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r-(τ_i, τ_j)-generalized fuzzy closed sets in smooth bitopological spaces

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Abstract. In this paper, we present the concept of r-(τ_i, τ_j)-generalized fuzzy closed (briefly, r-(τ_i, τ_j)-gfc) sets in the smooth bitopological spaces and we investigate some notions of these sets. By using r-(τ_i, τ_j)-gfc sets, we define a new fuzzy closure operator referred to as (i, j)-$\mathcal{GC}$ which generates a new smooth topology, $\tau_{(i,j)\text{-}\mathcal{GC}}$. An application of these sets the definition of (i, j)-$T_{\frac{1}{2}}$ spaces. Finally, (i, j)-GF-continuous mappings and (i, j)-GF-irresolute mappings are introduced and some of their properties studied.

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Keywords: r-(τ_i, τ_j)-generalized fuzzy closed sets, (i, j)-generalized fuzzy closure operator, (i, j)-$T_{\frac{1}{2}}$ spaces, (i, j)-generalized fuzzy continuous (irresolute) maps.

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1. Introduction

Šostak [18], introduced the fundamental concept of a ‘fuzzy topological structure’, as an extension of both crisp topology and Chang’s fuzzy topology [3], indicating that not only the object were fuzzified, but also the axiomatics. Subsequently, Badard [2], introduced the concept of ‘smooth topological space’. Chattopadhyay et al. [4] and Chattopadhyay and Samanta [5] re-introduced the same concept, calling it ‘gradation of openess’. Ramadan [16] and his colleagues introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [4, 5, 6, 8, 19]). Thus, the terms ‘fuzzy topology’, in Šostak’s sense, ‘gradation of openness’ and ‘smooth topology’ are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Lee et al. [13] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's fuzzy bitopological spaces [9].

Levine [14], introduced the concept of generalized closed sets of a topological space and a class of topological spaces called $T_{\frac{1}{2}}$ spaces. Subsequently, Fukutake [7], introduced the concept of generalized closed sets in bitopological spaces. Balasubramanian and Sundaram [1] introduced the concept of generalized fuzzy closed sets within Chang's fuzzy topology, as an extension of the generalized closed sets of Levine. Kim and Ko [11] defined r-generalized fuzzy closed sets in smooth topological spaces. Recently, we [20] introduced the concept of generalized fuzzy closed set in smooth bitopological space (X, τ_1, τ_2) by using smooth supra topological space (X, τ_{12}) induced from smooth bitopological space (X, τ_1, τ_2).

In this paper, we introduce r-(τ_i, τ_j)-gfc sets in smooth bitopological space (X, τ_1, τ_2), following the introduction of r-generalized fuzzy closed sets in smooth topological space of Kim and Ko [11] and Fukutake [7], and we show properties of these sets. Moreover, we introduce the concept (i, j)-$T_{\frac{1}{2}}$ space and strongly fuzzy pairwise $T_{\frac{1}{2}}$ space in a smooth bitopological space (X, τ_1, τ_2). Following Fukutake [7] (given that in a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$, if λ is $(\mathcal{T}_i, \mathcal{T}_j)$-generalized closed then $C_{\mathcal{T}_j}(\lambda) - \lambda$ contains no non empty $\mathcal{T}_i$-closed set), we show that this result is not true in a smooth bitopological space (see Remark 5.15). Consequently, many properties of $T_{\frac{1}{2}}$ space which depend on this fact (e.g. Proposition 2.13(ii), pp.21 and Theorem 2.15, pp.22 in [7]), have not varied. In addition, we define a new fuzzy closure operator (i, j)-$\mathcal{GC}$ by using r-(τ_i, τ_j)-gfc sets and consequently we obtain a new smooth topology, $\tau_{(i,j)-\mathcal{GC}}$ (see Theorem 4.5). Finally, we define and study (i, j)-GF-continuous (respectively, irresolute) mappings and investigate some of their properties.

2. Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$ and I^X be the family of all fuzzy sets on X. For any $\mu_1, \mu_2 \in I^X$, $\mu_1 \wedge \mu_2 = \min\{\mu_1(x), \mu_2(x) : x \in X\}$, $\mu_1 \vee \mu_2 = \max\{\mu_1(x), \mu_2(x) : x \in X\}$ and $\mu_1 - \mu_2 = \min\{\mu_1(x), \bar{1} - \mu_2(x) : x \in X\}$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha \ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with value 0 and 1, respectively. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. The fuzzy point x_t is said to be contained in a fuzzy set λ iff $\lambda(x) \geq t$. For $\lambda \in I^X$, $\bar{1} - \lambda$ denotes the complement of λ. FP stand for fuzzy pairwise. The indices are $i, j \in \{1, 2\}$ and $i \neq j$. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1 ([2, 4, 16, 18])**.** A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall \ \mu_1, \mu_2 \in I^X$,
(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1}-\mu) \geq r$. Note, Šostak [18] used the term 'fuzzy topology' and Chattopadhyay [4], the term 'gradation of openness' for a smooth topology τ.

Definition 2.2 ([13])**.** A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short). For $\lambda \in I^X$ and $r \in I_0$, r-τ_i-open (resp., closed) fuzzy set denotes the r-open (resp., closed) fuzzy set in (X, τ_i), for $i = 1, 2$.

Subsequently, the fuzzy closure for any fuzzy set in smooth topological space is given as follows:

Definition 2.3 ([5])**.** Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure of λ is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X |\ \mu \geq \lambda,\ \tau(\bar{1}-\mu) \geq r\}.$$

Definition 2.4 ([5])**.** A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$,
(C2) $\lambda \leq C(\lambda, r)$,
(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,
(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$,
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ given by

$$\tau_C(\lambda) = \bigvee\{r \in I |\ C(\bar{1}-\lambda, r) = \bar{1}-\lambda\}.$$

In a similar pattern, a fuzzy interior operator was defined.

Definition 2.5 ([10, 17])**.** A mapping $I : I^X \times I_0 \to I^X$ is called a fuzzy interior operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping I satisfies the following conditions:

(I1) $I(\bar{1}, r) = \bar{1}$,
(I2) $I(\lambda, r) \leq \lambda$,
(I3) $I(\lambda, r) \wedge I(\mu, r) = I(\lambda \wedge \mu, r)$,
(I4) $I(\lambda, r) \geq I(\lambda, s)$ if $r \leq s$,
(I5) $I(I(\lambda, r), r) = I(\lambda, r)$.

The fuzzy interior operator I generates a smooth fuzzy topology $\tau_I : I^X \longrightarrow I$ as follows:

$$\tau_I(\lambda) = \bigvee\{r \in I |\ I(\lambda, r) = \lambda\}.$$

Theorem 2.6 ([5, 12])**.** *Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a τ_i-fuzzy closure of λ is a mapping $C_{\tau_i} : I^X \times I_0 \longrightarrow I^X$, defined as*

$$C_{\tau_i}(\lambda, r) = \bigwedge\{\mu \in I^X |\ \mu \geq \lambda,\ \tau_i(\bar{1}-\mu) \geq r\}.$$

And, a τ_i-fuzzy interior of λ is a mapping $I_{\tau_i} : I^X \times I_0 \longrightarrow I^X$, defined as

$$I_{\tau_i}(\lambda, r) = \bigvee\{\mu \in I^X |\ \mu \leq \lambda,\ \tau_i(\mu) \geq r\}.$$

Then:

(1) C_{τ_i} *(resp.,* I_{τ_i}*) is a fuzzy closure (resp., interior) operator.*
(2) $\tau_{C_{\tau_i}} = \tau_{I_{\tau_i}} = \tau_i$.
(3) $I_{\tau_i}(\bar{1} - \lambda, r) = \bar{1} - C_{\tau_i}(\lambda, r)$, $\forall\, r \in I_0, \lambda \in I^X$.

Definition 2.7 ([11])**.** Let (X, τ) be a smooth topological space, let $\lambda, \mu \in I^X$ and $r \in I_0$. A fuzzy set λ is called an r-generalized fuzzy closed (r-gfc, for short) if $C_\tau(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq s$ for all $0 < s \leq r$. The complement of r-gfc is called an r-generalized fuzzy open (r-gfo, for short).

