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A common fixed point theorem for cyclic contractive mappings in fuzzy metric spaces

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Abstract. In this paper we prove a common fixed point theorem for mappings satisfying cyclic contraction. Main theorem of this paper is proved with the help of a control function. Some corollaries have been deduced. At last we give an example to validate our main result.

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1. Introduction

Fixed point theory plays an important role in functional analysis. The introduction of Banach's contraction mapping principle [2] gave further boost to fixed point theory in the development of functional analysis. Banach's contraction principle was later on generalized in various directions. A probabilistic generalization was proposed by Sehgal and Bharucha-Ried [29] in 1972, known as Sehgal contraction or B-contraction. Probabilistic metric spaces are probabilistic generalization of metric spaces. The inherent flexibility of these spaces allows us to extend the contraction mapping principle in more than one inequivalent ways. One of such extensions of contraction mapping was established in probabilistic metric spaces by Hicks [15], which is known as C-contraction. Subsequently, fixed point theory in probabilistic metric spaces has been developed in an extensive way. A comprehensive survey of this development up to 2001 described by Hadzic and Pap [14].

Fuzzy metric space is one of the generalization of metric spaces. Kramosil and Michalek [23] defined fuzzy metric spaces as a generalization of probabilistic metric spaces in 1975. George and Veeramani [10, 11] modified the definition of fuzzy metric spaces given by Kramosil and Michalek [23] in order to ensure the concept of Hausdorff topology in this setting of fuzzy metric spaces.

As Banach contraction is continuous, so a natural question arises whether there exists a class of mappings satisfying some contractive inequality which necessarily have fixed points in complete metric spaces but need not necessarily be continuous. Kannan type mappings [16, 17] are such mappings. A Banach contraction mapping may have a fixed point in a metric space which does not satisfy the condition of completeness. In [32] it has been established that the metric completeness is implied by the necessary existance of fixed points of the class of Kannan type mappings. Some of these works on Kannan type mappings may be seen in [20, 21, 31] .

Khan, Swaleh and Sessa [19] introduced a new type of contraction in metric space in 1984. They used a control function to prove their result. This control function is known as 'altering distance function' . After this paper many results have appeared in the literature of fixed point theory [25, 27, 28]. Choudhury and Das [4] extended the concept of altering distance function in the context of Menger spaces through a control function namely Φ-function. The basic properties of Φ-function is described in section 2.

In this paper we apply this type of function (Φ function) to find a fixed point result in a complete fuzzy metric space. The main results of our work alongwith the corollary and the example are described in section 3.

2. Preliminaries

In this section we give some mathematical preliminaries which are needed for our discussion.

Definition 2.1. (t - norm) A binary operation $* : [0,1]X[0,1] \to [0,1]$ is a t-norm if it satisfies the following conditions:

$$\begin{aligned}
&*(1,a) = a, \quad *(0,0) = 0,\\
&*(a,b) = *(b,a),\\
&*(c,d) \geq *(a,b) \quad \text{whenever } c \geq a \text{ and } d \geq b,\\
&*(*(a,b),c) = *(a,*(b,c)) \quad \text{where} \quad a,b,c,d \in [0,1].
\end{aligned}$$

Definition 2.2. (Fuzzy Metric Space [23]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X x X x $[0, \infty)$, satisfying the following conditions:

$$\begin{aligned}
&M(x,y,0) = 0;\\
&M(x,y,t) = 1 \quad \text{for all } t > 0 \text{ iff } x = y;\\
&M(x,y,t) = M(y,x,t);\\
&M(x,z,t+s) \;\geq\; M(x,y,t) * M(y,z,s);\\
&M(x,y,.) : [0,\infty) \to [0,1] \quad \text{is left continuous}\\
&\text{where} \quad x,y,z \in X \text{ and } t,s > 0.
\end{aligned}$$

Definition 2.3. (Fuzzy Metric Space [10]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X x X x $(0, \infty)$, satisfying the following conditions:

$$M(x, y, t) > 0;$$
$$M(x, y, t) = 1 \quad \text{for all } t > 0 \text{ iff } x = y;$$
$$M(x, y, t) = M(y, x, t);$$
$$M(x, z, t + s) \geq (M(x, y, t) * M(y, z, s));$$
$$M(x, y, .) : (0, \infty) \to [0, 1] \quad \text{is continuous}$$
$$\text{where} \quad x,y,z \in X \text{ and } t,s > 0$$

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to converge to $x \in X$ if and only if for each $\epsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \epsilon$ for all $n > n_0$.

