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On indicators of fuzzy relations

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Abstract. In the theory of fuzzy relations the properties of reflexivity, symmetry, antisymmetry, transitivity, etc. are intensely studied. The indicators of fuzzy relations measure the degree to which such a fuzzy relation verifies such a property. The purpose of this paper is to characterize indicators of fuzzy relations in terms of fuzzy modal operators.

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1. Introduction

In the theory of fuzzy relations properties as reflexivity, symmetry, antisymmetry, transitivity, etc. are often studied [7], [13], [16], [20], [25]. Such properties are intensely used in multicriteria decision–making [14], [20], in social choice [5], [6], in fuzzy revealed preference theory [21], [24], [15], in cluster analysis [32] etc. The indicators of fuzzy relations appeared in fuzzy logic where it is much more important to evaluate the degree of truth of a formula instead of saying whether the formula holds or not (see e.g. [7], [25]). An indicator of a fuzzy relation R is an element of the interval $[0, 1]$ which expresses the degree to which R verifies a property P. For example, it is more important to know the degree to which the relation R is transitive instead of knowing if R is transitive or not. In this way the indicators of fuzzy relations refine various results by the use of fuzzy relations ([7], [21], [24], [31], [33], [34], [35], [36]). At the same time they offer us a possibility to rank a set of alternatives according to the degree to which they verify one or more criteria (expressed by properties of a fuzzy relation).

In fuzzy choice functions theory their indicators have been introduced. They express the degree to which a fuzzy choice function fulfills a condition of rationality, revealed preference, consistency, etc. (see e.g. [21], [10], [15]). In [21], [24], [10], [15], [34] some connections between indicators of fuzzy choice functions and indicators of associated fuzzy preference relations are established.

On one hand, fuzzy modal operators appeared in fuzzy modal logic [18], [19], fuzzy morphology [12], [27] and fuzzy rough sets [17], [28].

In [29], properties of fuzzy relations such as reflexivity, symmetry, antisymmetry, transitivity, etc. have been characterized by fuzzy modal operators.

The aim of this paper is to generalize some results of [29] to the indicators of fuzzy relations. For the indicators of reflexivity, symmetry, transitivity, euclideanity and seriality we will prove characterization theorems in terms of fuzzy modal operators and in terms of the structure of residuated lattice of $[0, 1]$ associated with a left–continuous t–norm.

When these indicators take value 1, reflexivity, symmetry, transitivity, euclidieanity, respectively seriality of fuzzy relations and the results from [29] appear as particular cases of the mentioned characterization theorems.

Similarity relations are reflexive, symmetric and transitive fuzzy relations. They extend the crisp notion of equivalence relation and are mostly used in approximate reasoning [7], [8], [20], [21]. The notion of similarity indicator of a fuzzy relation appears naturally, as a refinement of the similarity relation.

For the similarity indicator one also proves superior approximation theorems by means of fuzzy modal operators. In the paper some order indicators are introduced and some perspectives of their study are discussed.

The results of the paper establish a bridge between the theory of fuzzy relations and fuzzy modal logic; they suggest how properties of fuzzy relations can be formulated in fuzzy modal logic and how they can be analyzed by its mechanisms.

2. Preliminaries

In the first part of this section we will recall some properties of the left–continuous t–norms and the associated residua [7], [20], [25]. The second part of the section introduces some indicators of fuzzy relations [7], [21], [31], [34] and two fuzzy modal operators [29].

If $\{a_i\}_{i\in I} \subseteq [0, 1]$ is a family of real numbers then we denote

$$\bigvee_{i\in I} a_i = \sup\{a_i \mid i \in I\} \text{ and } \bigwedge_{i\in I} a_i = \inf\{a_i \mid i \in I\}.$$

Let $*$ be a left–continuous t–norm [7], [20], [25] and $\rightarrow$ its residuum

$$a \rightarrow b = \bigvee\{c \in [0, 1] \mid a * c \leq b\}.$$

Recall the definition of the negation and the biresiduum associated with $\rightarrow$:
$\neg a = a \rightarrow 0$; $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$

In this paper two well-known continuous t–norms and their residua will be useful.
Łukasiewicz t-norm:

$$a *_L b = \max(0, a + b - 1); \qquad a \rightarrow_L b = \min(1, 1 - a + b)$$

Gödel t-norm:

$$a *_G b = a \wedge b; \qquad a \rightarrow_G b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

The following lemma is well-known (cf. [7], [20], [25]).

Lemma 2.1. *Let $a, b, c \in [0, 1]$ and $\{a_i\}_{i \in I} \subseteq [0, 1]$. Then for any continuous t–norm $*$*

(1) $a * b \leq c$ *iff* $a \leq b \rightarrow c$;
(2) $a * (a \rightarrow b) \leq b$;
(3) $a \leq b$ *iff* $a \rightarrow b = 1$;
(4) $1 \rightarrow a = a$;
(5) $a \rightarrow a = 1$;
(6) $a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c$;
(7) $a \leq b$ *implies* $b \rightarrow c \leq a \rightarrow c$ *and* $c \rightarrow a \leq c \rightarrow b$;
(8) $a \leq \neg b$ *iff* $a * b = 0$;
(9) $a * (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} a * a_i$;
(10) $a \wedge (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} (a \wedge a_i)$;
(11) $(\bigvee_{i \in I} a_i) \rightarrow a = \bigwedge_{i \in I} (a_i \rightarrow a)$;
(12) $a \rightarrow (\bigwedge_{i \in I} a_i) = (\bigwedge_{i \in I} a \rightarrow a_i)$

We fix a left–continuous t–norm $*$. Let X be a non–empty universe and $\mathcal{F}(X)$ the set of fuzzy subsets of X. If $x \in X$ then we denote by $\{1/x\}$ the characteristic function of the set $\{x\}$. A fuzzy relation is a function $R : X^2 \rightarrow [0, 1]$.

