two we get $A'_t \geq x_\lambda \not\leq U'$. That is $A'_t \not\leq U'$ and hence $A_t \neg q U$. This completes the proof. □

Similar to the Proposition 3.7 it can be shown that a $*$-point finite closure preserving collection is always $*$-locally finite.

Definition 3.8. Let (X,τ) be an L-ts, $A \in L^X$. A is strongly compact, if for every $\alpha \in M(L)$ every α net in A has a cluster point in A with height α.

Result 3.9. *Let (X, τ) be an L-ts, $A \in L^X$, A is strongly compact if and only if for every $\alpha \in M(L)$, every open α-Q-cover of A has a finite sub α-Q-cover.*

Theorem 3.10. *Let (X, τ) be an L-ts, $A \in L^X$. Then*
A is strongly compact $\Rightarrow$ A is $*$-paracompact $\Rightarrow$ A is $*$-subparacompact $\Rightarrow$ A is $*$-submetacompact.

Theorem 3.11. *Let (X, τ) be a weakly induced L-ts. Then the following conditions are equivalent*
(i) (X, τ) is submetacompact.
(ii) There exist $\alpha \in M(L)$ such that (X, τ) is α-submetacompact.
(iii) $(X, [\tau])$ is submetacompact.

Proof.
(i) $\Rightarrow$ (ii) Clear.
(ii) $\Rightarrow$ (iii) Let $U \subset [\tau]$ be an open cover of X. Then clearly $\{\chi_U : U \in \mathbf{U}\}$ is an open α-Q-cover of $\top$. Then by (ii), it has a θ-sequence of open α-Q-cover refinements say $\mathbf{V} = \{\mathbf{V}_n\}$. For each $V_n \in \mathbf{V}_n$ take $V_{n(\alpha')} = \{x \in X : V_n(x) \not\leq \alpha'\}$ and consider the collection $\mathbf{W}_n = \{V_{n(\alpha')} : V_n \in \mathbf{V}_n\}$. Then by the weakly induced property of (X, τ), $\mathbf{W}_n$ is an open cover of $(X, [\tau])$. Now clearly $\mathbf{W}_n$ is a point finite open refinement of $\mathbf{U}$ and it follows that $\mathbf{W}$ is a θ-sequence of $\mathbf{U}$. Hence (ii) $\Rightarrow$ (iii).
(iii) $\Rightarrow$ (i) Suppose that $\alpha \in M(L)$ and $\mathbf{U} \subset \tau$ be an open α-Q-cover of $\top$. Since (X, τ) is weakly induced $\{U_{(\alpha')} : U \in \mathbf{U}\}$ is an open cover of $(X, [\tau])$. Then there exists a θ-sequence of open refinements say $\mathbf{V} = \{\mathbf{V}_n\}$. For every $V_n \in \mathbf{V}_n$, let U_{V_n} be such that $V_n \subset U_{V_n(\alpha')}$. Let $\mathbf{W}_n = \{\chi_{Vn} \vee U_{Vn} : V_n \in \mathbf{V}_n \text{ and } V_n \subset U_{Vn(\alpha')}\}$. Now clearly $\mathbf{W}_n$ is an open α-Q-cover refinement of $\mathbf{U}$. Take $\mathbf{W} = \{\mathbf{W}_n\}$. Now we will prove that each $\mathbf{W}_n$ is point finite. Let $x_\lambda \in M(L^X)$. Then since $\mathbf{V}_n$ is point finite, it follows clearly that $x \in V_1, V_2, \ldots, V_n$ for some $n \in N$ and $V_i \in \mathbf{V}_n$ for $i = 1, 2, \ldots, n$. Now we will show that $x_\lambda \prec \chi_{Vi} \wedge U_{Vi}$ for at most finitely many i. For, if possible $x_\lambda \prec \chi_{Vi} \wedge U_{Vi}$ for infinitely many $V_i \in \mathbf{V}_n$. Then $x_\lambda \prec \chi_{Vi}$ or $x_\lambda \prec U_{Vi}$ for infinitely many $V_i \in \mathbf{V}_n$. In both cases $x \in V_i$ for infinitely many $V_i \in \mathbf{V}_n$. This is a contradiction and hence $\mathbf{W}_n$ is point finite. Therefore $\mathbf{W} = \{\mathbf{W}_n\}$ is a θ-sequence of $\mathbf{U}$ and thus (iii) $\Rightarrow$ (i). This completes the proof. □

Theorem 3.12. *Let (X, τ) be a weakly induced L-ts. Then the following conditions are equivalent*
(i) (X, τ) is $*$-submetacompact.
(ii) There exist $\alpha \in M(L)$ such that (X, τ) is α^*-submetacompact.
(iii) $(X, [\tau])$ is submetacompact.