Definition 2.5. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for each $\epsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.

In [13] another type of definition of Cauchy sequence given by M. Grabriec.

Definition 2.6. (Φ - function [4]) A function $\phi : R \to R$ is said to be a Φ - function if it satisfies the following conditions:

i) $\phi(t) = 0$ if and only if $t = 0$,
ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
iii) $\phi(t)$ is left continuous in $(0, \infty)$,
iv) ϕ is continuous at 0.

Some applications of this type of function may be seen in [5, 6, 7, 9, 24]

Definition 2.7. (Ψ-function) A function $\psi : [0, 1]\, X\, [0, 1] \to [0, 1]$ is said to be a Ψ function if it satisfies the following conditions :

i) ψ is monotone increasing and continuous function,
ii) $\psi(x, x) > x$ for all $0 < x < 1$,
iii) $\psi(1, 1) = 1$,
iv) $\psi(0, 0) = 0$.

An example of Ψ-function:

$$\psi(x, y) = \frac{p\sqrt{x} + q\sqrt{y}}{p + q}, \text{ p and q are positive numbers.}$$

An application of Ψ - function may be seen in [8]. The Ψ-function and the (Φ, Ψ) function are also used in the context of weak contraction results in fuzzy metric space [1] and intuitionistic fuzzy metric space [3] respectively.

In recent time another type of contraction appeared in the literature of fixed point theory. This type of contraction is known as Cyclic contraction.

Definition 2.8. (Cyclic Mapping) Let A and B be two non-empty sets. A cyclic mapping is a mapping $T : A \bigcup B \to A \bigcup B$ which satisfies :

$$TA \subseteq B \text{ and } TB \subseteq A$$

This line of research was initiated by Kirk, Srinivasan and Veeramani [22]. In this work amongst the other result the following generalization of the contraction mapping principle has been established.

Theorem 2.9 ([22])**.** *Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : X \to X$ satisfies :*

- *$fA \subseteq B$ and $fB \subseteq A$.*
- *$d(fx, fy) \leq kd(x, y)$ for all $x \in A$ and $y \in B$ where $k \in (0, 1)$.*

Then f has a unique fixed point in $A \cap B$.

This work has been extended further by different authors, some of which may be noted in [12, 18, 26, 30].

3. Main Result

Theorem 3.1. *Let (X,M,*) be a complete fuzzy metric space where $*$ is the minimum t-norm. Let there exist two non-empty closed subsets A and B of X such that mappings $T : A \to B$ and $f : B \to A$ satisfy following conditions :*

$$TA \subseteq B \text{ and } fB \subseteq A \tag{3.1}$$

$$M\left(Tx, fy, \phi\left(t\right)\right) \geq \psi\left(M\left(x, Tx, \phi\left(\frac{t_1}{a}\right)\right), M\left(y, fy, \phi\left(\frac{t_2}{b}\right)\right)\right) \tag{3.2}$$

for all $x \in A, y \in B$, where $t_1, t_2, t > 0$ with $t_1 + t_2 = t$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $A \cap B$ is non-empty and T and f have a unique common fixed point.

Proof. Let $x_0 \in A$ be any arbitrary point. As $x_0 \in A$, and $TA \subseteq B$, we can find $x_1 \in B$ such that $Tx_0 = x_1$.
Again, $x_1 \in B$, and $fB \subseteq A$ so that we can find $x_2 \in A$ such that $fx_1 = x_2$.
Continuing this process we can find $x_{2n} \in A$ and $x_{2n+1} \in B$ such that,

$$Tx_{2n} = x_{2n+1} \in B, \qquad fx_{2n+1} = x_{2n+2} \in A.$$

Now for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n be even, we have,

$$\begin{aligned} M\left(x_{n+1}, x_n, \phi\left(t\right)\right) &= M\left(Tx_n, fx_{n-1}, \phi\left(t\right)\right) \\ &\geq \psi\Big(\left(M\left(x_n, Tx_n, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_{n-1}, fx_{n-1}, \phi\left(\frac{t_2}{b}\right)\right)\right)\Big) \\ &= \psi\Big(\left(M\left(x_n, x_{n+1}, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_{n-1}, x_n, \phi\left(\frac{t_2}{b}\right)\right)\right)\Big) \end{aligned}$$

since $x_n \in A$ and $x_{n-1} \in B$.
Let us consider $t_1 = \frac{at}{a+b}$, $t_2 = \frac{bt}{a+b}$ and $c = a + b$, then obviously we have $0 < c < 1$, and

$$\frac{t_1}{a} = \frac{t}{c} = \frac{t_2}{b}. \tag{3.3}$$

Then from (3.3) we have,

$$M\left(x_{n+1}, x_n, \phi\left(t\right)\right) \geq \psi\left(M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right), M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right)\right). \tag{3.4}$$