A fuzzy relation R will be called:

- reflexive, if $R(x, x) = 1$ for all $x \in X$
- symmetric, if $R(x, y) = R(y, x)$ for all $x, y \in X$
- transitive, if $R(x, y) * R(y, z) \leq R(x, z)$ for all $x, y, z \in X$
- euclidian, if $R(z, x) * R(z, y) \leq R(x, y)$ for all $x, y, z \in X$
- serial, if $\bigvee_{y \in X} R(x, y) = 1$ for all $x \in X$

If R is a fuzzy relation on X then we define the following indicators [7], [31]:

- $Ref(R) = \bigwedge_{x \in X} R(x, x)$
- $Sym(R) = \bigwedge_{x,y \in X} (R(x, y) \rightarrow R(y, x))$
- $Tr(R) = \bigwedge_{x,y,z \in X} (R(x, y) * R(y, z) \rightarrow R(x, z))$
- $Eu(R) = \bigwedge_{x,y,z \in X} (R(z, x) * R(z, y) \rightarrow R(x, y))$
- $Ser(R) = \bigwedge_{x \in X} \bigvee_{y \in X} R(x, y)$

Lemma 2.2 ([7], [31]). (1) $Ref(R) = 1$ *iff R is reflexive;*
(2) $Sym(R) = 1$ *iff R is symmetric;*
(3) $Tr(R) = 1$ *iff R is transitive;*
(4) $Eu(R) = 1$ *iff R is euclidian;*

(5) *$Ser(R) = 1$ iff R is serial.*

By Lemma 2.2, the five indicators above refine the properties of reflexivity, symmetry, transitivity, euclideanity and seriality of fuzzy relations. For example, $Tr(R)$ indicates the transitivity degree of R.

Applying Lemma 2.1 (1) it follows immediately

Lemma 2.3. *For $x, y, z \in X$ the following inequalities hold:*

(1) $Sym(R) * R(x, y) \leq R(y, x)$;
(2) $Tr(R) * R(x, y) * R(y, z) \leq R(x, z)$;
(3) $Eu(R) * R(z, x) * R(z, y) \leq R(x, y)$.

For $A, B \in \mathcal{F}(X)$ let us denote

$$S(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$$

and

$$E(A, B) = S(A, B) \wedge S(B, A) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)).$$

It is clear that $A \subseteq B$ iff $S(A, B) = 1$ and $A = B$ iff $E(A, B) = 1$.

$S(A, B)$ is called the *subsethood degree* of A and B and $E(A, B)$ the *degree of equality* (degree of similarity) of A and B. Intuitively $S(A, B)$ expresses the truth value of the statement "A is included in B" and $E(A, B)$ the truth value of the statement "A and B contain the same elements" (see [7, p. 82]).

Let R be a fuzzy relation on X. Following [29] let us consider the fuzzy modal operators $[R] : \mathcal{F} \to \mathcal{F}$ and $< R >: \mathcal{F} \to \mathcal{F}$ defined by:

$$([R]A)(x) = \bigwedge_{y \in X} (R(x, y) \to A(y))$$

$$(< R > A)(x) = \bigvee_{y \in X} R(x, y) * A(y)$$

for all $A \in \mathcal{F}$ and $x \in X$.

3. Main results

In the study of various bivalent modal logic systems [11] properties of binary relations such as symmetry, transitivity, linearity, etc. appear. To define the fuzzy modal systems semantics it is necessary to employ corresponding properties of binary fuzzy relations [19]. But fuzzy logic semantics are mostly based on the idea of truth degree of sentences and less based on the fact that the sentence is true or false. Two conclusions follow from here:

(a) in the study of fuzzy modal logic semantics it is preferable to use indicators of fuzzy relations (expressing the degree of reflexivity, symmetry, transitivity, etc.) instead of these properties

(b) in order the indicators of fuzzy relations to be integrated into a fuzzy modal system, it is necessary that they should be expressed according to fuzzy modal operators.

The results of this section answer the goal (b). The theorems of the section will characterize the indicators $Ref(R)$, $Tr(R)$, $Sym(R)$, $Eu(R)$, $Ser(R)$ in terms of fuzzy modal operators introduced in the previous section. The main results of [29] are obtained as particular results of the theorems of this section.

We fix a left–continuous t–norm $*$ and a fuzzy relation R on X. For simplicity we denote $\mathcal{F} = \mathcal{F}(X)$.

Theorem 3.1. $Ref(R) = \bigwedge\limits_{A\in\mathcal{F}} S([R]A, A) = \bigwedge\limits_{A\in\mathcal{F}} S(A, <R> A)$

Proof. We prove only the first equality. Let $A \in \mathcal{F}$ and $x \in X$. By Lemma 2.1 (2)

$$Ref(R) * ([R]A)(x) = Ref(R) * \bigwedge_{y\in X} (R(x, y) \to A(y))$$
$$\leq R(x, x) * (R(x, x) \to A(x)) \leq A(x).$$

According to Lemma 2.1 (1), it follows that $Ref(R) \leq ([R]A)(x) \to A(x)$ for all $x \in X$ and $A \in \mathcal{F}$ hence

$$Ref(R) \leq \bigwedge_{A\in\mathcal{F}} \bigwedge_{x\in X} (([R]A)(x) \to A(x)) = \bigwedge_{A\in\mathcal{F}} S([R]A, A)$$

We prove the converse inequality. Let $x \in X$. We define $A_0 \in \mathcal{F}$ by $A_0(y) = R(x, y)$ for all $y \in X$. Remark that $([R]A_0)(x) = \bigwedge\limits_{u\in X} (R(x, u) \to A_0(u)) = 1$, therefore

$$S([R]A_0, A_0) = \bigwedge_{y\in X} (([R]A_0)(y) \to A_0(y)) \leq$$
$$\leq ([R]A_0)(x) \to A_0(x) = A_0(x) = R(x, x)$$

Then

$$\bigwedge_{A\in\mathcal{F}} S([R]A, A) \leq S([R]A_0, A_0) = \bigwedge_{x\in X} (([R]A_0)(x) \to A_0(x)) \leq$$
$$\leq \bigwedge_{x\in X} R(x, x)$$

□

Corollary 3.2 ([29]). *The following assertions are equivalent:*

(1) *R is reflexive;*
(2) *$[R]A \subseteq A$ for all $A \in \mathcal{F}$;*
(3) *$A \subseteq <R> A$ for all $A \in \mathcal{F}$.*

Theorem 3.3. $Sym(R) = \bigwedge\limits_{A\in\mathcal{F}} S(<R>[R]A, A) = \bigwedge\limits_{A\in\mathcal{F}} S(A, [R] <R> A)$

Proof. Let $A \in \mathcal{F}$. We prove that

(a) $Sym(R) \leq S(<R>[R]A, A)$
(b) $Sym(R) \leq S(A, [R] <R> A)$

Let $x \in X$. According to the definition of $[R]$ and $<R>$

$$(<R>[R]A)(x) = \bigvee_{y\in X} R(x, y) * ([R]A)(y)$$
$$= \bigvee_{y\in X} [R(x, y) * \bigwedge_{z\in X} (R(y, z) \to A(z))]$$