Proof.
(i) $\Rightarrow$ (ii) Clear.
(ii) $\Rightarrow$ (iii) Let $\mathbf{U} \subset [\tau]$ be an open cover of X. Then clearly $\{\chi_U : U \in \mathbf{U}\}$ is an open α-Q-cover of $\top$ and it has a $*$-θ-sequence of open α-Q-cover refinements say $V = \{V_n\}$. For each $V_n \in \mathbf{V}_n$ we take $\mathbf{W}_n = \{V_{n(\alpha')} : V_n \in \mathbf{V}_n\}$. Now clearly $\mathbf{W}_n$ is a refinement of $\mathbf{U}$ and a cover of X. Since (X, τ) is weakly induced, $\mathbf{W}_n \subset [\tau]$. Now take $\mathbf{W} = \{\mathbf{W}_n\}$. To prove $\mathbf{W}$ is a θ-sequence, it is enough if we prove that each $\mathbf{W}_n$ is a point finite collection.

We want to prove that for any $x \in X$, $x \in V_{ni(\alpha')}$ for at most finitely many i. By (ii) we have $x_\alpha \prec \chi_{Vni(0)}$ for at most finitely many i and hence we have $\alpha \not\leq \chi_{Vni(0)'}(x)$ for at most finitely many i. Now we know that $V_{ni(0)} \not\leq V_{ni(\alpha')}$ and hence $\chi V_{ni(0)} \not\leq \chi_{Vni(\alpha')}$. Therefore $\alpha' \not\geq \chi_{Vni(0)}(x) \not\leq \chi_{Vni(\alpha')}(x)$ for atmost finitely many i. That is $\chi_{Vni(\alpha')}(x) \not\leq \alpha'$ and thus $\chi_{Vni(\alpha')}(x) \neq \perp$ and hence it follows that $x \in V_{ni(\alpha')}$ for at most finitely many i.

(iii) $\Rightarrow$ (i) Suppose that $\alpha \in M(L)$ and $\mathbf{U} \subset \tau$ be an open α-Q-cover of $\top$. Since (X, τ) is weakly induced $\mathbf{U}^* = \{U_{(\alpha')} : U \in \mathbf{U}\}$ is an open cover of $(X, [\tau])$. Then by (iii) $\mathbf{U}^*$ has a θ-sequence of open cover refinements $\mathbf{V} = \{\mathbf{V}_n\}$. For every $V_n \in \mathbf{V}_n$, let $U_{Vn} \in \mathbf{U}$ such that $V_n \subset U_{Vn(\alpha')}$ and take $\mathbf{W}_n = \{\chi_{Vn} \vee U_{Vn} : V_n \in \mathbf{V}_n\}$. Clearly $\mathbf{W}_n$ is an open α-Q-cover refinement of $\top$. Now consider $\mathbf{W} = \{\mathbf{W}_n\}$. We will prove that $\mathbf{W}$ is a $*$-θ-sequence. It is enough if we show that each $\mathbf{W}_n$ is $*$-point finite.

Let $x_\alpha \in M(L^X)$. If possible let $x_\alpha \prec \chi(\chi_{Vn} \vee U_{Vn})_{(0)}$ for infinitely many $V_n \in \mathbf{V}_n$. That is $x_\alpha \prec \chi_{Vn} \wedge \chi_{UVn(0)}$ for infinitely many $V_n \in \mathbf{V}_n$. And hence $x_\alpha \prec \chi_{Vn}$ or $x_\alpha \prec \chi_{UVn(0)}$ for infinitely many $V_n \in \mathbf{V}_n$. In both cases $x \in V_n$ for infinitely many $V_n \in \mathbf{V}_n$. This is a contradiction that $\mathbf{V}_n$ is point finite. Hence $\mathbf{W}_n$ is $*$-point finite and this completes the proof. □

Theorem 3.13. *Let (X, τ) be a weakly induced L-ts. Then the following conditions are equivalent*
(i) (X, τ) is submetacompact.
(ii) For every $\alpha \in M(L)$, every well monotone open α-Q-cover of $\top$ has a θ-sequence of open α-Q-cover refinements.
(iii) There exists an $\alpha \in M(L)$ such that every well monotone open α-Q-cover of $\top$ has a θ-sequence of open α-Q-cover refinements.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Obvious.
(iii) $\Rightarrow$ (i) It is enough if we prove that $(X, [\tau])$ is metacompact. By a characterization of submetacompactness [3], it is enough to prove that every well monotone open cover of $(X, [\tau])$ has a θ-sequence of refinements.