Again, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n be odd, we have

$$\begin{aligned} M(x_{n+1}, x_n, \phi(t)) &= M(fx_n, Tx_{n-1}, \phi(t)) \quad [\text{as } x_{n+1} \in A,\ x_n \in B] \\ &= M(Tx_{n-1}, fx_n, \phi(t)) \\ &\geq \psi\left(M\left(x_{n-1}, Tx_{n-1}, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_n, fx_n, \phi\left(\frac{t_2}{b}\right)\right)\right) \\ &= \psi\left(M\left(x_{n-1}, x_n, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_n, x_{n+1}, \phi\left(\frac{t_2}{b}\right)\right)\right). \end{aligned}$$

By (3.3) we have from above

$$M(x_{n+1}, x_n, \phi(t)) \geq \psi\left(M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right), M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right)\right). \tag{3.5}$$

Combining (3.4) and (3.5) we have for all positive integer n,

$$M(x_{n+1}, x_n, \phi(t)) \geq \psi\left(M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right), M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right)\right). \tag{3.6}$$

Now we claim that for all $t > 0$,

$$M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right) \geq M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right). \tag{3.7}$$

If possible, let for some $t' > 0$,

$$M\left(x_{n+1}, x_n, \phi\left(\frac{t'}{c}\right)\right) < M\left(x_n, x_{n-1}, \phi\left(\frac{t'}{c}\right)\right)$$

Then we have from (3.6)

$$\begin{aligned} M(x_{n+1}, x_n, \phi(t')) &\geq \psi\left(M\left(x_{n+1}, x_n, \phi\left(\frac{t'}{c}\right)\right), M\left(x_n, x_{n-1}, \phi\left(\frac{t'}{c}\right)\right)\right) \\ &= \psi\left(M\left(x_{n+1}, x_n, \phi\left(\frac{t'}{c}\right)\right), M\left(x_{n+1}, x_n, \phi\left(\frac{t'}{c}\right)\right)\right) \\ &> M\left(x_{n+1}, x_n, \phi\left(\frac{t'}{c}\right)\right) [\text{by property of } \Psi \text{ - function}] \\ &\geq M(x_{n+1}, x_n, \phi(t')), \quad \text{which is a contradiction.} \end{aligned}$$

Therefore for all $t > 0$, (3.7) holds.
Hence using (3.7) we get from (3.6)

$$\begin{aligned} M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right) &\geq \psi\left(M\left(x_{n+1}, x_n, \phi\left(\frac{t}{c}\right)\right), M\left(x_{n-1}, x_n, \phi\left(\frac{t}{c}\right)\right)\right) \\ &\geq \psi\left(M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right), M\left(x_{n-1}, x_n, \phi\left(\frac{t}{c}\right)\right)\right) \\ &\geq M\left(x_n, x_{n-1}, \phi\left(\frac{t}{c}\right)\right). \end{aligned} \tag{3.8}$$

By repeated application of (3.8) we have,

$$M(x_{n+1}, x_n, \phi(t)) > M\left(x_1, x_0, \phi\left(\frac{t^n}{c}\right)\right). \tag{3.9}$$

Taking limit as $n \to \infty$ we have from (3.9), for all $t > 0$,

$$\lim_{n\to\infty} M\left(x_{n+1}, x_n, \phi\left(t\right)\right) = 1.$$

Next we show that the sequence $\{x_n\}$ is a Cauchy sequence.
If possible let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ for all positive integer k, such that

$$M\left(x_{m(k)}, x_{n(k)}, \epsilon\right) \leq 1 - \lambda. \tag{3.10}$$

We take $m(k)$ corresponding to $n(k)$ to be smallest integer satisfying (3.10) so that