Thus by Lemma 2.3 (1) and Lemma 2.1 (2), (9) we have

$Sym(R) * (< R > [R]A)(x) = Sym(R) * \bigvee_{y \in X} [R(x,y) * \bigwedge_{z \in X} (R(y,z) \to A(z))]$

$= \bigvee_{y \in X} [Sym(R) * R(x,y) * \bigwedge_{z \in X} (R(y,z) \to A(z))] \leq$

$\leq \bigvee_{y \in X} [(Sym(R) * R(x,y)) * (R(y,z) \to A(x))] \leq$

$\leq \bigvee_{y \in X} R(y,x) * (R(y,x) \to A(x)) \leq A(x)$

Then by Lemma 2.1 (1) we get

(c) $Sym(R) \leq (< R > [R]A)(x) \to A(x)$

We remark that

$([R] < R > A)(x) = \bigwedge_{y \in X} (R(x,y) \to (< R > A)(y))$

$= \bigwedge_{y \in X} [R(x,y) \to \bigvee_{z \in X} R(y,z) * A(z)]$

By Lemma 2.3 (1)

$Sym(R) * A(x) * R(x,y) \leq A(x) * R(y,x) \leq \bigvee_{z \in X} R(y,z) * A(z)$

therefore using Lemma 2.1 (1)

$Sym(R) * A(x) \leq \bigwedge_{y \in X} [R(x,y) \to \bigvee_{z \in X} R(y,z) * A(z)] = ([R] < R > A)(x)$

By Lemma 2.1 (1) we obtain

(d) $Sym(R) \leq A(x) \to ([R] < R > A)(x)$

Hence we obtain (a) and (b). It remains to prove the converse inequalities

(e) $\bigwedge_{A \in \mathcal{F}} S(< R > [R]A, A) \leq Sym(R)$

(f) $\bigwedge_{A \in \mathcal{F}} S(A, < R > [R]A) \leq Sym(R)$

Let $x, y \in X$. Consider the R–afterset A_0 of R defined by $A_0(z) = R(y,z)$ for all $z \in X$. In this case we have

$(< R > [R]A_0)(x) = \bigvee_{t \in X} [R(x,t) * \bigwedge_{v \in X} (R(t,v) \to A_0(v)] \geq$

$\geq R(x,y) * \bigwedge_{v \in X} (R(y,v) \to A_0(v)) = R(x,y)$

hence by Lemma 2.1 (7):

$\bigwedge_{A \in \mathcal{F}} S(< R > [R]A, A) \leq S(< R > [R]A_0, A_0) \leq$

$\leq (< R > [R])A_0(x) \to A_0(x) \leq R(x,y) \to A_0(x) = R(x,y) \to R(y,x)$

Therefore

$\bigwedge_{A \in \mathcal{F}} S(< R > [R]A, A) \leq \bigwedge_{x,y \in X} (R(x,y) \to R(y,x)) = Sym(R)$

Now we shall prove (f). Let $x, y \in X$ and $A_1 = \{1/x\}$. Since $A_1(x) = 1$ it follows

$\bigwedge_{A \in \mathcal{F}} S(< R > [R]A, A) \leq S(A_1, [R] < R > A_1) \leq$

$\leq A_1(x) \to ([R] < R > A_1)(x) =$

$= ([R] < R > A_1)(x) = \bigwedge_{u \in X} [R(x,u) \rightarrow \bigvee_{z \in X} R(u,z) * A_1(z)] \leq$

$\leq R(x,y) \rightarrow \bigvee_{z \in X} R(y,z) * A_1(z) = R(x,y) \rightarrow R(y,x).$

Hence we get

$\bigwedge_{A \in \mathcal{F}} S(A, < R > [R]A) \leq \bigwedge_{x,y \in A} (R(x,y) \rightarrow R(y,x)) = Sym(R)$

□

Corollary 3.4 ([29]). *The following assertions are equivalent:*

(1) *R is symmetric;*
(2) *$< R > [R]A \subseteq A$ for all $A \in \mathcal{F}$;*
(3) *$A \subseteq [R] < R > A$ for all $A \in \mathcal{F}$.*

Theorem 3.5. $Tr(A) = \bigwedge_{A \in \mathcal{F}} S([R]A, [R][R]A) = \bigwedge_{A \in \mathcal{F}} S(< R >< R > A, < R > A)$

Proof. Let $A \in \mathcal{F}$. First we establish the following inequality:

(a) $Tr(A) \leq S([R]A, [R][R]A)$
(b) $Tr(A) \leq S(< R >< R > A, < R > A)$

Let $x \in X$. Applying Lemma 2.1 (6) and (12) we obtain:

$$\begin{aligned}([R][R]A)(x) &= \bigwedge_{y \in X} [R(x,y) \rightarrow ([R]A)(y)] \\ &= \bigwedge_{y \in X} [R(x,y) \rightarrow \bigwedge_{z \in X} (R(y,z) \rightarrow A(z))] \\ &= \bigwedge_{y,z \in X} [R(x,y) \rightarrow (R(y,z) \rightarrow A(z))] \\ &= \bigwedge_{y,z \in X} [R(x,y) * R(y,z) \rightarrow A(z)]\end{aligned}$$

Let $y, z \in X$. Hence, by Lemma 2.3 (2) and Lemma 2.1 (2):

$Tr(R) * ([R]A)(x) * R(x,y) * R(y,z) \leq ([R]A)(x) * R(x,z) =$

$= R(x,z) * \bigwedge_{u \in X} (R(x,u) \rightarrow A(u)) \leq R(x,z) * (R(x,z) \rightarrow A(z)) \leq A(z)$

By Lemma 2.1 (1) for all $y, z \in X$ we have

$Tr(R) * ([R]A)(x) \leq R(x,y) * R(y,z) \rightarrow A(z)$

hence

$Tr(R) * ([R]A)(x) \leq \bigwedge_{y,z \in X} (R(x,y) * R(y,z) \rightarrow A(z)) = ([R][R]A)(x)$

Applying again Lemma 2.1 (1) $Tr(R) \leq ([R]A)(x) \rightarrow ([R][R]A)(x)$ for each $x \in X$ hence

$Tr(R) \leq \bigwedge_{x \in X} (([R]A)(x) \rightarrow ([R][R]A)(x)) = S([R]A, [R][R]A)$