Let $\{U_t : t \in T\}$ be a well monotone open cover of $(X, [\tau])$. Then clearly $\{\chi_{Ut} : t \in T\}$ is a well monotone open α-Q-cover of $\top$. So it has a θ-sequence of open α-Q-cover refinements say $\mathbf{A} = \{\mathbf{A}_n\}$ where $\mathbf{A}_n = \{A_{nt} : t \in T\}$. Let $\mathbf{B}_n = \{A_{nt(\alpha')} : t \in T\}$. Since (X, τ) is weakly induced, it follows that $\mathbf{B}_n \subset [\tau]$. Consider $\mathbf{B} = \{\mathbf{B}_n\}$. Now to show that $\mathbf{B}$ is the required θ-sequence, it is enough if we prove that each $\mathbf{B}_n$ is point finite. If possible let for any $x \in X$, $x \in B_n$ for infinitely many $B_n \in \mathbf{B}_n$. That is $A_{nt}(x) \not\leq \alpha'$ for infinitely many $t \in T$. Thus $x_\alpha \prec A_{nt}$ for infinitely many $t \in T$. This is a contradiction to that $\mathbf{A}_n$ is point finite.

Also $U_t \supset A_{nt(\alpha')}$. For, let $x \in A_{nt(\alpha')}$ for some $t \in T$. Now since $\{A_{nt} : t \in T\}$ refines $\{\chi_{Ut} : t \in T\}$ it follows that $\alpha' \not\geq A_t(x) \leq \chi_{Ut}(x)$ and this implies $\chi_{Ut}(x) \neq \perp$. Thus $x \in U_t$ and hence $\mathbf{B}_n$ is a refinement of $\{U_t : t \in T\}$ also .This completes the proof. □

4. Conclusion and future work

In this paper, we have introduced the notion of submetacompact spaces in L-topological spaces using the concept of θ-sequence of α-Q-covers. Moreover characterizations of fuzzy submetacompactness is obtained in terms of well monotone and directed α-Q-covers.

The relationship of the introduced concept of fuzzy submetacompactness with other types of non compact covering properties such as paracompactness, metacompactness and various implications is worth investigating.

References

[1] T. Baiju and S. J. John, Finitistic spaces in L-topological spaces, Proyecciones 28(1) (2009) 47–56.
[2] T. Baiju and S. J. John, Subparacompactness in L-topological spaces, J. Fuzzy Math. 19(2) (2011) 397–410.
[3] D. K. Burke, Covering properties, in K. Kunen, J. E. Vaughan (Eds.), Hand Book of Set Theoretic Topology, Elsevier Science Publishers 1984, 349–422.
[4] U. Hohle and S. E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets : Logic, Topology and Measure Theory, The Hand Book of Fuzzy Set Series 3, Kluwer Academic Pub. 1999.
[5] S. J. John and T. Baiju, Metacompactness in L-topological spaces, Iran. J. Fuzzy Syst. 5(3) (2008) 71–79.
[6] H. J. K. Junnila, On submetacompactness, Topology Proc. 3 (1978) 375–405.
[7] T. Kubiak, The topological modification of the L-fuzzy unit interval, in: S. E. Rodabaugh, E. P. Klement, U. Hohle (Eds.), Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht, (1992) 275–305.
[8] Ying-Ming Liu and Mao-Kang Luo, Fuzzy Topology, Advances in Fuzzy Systems-Applications and Theory Vol.9, World Scientific, 1997.
[9] F.-G. Shi and G.-Q. Yang, Countable fuzzy compactness in L-topological spaces, J. Harbin Univ. Sci. Tech. 2 (1992) 499–507.
[10] F. -G. Shi and C. -Y. Zheng, Paracompactness in L-topological Spaces, Fuzzy Sets and Systems 129 (2002) 29–37.
[11] G. J. Wang, On the structure of fuzzy lattices, Acta Math. Sinica 29(4) (1986) 539–543.
[12] G. J. Wang , Theory of L-fuzzy topological spaces, Shaanxi Normal University Pub., Xian 1988.
[13] J. M. Worrel and H. H. Wicke, Characterizations of developable spaces, Canad. J. Math. 17 (1965) 820–830.
[14] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

T. Baiju (baijutmaths@gmail.com)
Department of Mathematics, Manipal Institute of Technology, Manipal, Karnataka, India-576104

Sunil Jacob John (sunil@nitc.ac.in)
Department of Mathematics, National Institute of Technology Calicut, Kerala, India-673 601