$$M\left(x_{m(k)-1}, x_{n(k)}, \epsilon\right) > 1 - \lambda. \tag{3.11}$$

Now we claim that

$$M\left(x_{m(k)-2}, x_{n(k)}, \epsilon\right) > 1 - \lambda. \tag{3.12}$$

If possible let for $\epsilon > 0$

$$M\left(x_{m(k)-2}, x_{n(k)}, \epsilon\right) \leq 1 - \lambda \tag{3.13}$$

–which contradicts the fact that $m(k)$ is smallest integer satisfying (3.10).
Hence,

$$M\left(x_{m(k)-2}, x_{n(k)}, \epsilon\right) > 1 - \lambda.$$

If $\epsilon_1 < \epsilon$,then we have

$$M\left(x_{m(k)}, x_{n(k)}, \epsilon_1\right) \leq M\left(x_{m(k)}, x_{n(k)}, \epsilon\right).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $m\left(k\right) > n\left(k\right) > k$ and satisfying (3.10),(3.11),(3.12) whenever ϵ is replaced by a smallest positive value.
As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that, $\phi(\epsilon_2) < \epsilon$.
Then by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$M\left(x_{m(k)}, x_{n(k)}, \phi\left(\epsilon_2\right)\right) \leq 1 - \lambda, \tag{3.14}$$

$$M\left(x_{m(k)-1}, x_{n(k)}, \phi\left(\epsilon_2\right)\right) > 1 - \lambda, \tag{3.15}$$

$$M\left(x_{m(k)-2}, x_{n(k)}, \phi\left(\epsilon_2\right)\right) > 1 - \lambda. \tag{3.16}$$

By property of ϕ, we can choose $\rho_1 > 0$ and $\rho_2 > 0$ such that $\rho_1 + \rho_2 < \phi\left(\epsilon_2\right)$.
Again we have, for sufficiently large k,

$$M\left(x_{m(k)}, x_{m(k)-1}, \rho_1\right) > 1 - \lambda,$$

and

$$M\left(x_{m(k)-1}, x_{m(k)-2}, \rho_2\right) > 1 - \lambda.$$

As M is left continuous, we have

$$M\left(x_{m(k)-2}, x_{n(k)}, \phi\left(\epsilon_2\right) - \rho_1 - \rho_2\right) \geq 1 - \lambda.$$

Now from (3.14), we have when $m(k)$ = odd, $n(k)$ = even

$$1-\lambda \geq M\left(x_{m(k)}, x_{n(k)}, \phi\left(\epsilon_2\right)\right)=M\left(T x_{m(k)-1}, f x_{n(k)-1}, \phi\left(\epsilon_2\right)\right)$$
$$\geq \psi\Big(\left(M\left(x_{m(k)-1}, T x_{m(k)-1}, \phi\left(\frac{\epsilon_2'}{a}\right)\right), M\left(x_{n(k)-1}, f x_{n(k)-1}, \phi\left(\frac{\epsilon_2''}{b}\right)\right)\right)\Big)$$

$\epsilon_2=\epsilon_2'+\epsilon_2''$. Here ϵ_2' and ϵ_2'' are so chosen that $\frac{\epsilon_2'}{a}>\epsilon_2$ and $\frac{\epsilon_2''}{b} \geq \epsilon_2$, as $0<a+b<1$. Therefore,

$$\begin{aligned}
1-\lambda & \geq \psi\Big(M\left(x_{m(k)-1}, x_{m(k)}, \phi\left(\frac{\epsilon_2'}{a}\right)\right), M\left(x_{n(k)-1}, x_{n(k)}, \phi\left(\frac{\epsilon_2''}{b}\right)\right)\Big) \\
& \geq \psi\Big(M\left(x_{m(k)-1}, x_{m(k)}, \phi\left(\epsilon_2\right)\right), M\left(x_{n(k)-1}, x_{n(k)}, \phi\left(\epsilon_2\right)\right)\Big) \\
& \geq \psi(1-\lambda, 1-\lambda) \\
& > 1-\lambda, \quad \text{a contradiction.}
\end{aligned}$$

Again from (3.14) we have when $m(k)$ = even, $n(k)$ = even.