Now we shall prove (b). Let $x \in X$. Thus

$(< R >< R > A)(x) = \bigvee_{y \in X} R(x,y) * (< R > A)(y) = \bigvee_{y,z \in X} R(x,y) * R(y,z) * A(z)$

therefore by using Lemma 2.3 (2)

$Tr(R) * (< R >< R > A)(x) = \bigvee_{y,z \in X} Tr(R) * R(x, y) * R(y, z) * A(z) \leq$

$\leq \bigvee_{y,z \in X} R(x, z) * A(z) = \bigvee_{z \in X} R(x, z) * A(z) = (< R > A)(x)$

By Lemma 2.1 (1) it follows that $Tr(R) \leq (< R >< R > A)(x) \to (< R > A)(x)$ for each $x \in X$, hence $Tr(R) \leq S(< R >< R > A, < R > A)$

From (a) and (b) it follows immediately

$Tr(R) \leq \bigwedge_{A \in \mathcal{F}} S([R]A, [R][R]A)$; $Tr(R) \leq \bigwedge_{A \in \mathcal{F}} S(< R >< R > A, < R > A)$

We establish now the converse inequalities. First we prove that

(c) $\bigwedge_{A \in \mathcal{F}} S([R]A, [R][R]A) \leq Tr(R)$

Let $x, y, z \in X$. We define $A_0 \in \mathcal{F}$ by $A_0(u) = R(x, u)$ for all $u \in X$. We remark that $([R]A_0)(x) = 1$, hence, by Lemma 2.1 (2):

$S([R]A_0, [R][R]A_0) * R(x, y) * R(y, z) \leq$

$R(x, y) * R(y, z) * (([R]A_0)(x) \to ([R][R]A_0)(x)) =$

$= R(x, y) * R(y, z) * ([R][R]A_0)(x) =$

$= R(x, y) * R(y, z) * \bigwedge_{u,v \in X} (R(x, u) * R(u, v) \to A_0(v)) \leq$

$\leq R(x, y) * R(y, z) * (R(x, y) * R(y, z) \to A_0(z)) \leq A_0(z) = R(x, z)$

Applying Lemma 2.1 (1) it follows $S([R]A_0, [R][R]A_0) \leq R(x, y) * R(y, z) \to R(x, z)$ for all $x, y, z \in X$, thus

$\bigwedge_{A \in \mathcal{F}} S([R]A, [R][R]A) \leq S([R]A_0, [R][R]A_0) \leq$

$\leq \bigwedge_{x,y,z \in X} (R(x, y) * R(y, z) \to R(x, z)) = Tr(R)$

It remains to prove the inequality

(d) $\bigwedge_{A \in \mathcal{F}} S(< R >< R > A, < R > A) \leq Tr(R)$

Let $x, y, z \in X$ and take $A_1 = \{1/z\}$. By Lemma 2.1 (1) one obtains

$S(< R >< R > A_1, < R > A_1) * R(x, y) * R(y, z) \leq$

$\leq [(< R >< R > A_1)(x) \to (< R > A_1)(x)] * R(x, y) * R(y, z) =$

$= [\bigvee_{u,v \in X} R(x, u) * R(u, v) * A_1(v)) \to (< R > A_1)(x)] * R(x, y) * R(y, z) \leq$

$\leq R(x, y) * R(y, z) * [R(x, y) * R(y, z) * A_1(z) \to (< R > A_1)(x)] =$

$= R(x, y) * R(y, z) * [R(x, y) * R(y, z) \to (< R > A_1)(x)] \leq$

$\leq (< R > A_1)(x) = \bigvee_{t \in X} R(x, t) * A_1(t) = R(x, z)$

Then $S(< R >< R > A_1, < R > A_1) \leq R(x, y) * R(y, z) \to R(x, z)$
hence

$\bigwedge_{A \in \mathcal{F}} S(< R >< R > A, < R > A) \leq S(< R >< R > A_1, < R > A_1) \leq$

$\leq \bigwedge_{x,y,z \in X} (R(x, y) * R(y, z) \to R(x, z)) = Tr(R)$ □

Corollary 3.6 ([29]). *The following assertions are equivalent:*

(1) *R is transitive;*
(2) *$[R]A \subseteq [R][R]A$ for all $A \in \mathcal{F}$;*
(3) *$< R >< R > A \subseteq < R > A$ for all $A \in \mathcal{F}$.*

Theorem 3.7. $Eu(R) = \bigwedge_{A\in\mathcal{F}} S(< R > [R]A, [R]A) = \bigwedge_{A\in\mathcal{F}} S(< R > A, [R] < R > A)$

Proof. First we shall prove the inequality

(a) $Eu(R) \leq \bigwedge_{A\in\mathcal{F}} S(< R > [R]A, [R]A)$

Let $A \in \mathcal{F}$ and $x, y, z \in X$. By Lemma 2.3 (3) and Lemma 2.1 (2):

$Eu(R) * R(x, y) * R(x, z) * \bigwedge_{t\in X} (R(y, t) \to A(t)) \leq$
$\leq R(y, z) * (R(y, z) \to A(z)) \leq A(z)$

hence, according to Lemma 2.1 (1) one gets

$Eu(R) * R(x, y) * \bigwedge_{t\in X} (R(y, t) \to A(t)) \leq R(x, z) \to A(z)$.

Since $(< R > [R]A)(x) = \bigvee_{y\in X} [R(x, y) * \bigwedge_{t\in X} (R(y, t) \to A(t))]$

for each $z \in X$ it follows that

$$Eu(R) * (< R > [R]A)(x) = \bigvee_{y\in X} [Eu(R) * R(x, y) * \bigwedge_{t\in X} (R(y, t) \to A(t))] \leq$$
$$\leq \bigwedge_{z\in X} (R(x, z) \to A(z)) = ([R]A)(x)$$

Thus, by Lemma 2.1(1), $Eu(R) \leq (< R > [R]A)(x) \to ([R]A)(x)$ for all $A \in \mathcal{F}$ and $x \in X$, hence

$$Eu(R) \leq \bigwedge_{A\in\mathcal{F}} \bigwedge_{x\in X} [(< R > [R]A)(x) \to ([R]A)(x)]$$
$$= \bigwedge_{A\in\mathcal{F}} S(< R > [R]A, [R]A)$$