Lemma 2.8 ([15])**.** *Let $f : X \longrightarrow Y$ be a mapping and let λ and μ be fuzzy sets in X and Y, respectively, then the following properties hold:*

(1) $\lambda \leq f^{-1}(f(\lambda))$ *and equality holds if f is injective.*
(2) $f(f^{-1}(\mu)) \leq \mu$ *and equality holds if f is surjective.*
(3) *For any fuzzy point x_t in X, $f(x_t)$ is a fuzzy point in Y and $f(x_t) = (f(x))_t$.*
(4) *When $f(\lambda) \leq \mu$, $\lambda \leq f^{-1}(\mu)$.*

Definition 2.9 ([4])**.** Let (X, τ) and (Y, σ) be smooth topological spaces. A mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be fuzzy continuous if $\tau(f^{-1}(\mu)) \geq \sigma(\mu)$, for each $\mu \in I^Y$.

Definition 2.10 ([10])**.** Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be smooth bitopological spaces. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be FP-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \sigma_i(\mu)$, for each $\mu \in I^Y$ and $i = 1, 2$.

3. r-(τ_i, τ_j)-generalized fuzzy closed sets

In this section we introduce and investigate the concept of r-(τ_i, τ_j)-generalized fuzzy closed sets in smooth bts (X, τ_1, τ_2).

Definition 3.1. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then λ is called:

(1) an r-(τ_i, τ_j)-generalized fuzzy closed (r-(τ_i, τ_j)-gfc, for short), if $C_{\tau_j}(\lambda, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for all $0 < s \leq r$.
(2) an r-(τ_i, τ_j)-generalized fuzzy open (r-(τ_i, τ_j)-gfo, for short), if $\bar{1} - \lambda$ is an r-(τ_i, τ_j)-gfc.

Remark 3.2. If $\tau_1 = \tau_2$ in Definition 3.1, then r-(τ_i, τ_j)-gfc is an r-gfc in Definition 2.7 in the sense of Kim [11].

Remark 3.3. We denote to the family of all r-(τ_i, τ_j)-gfc sets in smooth bts (X, τ_1, τ_2) by $r_{\mathfrak{gc}}(\tau_i, \tau_j)$.

The next proposition shows the relationship between r-τ_j-closed (resp., open) fuzzy sets and r-(τ_i, τ_j)-gfc (resp., gfo) sets in smooth bts (X, τ_1, τ_2).

Proposition 3.4. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:*

(1) *If λ is an r-τ_j-closed fuzzy set, then λ is an r-(τ_i, τ_j)-gfc.*
(2) *If λ is an r-τ_j-open fuzzy set, then λ is an r-(τ_i, τ_j)-gfo.*

Proof. To show (1), let $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. Since $\tau_j(\bar{1} - \lambda) \geq r$, then $C_{\tau_j}(\lambda, r) = \lambda$. In view of Theorem 2.6(1) and Definition 2.4(C4), we get $C_{\tau_j}(\lambda, s) \leq C_{\tau_j}(\lambda, r) = \lambda$ for all $s \leq r$. Thus, $C_{\tau_j}(\lambda, s) \leq \mu$. Hence, λ is an r-(τ_i, τ_j)-gfc. To prove (2), clearly $\bar{1} - \lambda$ is an r-τ_j-closed fuzzy set. By using (1), we get that λ is an r-(τ_i, τ_j)-gfo. □

The converse of the above proposition is not true as seen from the following example.

Example 3.5. Let $X = \{a, b\}$. We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.5} \vee b_{0.8}, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.7} \vee b_{0.5}, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{2}$ the fuzzy set $\lambda = a_{0.3} \vee b_{0.3}$ is a $\frac{1}{2}$-(τ_1, τ_2)-gfc but λ is not a $\frac{1}{2}$-τ_2-closed fuzzy set.