$$\begin{aligned}
1-\lambda & \geq M\Big(x_{m(k)}, x_{n(k)}, \phi\Big(\epsilon_2\Big)\Big) \\
& \geq \min\{M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon_2'\Big)\Big), M\Big(x_{m(k)-1}, x_{n(k)}, \phi\Big(\eta_1+\eta_2\Big)\Big)\}
\end{aligned}$$
(3.17)

where $\phi\Big(\epsilon_2\Big) \geq \phi\Big(\epsilon_2'\Big)+\phi\Big(\eta_1+\eta_2\Big)$ and η_1 and η_2 are so chosen that $\frac{\eta_1}{a} \geq \epsilon_2$ and $\frac{\eta_2}{b} \geq \epsilon_2$.
Therefore,

$$\begin{aligned}
1-\lambda \geq & \min\Big(\{\Big(M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon_2'\Big)\Big), \\
& \qquad \psi\Big(M\Big(T x_{m(k)-2}, f x_{n(k)-1}, \phi\Big(\eta_1+\eta_2\}\Big)\Big)\Big)\Big), \\
\geq & \min\Big\{M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon_2'\Big)\Big), \\
& \psi\Big(M\Big(x_{m(k)-2}, T x_{m(k)-2}, \phi\Big(\frac{\eta_1}{a}\Big)\Big), M\Big(x_{n(k)-1}, f x_{n(k)-1}, \phi\Big(\frac{\eta_2}{b}\Big)\Big)\Big)\Big\} \\
> & \min\Big\{1-\lambda, \\
& \psi\Big(M\Big(x_{m(k)-2}, x_{m(k)-1}, \phi(\epsilon_2)\Big), M\Big(x_{n(k)-1}, x_{n(k)}, \phi(\epsilon_2)\Big)\Big)\Big\} \\
> & \min\Big\{1-\lambda, \psi\Big(1-\lambda, 1-\lambda\Big)\Big\} \\
> & 1-\lambda \quad \text{a contradiction.}
\end{aligned}$$

Again,from (3.14), we have when $m(k)$ = odd, $n(k)$ = odd,

$$\begin{aligned}
1-\lambda & \geq M\Big(x_{m(k)}, x_{n(k)}, \phi\Big(\epsilon_2\Big)\Big) \\
& = \min\{M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon_2'\Big)\Big), M\Big(x_{m(k)-1}, x_{n(k)}, \phi\Big(\eta_1+\eta_2\Big)\Big)\}
\end{aligned}$$
(3.18)

where $\phi\Big(\epsilon_2\Big) \geq \phi\Big(\epsilon'_2\Big)+\phi\Big(\eta_1+\eta_2\Big)$ and η_1 and η_2 are so choosen that $\frac{\eta_1}{a} \geq \epsilon_2$ and $\frac{\eta_2}{b} \geq \epsilon_2$.

Therefore,

$$\begin{aligned}
1-\lambda &\geq \min\Big(\{\Big(M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon'_2\Big)\Big), \\
&\qquad \Big(M\Big(Tx_{m(k)-2}, fx_{n(k)-1}, \phi\Big(\eta_1+\eta_2\}\Big)\Big)\Big)\Big), \\
&\geq \min\Big\{M\Big(x_{m(k)}, x_{m(k)-1}, \phi\Big(\epsilon'_2\Big)\Big), \\
&\quad \psi\Big(M\Big(x_{n(k)-1}, Tx_{n(k)-1}, \phi\Big(\frac{\eta_1}{a}\Big)\Big), M\Big(x_{m(k)-2}, fx_{m(k)-2}, \phi\Big(\frac{\eta_2}{b}\Big)\Big)\Big)\Big\} \\
&> \min\Big\{1-\lambda, \\
&\quad \psi\Big(M\Big(x_{n(k)-1}, x_{n(k)}, \phi(\epsilon_2)\Big), M\Big(x_{m(k)-2}, x_{m(k)-1}, \phi(\epsilon_2)\Big)\Big)\Big\} \\
&> \min\Big\{1-\lambda, \psi\Big(1-\lambda, 1-\lambda\Big)\Big\} \\
&> 1-\lambda \quad \text{a contradiction.}
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence irrespective of $m(k)$, $n(k)$ are even or odd. Since X is complete, we have $x_n \to z \in X$ for $n \to \infty$, that is , $\lim_{n\to\infty} x_n = z$. The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of x_n also converges to z. Now $\{x_{2n}\} \subseteq A$ and A is closed. Therefore $z \in A$. Similarly $\{x_{2n+1}\} \subseteq B$ and B is closed. Therefore $z \in B$. Thus we have $z \in A \cap B$.