Now we shall establish the converse inequality

(b) $\bigwedge_{A\in\mathcal{F}} S(< R > [R]A, [R]A) \leq Eu(R)$

Let $x, y, z \in X$ and $A_0 \in \mathcal{F}$ defined by $A_0(u) = R(y, u)$ for all $u \in X$. Then

$$S(< R > [R]A_0, [R]A_0) \leq (< R > [R]A_0)(x) \to ([R]A_0)(x) =$$
$$= [\bigvee_{u\in X} R(x, u) * \bigwedge_{v\in X} (R(u, v) \to A_0(v))] \to \bigwedge_{t\in X} (R(x, t) \to A_0(t)) =$$
$$= \bigwedge_{u,t\in X} [R(x, u) * \bigwedge_{v\in X} (R(u, v) \to A_0(v)) \to (R(x, t) \to A_0(t))] =$$
$$= \bigwedge_{u,t\in X} [(R(x, t) * R(x, u) * \bigwedge_{v\in X} (R(u, v) \to A_0(y)) \to A_0(t)] \leq$$
$$\leq (R(x, y) * R(x, z) * \bigwedge_{v\in X} (R(y, v) \to R(y, v))) \to A_0(z) =$$
$$= R(x, y) * R(x, z) \to R(y, z)$$

We conclude that

$$\bigwedge_{A\in\mathcal{F}} S(<R>[R]A,[R]A) \leq S(<R>[R]A_0,[R]A_0) \leq$$
$$\leq \bigwedge_{x,y,z\in X} (R(x,y)*R(x,z)\rightarrow R(y,z)) = Eu(R)$$

Next we prove the following inequality

(c) $Eu(R) \leq \bigwedge_{A\in\mathcal{F}} S(<R>A,[R]<R>A)$

Let $A\in\mathcal{F}$ and $x,y\in X$. By Lemma 2.3 (3)

$$Eu(R)*(<R>A)(x)*R(x,y) = \bigvee_{z\in X} Eu(R)*R(x,y)*R(z,z)*A(z) \leq$$
$$\leq \bigvee_{z\in X} R(y,z)*A(z)$$

therefore, by Lemma 2.1 (1)

$$Eu(R)*(<R>A)(x) \leq R(x,y)\rightarrow \bigvee_{z\in X} R(y,z)*A(z)$$

Hence

$$Eu(R)*(<R>A)(x) \leq \bigwedge_{y\in X}[R(x,y)\rightarrow \bigvee_{z\in X} R(x,z)*A(z)] =$$
$$= ([R]<R>A)(x)$$

Applying again Lemma 2.1 (1) one gets $Eu(R) \leq (<R>A)(x) \rightarrow ([R]<R>A)(x)$

therefore

$$Eu(R) \leq \bigwedge_{A\in\mathcal{F}}\bigwedge_{x\in X}[(<R>A)(x)\rightarrow ([R]<R>A)(x) =$$
$$= \bigwedge_{A\in\mathcal{F}} S(<R>A,[R]<R>A)$$

The following inequality has remained to be proved:

(d) $\bigwedge_{A\in\mathcal{F}} S(<R>A,[R]<R>A) \leq Eu(R)$

Let $x,y,z\in X$. We denote $A_1=\{1/y\}$. Then

$$(<R>A_1)(z) = \bigvee_{v\in X} R(z,v)*A_1(v) = R(z,y)$$
$$([R]<R>A_1)(z) = \bigwedge_{u\in X}[R(z,u)\rightarrow \bigvee_{v\in X} R(u,v)*A_1(v)]$$
$$= \bigwedge_{u\in X}(R(z,u)\rightarrow R(u,y))$$

therefore

$$(<R>A_1)(z)\rightarrow ([R]<R>A_1)(z) = R(z,y)\rightarrow \bigwedge_{u\in X}(R(z,u)\rightarrow R(u,y)) \leq$$
$$\leq R(z,y)\rightarrow (R(z,x)\rightarrow R(x,y)) = R(z,x)*R(z,y)\rightarrow R(x,y)$$

Hence

$$\bigwedge_{A\in\mathcal{F}} S(<R>A,[R]<R>A) \leq S(<R>A_1,[R]<R>A_1) \leq$$
$$\leq \bigwedge_{x,y,z\in X} (R(z,x)*R(z,y)\rightarrow R(x,y)) = Eu(R)$$ □

Corollary 3.8 ([29]). *The following assertions are equivalent:*

(1) *R is euclidian;*
(2) *$< R > [R]A \subseteq [R]A$ for all $A \in \mathcal{F}$;*
(3) *$< R > A \subseteq [R]$ for all $A \in \mathcal{F}$.*

As far as $Ser(R)$ is concerned, in the general case of a left–continuous t–norm one can give only the following partial result:

Theorem 3.9. $\bigwedge_{A\in\mathcal{F}} S([R]A, < R > A) \leq Ser(R)$

Proof. Let $x \in X$. We define $A_0 \in \mathcal{F}$ by $A_0(y) = R(x, y)$ for any $y \in X$. Then

$([R]A_0)(x) = \bigwedge_{y\in X} (R(x, y) \to A_0(y)) = 1$ hence

$$\bigwedge_{A\in\mathcal{F}} S([R]A, < R > A) \leq S([R]A_0, < R > A_0) \leq$$
$$\leq ([R]A_0)(x) \to (< R > A_0)(x) = 1 \to (< R > A_0)(x) = (< R > A_0)(x) =$$
$$= \bigvee_{y\in X} R(x, y) * A_0(y) \leq \bigvee_{y\in X} R(x, y)$$

Hence

$$\bigwedge_{A\in\mathcal{F}} S([R]A, < R > A) \leq S([R]A_0, < R > A_0) \leq$$
$$\leq \bigwedge_{x\in X} \bigvee_{y\in X} R(x, y) = Ser(R)$$

□

For the Gödel t-norm one can obtain the following characterization of $Ser(R)$.