Theorem 3.6. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is both r-τ_i-open fuzzy set and r-(τ_i, τ_j)-gfc, then λ is an r-τ_j-closed fuzzy set.*

Proof. Since λ is an r-τ_i-open fuzzy set. i.e., $\tau_i(\lambda) \geq r$, then $\tau_i(\lambda) \geq s$ for $0 < s \leq r$. Since $\lambda \leq \lambda$ and λ is an r-(τ_i, τ_j)-gfc, then from Definition 3.1(1), $C_{\tau_j}(\lambda, s) \leq \lambda$ for $0 < s \leq r$. However, $\lambda \leq C_{\tau_j}(\lambda, s)$. Thus, $C_{\tau_j}(\lambda, s) = \lambda$ for $0 < s \leq r$. Consequently, $C_{\tau_j}(\lambda, r) = \lambda$. Hence, λ is an r-τ_j-closed fuzzy set. □

Remark 3.7. The notions of r-(τ_i, τ_j)-gfc sets and r-gfc set in (X, τ_i) are independent. For, in Example 3.5, the fuzzy set $\lambda = a_{0.3} \vee b_{0.3}$ is a $\frac{1}{2}$-(τ_1, τ_2)-gfc but λ is not a $\frac{1}{2}$-gfc in (X, τ_1), and the fuzzy set $\lambda = a_{0.4} \vee b_{0.2}$ is a $\frac{1}{2}$-gfc in (X, τ_1) but λ is not a $\frac{1}{2}$-(τ_1, τ_2)-gfc in (X, τ_1, τ_2).

Proposition 3.8. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:*

(1) *If λ_1, λ_2 are r-(τ_i, τ_j)-gfc sets, then $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-gfc.*
(2) *If λ_1, λ_2 are r-(τ_i, τ_j)-gfo sets, then $\lambda_1 \wedge \lambda_2$ is an r-(τ_i, τ_j)-gfo.*

Proof. To prove part (1), let $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. This yields, $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. Since λ_1, λ_2 are r-(τ_i, τ_j)-gfc sets, then $C_{\tau_j}(\lambda_1, s) \leq \mu$ and $C_{\tau_j}(\lambda_2, s) \leq \mu$, imply $C_{\tau_j}(\lambda_1, s) \vee C_{\tau_j}(\lambda_2, s) \leq \mu$. In view of Definition 2.4(C3), $C_{\tau_j}(\lambda_1 \vee \lambda_2, s) = C_{\tau_j}(\lambda_1, s) \vee C_{\tau_j}(\lambda_2, s) \leq \mu$. Hence, $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-gfc. The proof of (2), follows from the duality of (1). □

Remark 3.9. The intersection (resp., union) of two r-(τ_i, τ_j)-gfc (resp., gfo) sets cannot to be an r-(τ_i, τ_j)-gfc (resp., gfo) set as seen from the following example.

Example 3.10. Let $X = \{a, b, c\}$. We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.5}, \\ \frac{1}{3} & \text{if } \lambda = b_{0.4} \vee c_{0.4}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.5} \vee b_{0.4} \vee c_{0.4}, \\ 0 & \textit{otherwise}; \end{cases} \qquad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = b_{0.4}, \\ \frac{1}{3} & \text{if } \lambda = c_{0.4}, \\ \frac{1}{2} & \text{if } \lambda = b_{0.4} \vee c_{0.4}, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = 1$ the fuzzy sets $\nu_1 = a_{0.5} \vee b_{0.4}$ and $\nu_2 = b_{0.4} \vee c_{0.6}$ are 1-(τ_2, τ_1)-gfc sets but $\nu_1 \wedge \nu_2$ is not a 1-(τ_2, τ_1)-gfc set.

Next we introduce some prosperities of r-(τ_i, τ_j)-gfc (resp., gfo) sets.

Proposition 3.11. *Let (X, τ_1, τ_2) be a smooth bts. If $\tau_1 \leq \tau_2$, then $r_{\mathfrak{gc}}(\tau_2, \tau_1) \subseteq r_{\mathfrak{gc}}(\tau_1, \tau_2)$.*

Proof. Let $\lambda \in r_{\mathfrak{gc}}(\tau_2, \tau_1)$, i.e., λ is an r-(τ_2, τ_1)-gfc. Let $\lambda \leq \mu$ such that $\tau_1(\mu) \geq s$ for $0 < s \leq r$. Since $\tau_1 \leq \tau_2$, then $\tau_2(\mu) \geq s$ for $0 < s \leq r$. Since λ is an r-(τ_2, τ_1)-gfc, we have $C_{\tau_1}(\lambda, s) \leq \mu$. Again since $\tau_1 \leq \tau_2$, then $C_{\tau_2}(\lambda, s) \leq C_{\tau_1}(\lambda, s) \leq \mu$. So, $C_{\tau_2}(\lambda, s) \leq \mu$. Hence, $\lambda \in r_{\mathfrak{gc}}(\tau_1, \tau_2)$. □

Remark 3.12. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) $r_{\mathfrak{gc}}(\tau_1, \tau_2)$ is generally not equal to $r_{\mathfrak{gc}}(\tau_2, \tau_1)$. To show this consider Example 3.5. Let $\lambda = a_{0.3} \vee b_{0.5}$, then λ is a $\frac{1}{2}$-(τ_1, τ_2)-gfc but not a $\frac{1}{2}$-(τ_2, τ_1).