We now show that $fz = z = Tz$. If possible, let $0 < M(z, fz, \phi(t)) < 1$ for some $t > 0$. Now,

$$\begin{aligned}
& M(x_{2n+1}, fz, \phi(t)) = M(Tx_{2n}, fz, \phi(t)) \\
\geq\ & \psi\left(M\left(x_{2n}, Tx_{2n}, \phi\left(\frac{t_1}{a}\right)\right), M\left(z, fz, \phi\left(\frac{t_2}{b}\right)\right)\right) \quad [\text{where } t_1+t_2=t.] \\
=\ & \psi\left(M\left(x_{2n}, x_{2n+1}, \phi\left(\frac{t_1}{a}\right)\right), M\left(z, fz, \phi\left(\frac{t_2}{b}\right)\right)\right) \\
=\ & \psi\left(M\left(x_{2n}, x_{2n+1}, \phi\left(\frac{t}{c}\right)\right), M\left(z, fz, \phi\left(\frac{t}{c}\right)\right)\right) \\
& [\text{ taking } t_1 = \frac{at}{a+b},\ t_2 = \frac{bt}{a+b} \text{ and } c = a+b].
\end{aligned}$$

Taking limits on both sides,

$$\lim_{n\to\infty} M(x_{2n+1}, fz, \phi(t)) \geq \lim_{n\to\infty} \psi\left(M\left(x_{2n}, x_{2n+1}, \phi\left(\frac{t}{c}\right)\right)\right),$$

$$\begin{aligned} M(z, fz, \phi(t) &\geq \psi\left(M\left(z, z, \phi\left(\frac{t}{c}\right)\right), M\left(z, fz, \phi\left(\frac{t}{c}\right)\right)\right) \\ &= \psi\left(1, M\left(z, fz, \phi\left(\frac{t}{c}\right)\right)\right) \\ &\geq \psi\left(M\left(z, fz, \phi\left(\frac{t}{c}\right)\right) M\left(z, fz, \phi\left(\frac{t}{c}\right)\right)\right) \\ &> M\left(z, fz, \phi\left(\frac{t}{c}\right)\right), \quad \text{[By property of } \Psi \text{ - function]} \\ &\geq M\left(z, fz, \phi(t)\right), \quad \text{which is a contradiction.} \end{aligned}$$

$$\therefore M\left(z, fz, \phi(t)\right) = 1, \text{for all } t > 0,$$

That is $z = fz$.
Again if possible let $0 < M\left(z, Tz, \phi(t)\right) < 1$, for some $t > 0$. Now, for that t we have,

$$\begin{aligned} & M\left(Tz, x_{2n+2}, \phi(t)\right) = M\left(Tz, fx_{2n+1}, \phi(t)\right) \\ \geq\ & \psi\left(M\left(z, Tz, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_{2n+1}, fx_{2n+1}, \phi\left(\frac{t_2}{b}\right)\right)\right) \quad \text{[where } t_1 + t_2 = t\text{]} \\ =\ & \psi\left(M\left(z, Tz, \phi\left(\frac{t_1}{a}\right)\right), M\left(x_{2n+1}, x_{2n+2}, \phi\left(\frac{t_2}{b}\right)\right)\right) \\ =\ & \psi\left(M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right), M\left(x_{2n+1}, x_{2n+2}, \phi\left(\frac{t}{c}\right)\right)\right). \\ & \text{[taking } t_1 = \frac{at}{a+b},\ t_2 = \frac{bt}{a+b} \text{ and } c = a+b\text{]} \end{aligned}$$

Taking limits on both sides,

$$\begin{aligned} & M\left(Tz, z, \phi(t)\right) \geq \psi\left(M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right), M\left(z, z, \phi\left(\frac{t}{c}\right)\right)\right). \\ \geq\ & \psi\left(M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right), 1\right) \\ \geq\ & \psi\left(M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right), M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right)\right). \\ >\ & M\left(z, Tz, \phi\left(\frac{t}{c}\right)\right) \quad \text{[By property of } \Psi \text{ - function]} \\ \geq\ & M\left(z, Tz, \phi(t)\right), \end{aligned}$$

which is a contradiction. Therefore $M\left(z, Tz, \phi(t)\right) = 1$ for all $t > 0$, That is $z = Tz$. Hence we have $z = fz = Tz$.
For uniqueness , let w be another fixed point of T and f.