Theorem 3.10. *If $*$ is the Gödel t-norm $\wedge$ then*

$$Ser(R) = \bigwedge_{A\in\mathcal{F}} S([R]A, < R > A)$$

Proof. Let $A \in \mathcal{F}$ and $x \in X$. By Lemma 2.1 (10) and (2) it follows

$$Ser(R) \wedge ([R]A)(x) = [\bigwedge_{v\in X} \bigvee_{y\in X} R(v, y)] \wedge ([R]A)(x) \leq$$
$$\leq [\bigvee_{y\in X} R(x, y)] \wedge ([R]A)(x) =$$
$$= \bigvee_{y\in X} [R(x, y) \wedge ([R]A)(x)] =$$
$$= \bigvee_{y\in X} [R(x, y) \wedge \bigwedge_{z\in X} R(x, z) \to A(z)] \leq$$
$$\leq \bigvee_{y\in X} [R(x, y) \wedge (R(x, y) \to A(y))] =$$
$$= (< R > A)(x)$$

By Lemma 2.1 (1) from the previous inequalities it follows

$Ser(R) \leq ([R]A)(x) \to (< R > A)(x)$ for any $x \in X$ and $A \in \mathcal{F}$.

$$Ser(R) \leq \bigwedge_{A\in\mathcal{F}} S([R]A, < R > A)$$

The converse inequality follows by Theorem 3.9.

□

Example 3.11. Let $*$ be the Lukasiewicz t–norm. Assume that $X = \{a, b\}$. If R is a fuzzy relation on X then

$Ser(R) = [R(a,a) \vee R(a,b)] \wedge [R(b,a) \vee R(b,b)]$

We consider the following fuzzy relation R on X:

$R(a,a) = \frac{1}{2}$; $R(a,b) = \frac{1}{3}$; $R(b,a) = 1$; $R(b,b) = 0$

Replacing above one obtains $Ser(R) = \frac{1}{2}$. We consider the fuzzy set $A_0 : X \to [0,1]$ defined by $A_0(a) = \frac{1}{2}$; $A_0(b) = \frac{1}{3}$. Then

$$([R]A_0)(a) = [R(a,a) \to A_0(a)] \wedge [R(a,b) \to A_0(b)] = (\tfrac{1}{2} \to \tfrac{1}{2}) \wedge (\tfrac{1}{3} \to \tfrac{1}{3}) = 1$$

$$(< R > A_0)(a) = [R(a,a) * A_0(a)] \vee [R(a,b) * A_0(b)] = = \tfrac{1}{2} * \tfrac{1}{2} \vee \tfrac{1}{3} * \tfrac{1}{3} = 0$$

From here we deduce

$$\bigwedge_{A \in \mathcal{F}} S([R]A, < R > A) \leq S([R]A_0, < R > A_0) \leq$$

$[R]A_0(a) \to < R > A_0(a) = \neg \to 0 = 0$

It follows that for the Lukasiewicz t–norm

$$Ser(R) \neq \bigwedge_{A \in \mathcal{F}} S([R]A, < R > A)$$

4. The similarity indicator

Let R be a fuzzy relation on the universe X and $*$ a fixed left–continuous t–norm. We recall that R is called *similarity relation* (w.r.t. the t–norm $*$) if it is reflexive, symmetric and transitive (see [7], [28]). Then the indicators of reflexivity, symmetry and transitivity allow to introduce the similarity indicator associated with a fuzzy relation.

The similarity indicator of the fuzzy relation R is defined as

$Sim(R) = Ref(R) * Sym(R) * Tr(R)$

Lemma 4.1. *The fuzzy relation R is a similarity relation if $Sim(R) = 1$.*

Proof. It follows from the definition of $Sim(R)$ and Lemma 2.2 (a)-(c). □

Example 4.2. Let $X = \mathbf{N} = \{0, 1, 2, \ldots, \}$ and $\alpha \in [0, 1]$. We consider the binary relation on X defined by the infinite matrix:

$$R = \begin{bmatrix} \alpha & \frac{1}{2} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \alpha + \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \alpha + \frac{1}{2^\alpha} & \frac{1}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \alpha + \frac{1}{2^{n-1}} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \ldots & 0 & \alpha + \frac{1}{2^n} \end{bmatrix}$$

Therefore $R(0,0) = \alpha$, $R(0,1) = \frac{1}{2}$, $R(n,n) = \alpha + \frac{1}{2^n}$, $R(n, n+1) = \frac{1}{2}$ for $n \geq 1$ and for the other entries $R(n,m) = 0$.

We compute $Ref(R)$:

$$Ref(R) = \bigwedge_{n \geq 0} R(n,n) = \alpha \wedge \bigwedge_{n \geq 1} (\alpha + \frac{1}{2^n}) = \alpha \wedge \alpha = \alpha$$

According to the definition, $Sym(R)$ can be written:

$Sym(R) = \bigwedge_{n\geq 0}\bigwedge_{m\geq 0}(R(n,m) \rightarrow R(m,n))$

For any $n \geq 0$ one has

$$\bigwedge_{m\geq 0}(R(n,m) \rightarrow R(m,n)) = (R(n,n) \rightarrow R(n,n)) \wedge (R(n,n+1) \rightarrow R(n+1,n))$$
$$= R(n,n+1) \rightarrow 0 = \neg R(n,n+1) = \neg\tfrac{1}{2}$$

Then

$Sym(R) = \neg\frac{1}{2}$

With a similar reasoning for the computation of $Trans(R)$ one has:

$$Trans(R) = \bigwedge_{m,n,k\geq 0}(R(n,k) * R(k,m) \rightarrow R(n,m)) =$$
$$= \bigwedge_{n,m\geq 0}[(R(n,n) * R(n,m) \rightarrow R(n,m)) \wedge (R(n,n+1) * R(n+1,m) \rightarrow R(n,m))]$$
$$= \bigwedge_{n,m\geq 0}[\frac{1}{2} * R(n+1,m) \rightarrow R(n,m)] =$$
$$= \bigwedge_{n\geq 0}[(\frac{1}{2} * R(n+1,n+1) \rightarrow R(n,n+1)) \wedge (\frac{1}{2} * R(n+1,n+2) \rightarrow R(n,n+2))]$$
$$= \bigwedge_{n\geq 0}[(\frac{1}{2} * (\alpha + \frac{1}{2^{n+1}}) \rightarrow \frac{1}{2}) \wedge (\frac{1}{2} * \frac{1}{2} \rightarrow 0)] =$$
$$= \neg(\tfrac{1}{2} * \tfrac{1}{2})$$

Thus $Sim(R) = \alpha * \neg\frac{1}{2} * \neg(\frac{1}{2} * \frac{1}{2})$.

If $*$ is the Gödel t-norm then $\neg\frac{1}{2} = 0$, thus $Sym(R) = 0$; if $*$ is the Lukasiewicz t–norm then $\neg\frac{1}{2} = \frac{1}{2}$, thus $Sym(R) = \alpha *_L \frac{1}{2} = \max(0, \alpha + \frac{1}{2} - 1) = \max(0, \alpha - \frac{1}{2})$.