(2) If $\lambda \in r_{\mathfrak{gc}}(\tau_1, \tau_2) \cap r_{\mathfrak{gc}}(\tau_2, \tau_1)$, then λ is called pairwise gfc.

Theorem 3.13. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \mu \in I^X$ and $r \in I_0$. Then:*

(1) If λ is an r-(τ_i, τ_j)-gfc such that $\lambda \leq \mu \leq C_{\tau_j}(\lambda, r)$, then μ is an r-(τ_i, τ_j)-gfc.

(2) λ is an r-(τ_i, τ_j)-gfo if and only if $\mu \leq I_{\tau_j}(\lambda, r)$, whenever $\mu \leq \lambda$ and μ is an r-τ_i-closed fuzzy set.

(3) If λ is an r-(τ_i, τ_j)-gfo such that $I_{\tau_j}(\lambda, r) \leq \mu \leq \lambda$, then μ is an r-(τ_i, τ_j)-gfo.

Proof. To prove (1), let $\mu \leq \nu$ such that $\tau_i(\nu) \geq s$ for $0 < s \leq r$. Since $\lambda \leq \mu$, we obtain $\lambda \leq \nu$. Since λ is an r-(τ_i, τ_j)-gfc, this yields $C_{\tau_j}(\lambda, s) \leq \nu$ for $0 < s \leq r$. From Definition 3.1(1) and Definition 2.4(C5), we have

$$C_{\tau_j}(\mu, s) \leq C_{\tau_j}(C_{\tau_j}(\lambda, s), s) = C_{\tau_j}(\lambda, s) \leq \nu.$$

Thus, $C_{\tau_j}(\mu, s) \leq \nu$ and consequently, μ is an r-(τ_i, τ_j)-gfc.

Next to prove (2), for the necessity, let $\bar{1} - \lambda \leq \bar{1} - \mu$ and $\tau_i(\bar{1} - \mu) \geq s$ for $0 < s \leq r$ and apply Definition 3.1(1) and Theorem 2.6(3), giving the required result.

Conversely, let $\bar{1} - \lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. i.e., $\bar{1} - \mu \leq \lambda$ such that $\bar{1} - \mu$ is an s-closed fuzzy set for $0 < s \leq r$. Assuming we have $\bar{1} - \mu \leq I_{\tau_j}(\lambda, s)$, this implies $\bar{1} - I_{\tau_j}(\lambda, s) \leq \mu$. In view of Theorem 2.6(3), we then have $C_{\tau_j}(\bar{1} - \lambda, s) \leq \mu$. Thus, $\bar{1} - \lambda$ is an r-(τ_i, τ_j)-gfc. Hence, λ is an r-(τ_i, τ_j)-gfo. Finally, to prove (3), taking $\bar{1} - \lambda$ as an r-(τ_i, τ_j)-gfc and then applying (1), we have the required result. □

Theorem 3.14. *Let (X, τ_1, τ_2) be a smooth bts. Then for each $x \in X$ and $t = 1$, x_t is an r-τ_i-closed fuzzy set or $\bar{1} - x_t$ is an r-(τ_i, τ_j)-gfc.*

Proof. If x_t is not an r-τ_i-closed fuzzy set, then $\bar{1}-x_t$ is not an r-τ_i-open fuzzy set, implying that the only r-τ_i-open fuzzy set in X which containing $\bar{1}-x_t$ is $\bar{1}$. Thus, $C_{\tau_j}(\bar{1}-x_t,s)\leq \bar{1}$ for all $0<s\leq r$. Therefore, $\bar{1}-x_t$ is an r-(τ_i,τ_j)-gfc. □

4. Characterization of (i,j)-generalized fuzzy closure operator

In this section, we introduce a new fuzzy closure operator by using r-(τ_i,τ_j)-gfc sets and study some of their properties. Also, we introduce a new smooth topology by using the fuzzy closure operator.