Then we have for all $t > 0$,

$$\begin{aligned} M(z, w, \phi(t)) &= M(Tz, fw, \phi(t)) \\ &\geq \psi\left(M\left(z, Tz, \phi\left(\frac{t_1}{a}\right)\right), M\left(w, fw, \phi\left(\frac{t_2}{b}\right)\right)\right). \\ &= \psi\left(M\left(z, z, \phi\left(\frac{t_1}{a}\right)\right), M\left(w, w, \phi\left(\frac{t_2}{b}\right)\right)\right) \\ &\geq \psi(1, 1). \\ &= 1. \end{aligned}$$

$\therefore z = w.$

This proves the uniqueness of fixed point and completes the proof of the theorem. □

Taking $t_1 = t_2 = \frac{t}{3}$ in Theorem 3.1, we get the corollary below,

Corollary 3.2. *Let (X,M,*) be a complete metric space where $*$ is minimum t-norm. Let there exist two non-empty closed subsets A and B of X such that mappings $T : A \to B$ and $f : B \to A$ satisfy following conditions :*

(3.19) $\quad TA \subseteq B \quad$ *and* $\quad fB \subseteq A$

(3.20) $$M\Big(Tx, fy, \phi\big(t\big)\Big) \geq \psi\Big(M\Big(x, Tx, \phi\Big(\frac{t}{2a}\Big)\Big), M\Big(y, fy, \phi\Big(\frac{t}{2b}\Big)\Big)\Big)$$

for all $x \in A, y \in B$, where $t > 0$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $A \cap B$ is non-empty and T and f have a unique fixed point.

Taking $a = b$ in Theorem 3.1 we have the following corollary,

Corollary 3.3. *Let (X,M,*) be a complete metric space where $*$ is the 3rd order minimum t-norm. Let there exist two non-empty closed subsets A and B of X*s.t. *mappings $T : A \to B$ and $f : B \to A$ satisfy following conditions :*

(3.21) $\quad TA \subseteq B \quad$ *and* $\quad fB \subseteq A$

(3.22) $$M\Big(Tx, fy, \phi\big(t\big)\Big) \geq \psi\Big(M\Big(x, Tx, \phi\Big(\frac{t_1}{a}\Big)\Big), M\Big(y, fy, \phi\Big(\frac{t_2}{a}\Big)\Big)\Big)$$

$\forall x \in A, y \in B$, where $t_1, t_2, t > 0$,with $t_1 + t_2 = t$ and $0 < a < 1$, ψ is a ψ-function and ϕ is a ϕ-function. Then $A \cap B$ is non-empty and T and f have a unique fixed point.

Following corollary comes by taking $T = f$ in Theorem 3.1,

Corollary 3.4. *Let $(X, M, *)$ be a complete metric space where $*$ is a minimum t-norm. Let there exist two non-empty closed subsets A and B of X such that T is a self mapping on X satisfies following conitions :*

(3.23) $\quad TA \subseteq B \quad$ *and* $\quad TB \subseteq A$

(3.24) $$M\Big(Tx, Ty, \phi\big(t\big)\Big) \geq \psi\Big(M\Big(x, Tx, \phi\Big(\frac{t_1}{a}\Big)\Big), M\Big(y, Ty, \phi\Big(\frac{t_2}{b}\Big)\Big)\Big)$$

$\forall x \in A, y \in B$, where $t_1, t_2, t > 0$ with $t_1 + t_2 = t$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $A \cap B$ is non-empty and T has a unique fixed point.

Now we give an example to validate theorem 3.1.

Example 3.5. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{x_1, x_2, x_4\}$, $B = \{x_2, x_3\}$ and $M(x, y, t)$ is defined by

$$\begin{aligned} M(x_1, x_2, t) &= M(x_1, x_3, t) = M(x_1, x_4, t) = M(x_2, x_4, t) \\ &= M(x_3.x_4, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 0.4 & \text{if } 0 < t < 4 \\ 1 & \text{if } t \geq 4 \end{cases} \end{aligned} \tag{3.25}$$

$$M(x_2, x_3, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 0.75 & \text{if } 0 < t \leq 7 \\ 1 & \text{if } t > 7. \end{cases} \tag{3.26}$$

It is easy to verify that $(X, M, *)$ is a complete fuzzy metric space.If we define

$$T : A \to B, f : B \to A \tag{3.27}$$

as follows : $Tx_1 = x_2$, $Tx_2 = x_2$, $Tx_4 = x_3$ and $fx_2 = x_2$, $fx_3 = x_2$ then it satisfies all conditions of the Theorem 3.1 where $\phi(t) = 2t$, $\psi(x, y) = \frac{(\sqrt{x}+\sqrt{y})}{6}$. Then T and f have a unique fixed point x_2.