Theorem 4.3. *If $A \in \mathcal{F}$ then the following inequalities hold:*
(i) $Sim(R) \leq E([R] < R > A, < R > A)$
(ii) $Sim(R) \leq E(< R > [R]A, [R]A)$

Proof. By Theorem 3.1

(a) $Sim(R) \leq Ref(R) \leq S([R] < R > A, < R > A)$

(b) $Sim(R) \leq Ref(R) \leq S([R]A, < R > [R]A)$

Next we prove the following inequalities:

(c) $Sim(R) \leq S(< R > A, [R] < R > A)$

(d) $Sim(R) \leq S(< R > [R]A, [R]A)$

Let $x, y \in X$. Then by Lemma 2.3 (1) and (2):

$$Sim(R) * (< R > A)(x) * R(x,y) =$$
$$= Sim(R) * [\bigvee_{z\in X} R(x,z) * A(z)] * R(x,y) =$$
$$= \bigvee_{z\in X} Sim(R) * R(x,y) * R(x,z) * A(z) \leq$$
$$\leq \bigvee_{z\in X} Tr(R) * (Sym(R) * R(x,y)) * R(x,z) * A(z)$$
$$\leq \bigvee_{z\in X} Tr(R) * R(y,x) * R(x,z) * A(z) \leq$$

$\leq \bigvee_{z\in X} R(y,z) * A(z)$

Then by Lemma 2.1 (1)

$Sim(R) * (< R > A)(x) \leq R(x,y) \to \bigvee_{z\in X} R(y,z) * A(z)$

Then

$Sim(R) * (< R > A)(x) \leq \bigwedge_{y\in X} [R(x,y) \to \bigvee_{z\in X} R(y,z) * A(z)] =$

$= ([R] < R > A)(x)$

Applying again Lemma 2.1 (1) for any $x \in X$ one obtains:

$Sim(R) \leq (< R > A)(x) \to ([R] < R > A)(x)$

from where it follows

$Sim(R) \leq \bigwedge_{x\in X} [(< R > A)(x) \to ([R] < R > A)(x)] =$

$= S(< R > A, [R] < R > A)$

We prove now inequality (d). Let $x, y \in X$. Applying Lemma 2.3 (1), (2) and Lemma 2.1 (2) it follows

$Sim(R) * (< R > [R]A)(x) * R(x,y) =$

$= Sim(R) * [\bigvee_{u\in X} (R(x,u) * \bigwedge_{v\in X} (R(u,v) \to A(v)))] * R(x,y)$

$= \bigvee_{u\in X} [Sim(R) * R(x,u) * R(x,y) * \bigwedge_{v\in X} (R(u,v) \to A(u))] \leq$

$\leq \bigvee_{u\in X} [Tr(R) * (Sym(R) * R(x,u)) * R(x,y) * (R(u,y) \to A(y))] \leq$

$\leq \bigvee_{u\in X} [Tr(R) * R(u,x) * R(x,y) * (R(u,y) \to A(y))] \leq$

$\leq \bigvee_{u\in X} [R(u,y) * (R(u,y) \to A(y))] \leq A(y)$

By Lemma 2.1 (1)

$Sim(R) * (< R > [R]A)(x) \leq R(x,y) \to A(y)$ for all $y \in X$

therefore

$Sim(R) * (< R > [R]A)(x) \leq \bigwedge_{y\in X} (R(x,y) \to A(y)) = ([R]A)(x)$

Applying again Lemma 2.1 one obtains

$Sim(R) \leq (< R > [R]A)(x) \to ([R]A)(x)$ for any $x \in X$

from where

$Sim(R) \leq \bigwedge_{x\in X} [(< R > [R]A)(x) \to ([R]A)(x)] =$

$= S(< R > [R]A, [R]A)$

From (a) and (c) it follows

$Sim(R) \leq S([R] < R > A, < R > A) \wedge S(< R > A, [R] < R > A) =$

$= E([R] < R > A, [R]A)$

Similarly from (b) and (d) it follows (ii). □

Corollary 4.4 ([31])**.** *Let R be a similarity relation on X. Then for any $A \in \mathcal{F}$ the following equalities are true:*

$[R] < R > A =< R > A, < R > [R]A = [R]A$

Proof. By Lemma 4.1, $Sim(R) = 1$, thus from Theorem 4.3 (i) it follows
$E([R] < R > A, < R > A) = 1$. Thus $[R] < R > A =< R > A$.
The second equality is proved similarly. □

Theorem 4.5. *Assume that* $*$ *is the Gödel t-norm* $\wedge$. *Then for any* $A \in \mathcal{F}$ *the following inequalities hold:*
(i) $Sim(R) \leq E([R][R]A, [R]A)$
(ii) $Sim(R) \leq E(< R >< R > A, < R > A)$

Proof. By Theorems 3.1 and 3.5:
$Sim(R) \leq Ref(R) \leq S(R][R]A, [R]A)$
$Sim(R) \leq Ref(R) \leq S(< R > A, < R >< R > A)$
$Sim(R) \leq Tr(R) \leq S([R]A, [R][R]A)$
$Sim(R) \leq Tr(R) \leq S(< R >< R > A, < R > A)$
Then, from the first and the third inequality it follows (i):
$Sim(R) \leq S([R]R]A, [R]A) \wedge S([R]A, [R][R]A) = E([R][R]A, [R]A)$
Similarly, (ii) follows from the other two inequalities. □

Remark 4.6. If R is a similarity relation then, according to Lemma 2.2, $Sim(R) = 1$, from where it follows that in Theorems 4.3 and 4.5 we have equalities instead of inequalities. The comparison of the left side member with the right side member in the inequalities of the two theorems depends on the form of the fuzzy set A; to find other sufficient conditions in order to have equalities seems a difficult problem.

5. Order indicators

In social choice theory [3] and consumer theory [26] individual preferences are modeled by binary relations called preference relations. Usually, a preference relation needs to be reflexive and transitive.

Reflexivity and transitivity properties appear also in case of fuzzy preference relations [20], [21]. A *fuzzy preorder* is a reflexive and transitive fuzzy relation.