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References

[1] M. Abbas, M. Imdad and D. Gopal, Ψ-weak contractions in fuzzy metric spaces, Iran. J. Fuzzy Syst. 8 (2011) 141–148.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leurs applications auxequations integrales, Fund. Math. 3 (1922) 133–181.

[3] I. Beg, C. Vetro, D. Gopal and M. Imdad, (Φ, Ψ)-weak contractions in intuitionistic fuzzy metric spaces, J. Intell. Fuzzy Systems 26 (2014) 2497–2504.

[4] B. S. Choudhury and K. P. Das, A new contraction principle in Menger spaces, Acta Math. Sin. (Engl. Ser.) 24 (2008) 1379–1386.

[5] B. S. Choudhury, K. P. Das and P. N. Dutta, A fixed point results in Menger spaces using a real function, Acta Math. Hungar. 122 (2008) 203–216.

[6] B. S. Choudhury and K. P. Das, A coincidence point result in Menger spaces using a control function, Chaos Solitons Fractals 42 (2009) 3058–3063.

[7] B. S. Choudhury, S. Kutukcu and K. P. Das, On fixed points in non-Archimedean Menger PM - spaces, Kochi J. Math. 7 (2012) 41–50 .

[8] K. P. Das, B. S. Choudhury and P. Bhattacharyya, A common fixed point theorem for weakly compatible mappings in complete fuzzy metric space, Rev. Bull. Cal. Math. Soc. 21 (2013) 181–192.

[9] P. N. Dutta ,B. S. Choudhury and K. P. Das, Some fixed point results in Menger spaces using a control function, Surv. Math. Appl. 4 (2009) 41–52.

[10] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[11] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.

[12] D. Gopal, M. Imdad, C. Vetro and M. Hasan, Fixed point theory for cyclic weak ϕ-contraction in fuzzy metric spaces, J. Nonlinear Anal. Appl. Volume 2012 (2012), Article ID jnaa-00110, 11 Pages.

[13] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385–389.

[14] O. Hadzic and E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers 2001.

[15] T. L. Hicks, Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 13 (1983) 63–72.

[16] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71–76.

[17] R. Kannan, Some results on fixed points - II, Amer. Math. Monthly 76 (1969) 405–408.

[18] E. Karapinar, Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett. 24 (2011) 822–825.

[19] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984) 1–9.

[20] M. Kikkawa and T. Suzuki, Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl. 2008, Art. ID 649749, 8 pp.

[21] M. Kikkawa and T. Suzuki, Some similarity between contractions and Kannan mappings II, Bull. Kyushu Inst. Technol. Pure Appl. Math. 55 (2008) 1–13.

[22] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4 (2003) 79–89.

[23] I. Kramosil and J. Michaelek, Fuzzy metric and statistical metric spaces, Kybernetika (Prague) 11(5) (1975) 336–344.

[24] D. Mihet, Altering distance in probabilistic Menger spaces, Nonlinear Anal. 71 (2009) 2734–2738.

[25] S. V. R. Naidu, Some fixed point theorem in metric spaces by altering distances, Czech. Math. J. 53 (128) (2003) 205–212 .

[26] M. Pacurar and I. A. Rus, Fixed point theory for ϕ-contractions, Nonlinear Anal. 72 (2010) 1181–1187.

[27] K. P. R. Sastry and G. V. R. Babu, Some fixed point theorems by alteringdistamces between the points, Ind. J. Pure Appl. Math. 30 (1999) 641–647

[28] K. P. R. Sastry, S. V. R. Naidu, G. V. R. Babu and G. A. Naidu, Generalisation of common fixed point theorems for weakly commuting maps by altering distances, Tamkang J. Math. 31 (2000) 243–250.

[29] V. M. Sehgal and A. T. Bharucha-Reid, Fixed point of contraction mappings on PM space, Math. Sys. Theory 6 (1972) 97–100.

[30] Y. H. Shen, D. Qiu and W. Chen, Fixed point theory for cyclic ϕ-contractions in fuzzy metric spaces. Iran. J. Fuzzy Syst. 10 (2013) 125–133.

[31] N. Shioji, T. Suzuki and W. Takahashi, Contractive mappings, Kannan mappings and metric completeness, Proc. Amer. Math. Soc. 126 (1998) 3117–3124.

[32] P. V. Subrahmanyan, Completeness and fixed points, Monatsh Math. 80 (1975) 325–330.

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