In fuzzy decision making it is accepted the idea that a fuzzy criterion is mathematically represented by a fuzzy preorder [20]. Nevertheless when the decisions are multicriterial, the fuzzy relation R obtained by aggregating criteria $R_1, \ldots, R_n$ is not always transitive. Then the fuzzy relation R turns into a transitive fuzzy relation $\hat{R}$ (usually, the transitive closure of R) (see [20], p. 176). A fuzzy criterion $\hat{R}$ is obtained, whose effect can be far away from what criteria $R_1, \ldots, R_n$ express.

A way to avoid this inconvenience is, instead of reflexivity and transitivity conditions, an indicator of fuzzy preorder to be used. If R is a fuzzy relation on a set of alternatives X then we define
$FPreOrd(R) = Ref(R) \wedge Tr(R)$.

By Lemma 2.2 (a) and (c), R is a fuzzy preorder iff $FPreOrd(R) = 1$. $FPreOrd(R)$ is called the *fuzzy preorder indicator* of R; it measures the degree to which the fuzzy relation R is a fuzzy preorder.

Then we will no longer require that a criterion to be a fuzzy preorder; instead we will require conditions on the fuzzy preorder indicator. For example, we will impose the condition $FPreOrd(R) \geq \alpha$ where α is a pre-chosen level. The inequality

$FPreOrd(R) \geq \alpha$ can be read: relation R satisfies the condition of being criterion to a larger extent than the threshold α.

By Theorems 3.1 and 3.5, the fuzzy preorder indicator can be expressed through fuzzy modal operators $<>$ and $[\ \ ]$, therefore it can be formalized in a fuzzy modal logic system.

(Partial or total) order relations play an important role in social choice theory and consumer theory [3], [26]. In defining a notion of fuzzy order the property of antisymmetry of a fuzzy relation should appear. There exist several proposals to define antisymmetry of a fuzzy relation (one of them is in [20], p. 50). The most satisfying definition of antisymmetry of a fuzzy relation seems to be in [8], based on similarity relations. We recall the definition of fuzzy order from [8].

Let X be a set of alternatives and Ω a similarity relation on X (w.r.t. a left–continuous t–norm $*$). A fuzzy relation R on X is said to be

- Ω–reflexive, if $\Omega(x, y) \leq R(x, y)$, for all $x, y \in X$;
- Ω–antisymmetric, if $R(x, y) * R(y, x) \leq \Omega(x, y)$, for all $x, y \in X$.

A fuzzy relation R on X is called Ω*–order* if it is Ω–reflexive, Ω–antisymmetric and transitive.

Given a similarity relation Ω on X, in [22] the following indicators were introduced:

$$Ref_{\Omega}(R) = \bigwedge_{x,y \in X} (\Omega(x, y) \to R(x, y))$$

$$Ant_{\Omega}(R) = \bigwedge_{x,y \in X} (R(x, y) * R(y, x) \to \Omega(x, y)).$$

Using these definitions and Lemma 2.1 (3) the equivalences follow immediately:

- $Ref_{\Omega}(R) = 1$ iff R is Ω–reflexive.
- $Ant_{\Omega}(R) = 1$ iff R is Ω–antisymmetric.

Then $Ref_{\Omega}(R)$ will be called the Ω–reflexivity indicator, and $Ant_{\Omega}(R)$ the Ω–antisymmetry indicator.

Naturally, the fuzzy order indicator $FOrd_{\Omega}(R)$ will be defined by

$$FOrd_{\Omega}(R) = Ref_{\Omega}(R) \wedge Ant_{\Omega}(R) \wedge Tr(R).$$

An *open problem* is the characterization of $Ref_{\Omega}(R)$, $Ant_{\Omega}(R)$ and $FOrd_{\Omega}(R)$ indicators by means of fuzzy modal operators $<>$ and $[\ \ ]$. These will allow their formalization in a fuzzy modal logic.

Szpilrajn theorem (every partial order can be extended to a linear order) is an important result in crisp binary relation theory. It has several applications in social choice theory, economics, game theory, etc. (see the introduction of [1] for references on the applications of Szpilrajn theorem). Extensions of Szpilrajn theorem to fuzzy relations can be found in [9], [21], [1], [2]. In [22] it is proved a Szpilrajn–type theorem formulated in terms of $FOrd_{\Omega}(R)$ indicator. An *open problem* is to formulate this result by fuzzy modal operators $<>$ and $[\ \ ]$ and to prove it in the framework of a fuzzy modal logic system.

6. Concluding remarks

This paper connects two important notions from the theory of fuzzy relations: indicators and fuzzy modal operators. Some belong to a more intuitive level: they measure the degree to which a fuzzy relation verifies a property. The other can be regarded in the formal context of fuzzy modal logic. The main propositions characterize the indicators of reflexivity, symmetry, transitivity, euclideanity, seriality w.r.t. fuzzy modal operators.

Such results give an idea on the way important properties of fuzzy relations can be analyzed by formal mechanisms of fuzzy modal logic. Of course this desideratum is a problem to be solved.

Also the proved formulas can lead to computation methods in the analysis of attributes expressed by properties of fuzzy relations, accordingly in multicriterial decisions.

We remark some perspectives to continue the findings of the paper by enumerating the following open problems:

(a) To prove similar characterization theorems for other important indicators of fuzzy relations: irreflexivity, asymmetry, transitivity, semitransitivity, Ferrers, etc.

(b) To analyze the way similar results are reflected in the properties of the indicators of rationality, revealed preference and consistency of fuzzy choice functions.

Theorem 4.9 of [23] shows that the Arrow index of a fuzzy choice function C coincides with the congruence indicators of C and all three indicators can be expressed with respect to the transitivity indicator $Tr(R_C)$ of the fuzzy revealed preference relation R_C (see [21], p. 92). The mentioned result is a fuzzy version of Arrow–Sen theorem from classic revealed preference theory [30]. By Theorem 3.5, this fuzzy Arrow–Sen theorem can be expressed in terms of the fuzzy modal operators $<>$ and $[\ ]$. This remark could be a first step to build a fuzzy modal logic system where fuzzy Arrow–Sen theorem should be formalized.

(c) Which is the impact of the theorems of the paper on the applications of fuzzy modal operators in fuzzy morphology and fuzzy rough sets?

(d) Fuzzy social choice theory (in particular, various fuzzy versions of Arrow's impossibility theorem) uses essentially properties of fuzzy preference relations (see e.g. [4], [5]). How can the indicators of fuzzy relations and fuzzy modal operators be applied to fuzzy social choice? Can one formulate and prove a form of fuzzy Arrow's impossibility theorem in terms of the indicators of fuzzy relations and of fuzzy modal operators?

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