Definition 4.1. Let (X,τ_1,τ_2) be a smooth bts, $\lambda\in I^X$ and $r\in I_0$. The (i,j)-generalized fuzzy closure of λ is a map, (i,j)-$\mathcal{GC}: I^X\times I_0\longrightarrow I^X$ defined by

$$(i,j)\text{-}\mathcal{GC}(\lambda,r)=\bigwedge\{\rho\in I^X\mid \rho\geq\lambda,\ \rho\ is\ r\text{-}(\tau_i,\tau_j)\text{-}gfc\},$$

and the (i,j)-generalized fuzzy interior of λ is a map, (i,j)-$\mathcal{GI}: I^X\times I_0\longrightarrow I^X$ defined by

$$(i,j)\text{-}\mathcal{GI}(\lambda,r)=\bigvee\{\rho\in I^X\mid \rho\leq\lambda,\ \rho\ is\ r\text{-}(\tau_i,\tau_j)\text{-}gfo\}.$$

Proposition 4.2. *Let (X,τ_1,τ_2) be a smooth bts, $\lambda\in I^X$ and $r\in I_0$. Then,*

$$I_{\tau_j}(\lambda,r)\leq(i,j)\text{-}\mathcal{GI}(\lambda,r)\leq\lambda\leq(i,j)\text{-}\mathcal{GC}(\lambda,r)\leq C_{\tau_j}(\lambda,r).$$

Proof. Since every r-τ_j-closed (resp., open) fuzzy set is an r-(τ_i,τ_j)-gfc (resp., gfo) set, the proof is established. □

Next, we state some basic properties of (i,j)-$\mathcal{GC}$ and (i,j)-$\mathcal{GI}$ in the following proposition.

Proposition 4.3. *Let (X,τ_1,τ_2) be a smooth bts, λ,λ_1 and $\lambda_2\in I^X$ and $r\in I_0$. Then:*

(1) (i,j)-$\mathcal{GI}(\bar{1}-\lambda,r)=\bar{1}-(i,j)$-$\mathcal{GC}(\lambda,r)$.
(2) *If $\lambda_1\leq\lambda_2$, then (i,j)-$\mathcal{GC}(\lambda_1,r)\leq(i,j)$-$\mathcal{GC}(\lambda_2,r)$.*
(3) *If λ is an r-(τ_i,τ_j)-gfc, then (i,j)-$\mathcal{GC}(\lambda,r)=\lambda$.*
(4) *If $\lambda_1\leq\lambda_2$, then (i,j)-$\mathcal{GI}(\lambda_1,r)\leq(i,j)$-$\mathcal{GI}(\lambda_2,r)$.*
(5) *If λ is an r-(τ_i,τ_j)-gfo, then (i,j)-$\mathcal{GI}(\lambda,r)=\lambda$.*

Proof. We prove (1) using Definition 4.1:

$$\begin{aligned}\bar{1}-(i,j)\text{-}\mathcal{GC}(\lambda,r) &= \bar{1}-\bigwedge\{\rho\in I^X|\ \rho\geq\lambda,\ \rho\ is\ r\text{-}(\tau_i,\tau_j)\text{-}gfc\}\\ &= \bigvee\{\bar{1}-\rho\in I^X|\ \bar{1}-\rho\leq\bar{1}-\lambda,\bar{1}-\rho\ is\ r\text{-}(i,j)\text{-}gfo\}\\ &= (i,j)\text{-}\mathcal{GI}(\bar{1}-\lambda,r).\end{aligned}$$

The proof of (2), follows from Definition 4.1 while the proof of (3), follows from Definition 4.1 and Proposition 4.2. The proof of (4), comes by taking the complement of (2) and from (1). Finally, the proof of (5) is from the same elements as are in (3). □

In Proposition 4.3 the converse of (3) and (5) is not true as the following example show. The example in inspired by the one introduced in ([11], pp.333).

Example 4.4. Let $X = \{a, b\}$. Define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.7}, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.8}, \\ 0 & \textit{otherwise}. \end{cases}$$

Then (X, τ_1, τ_2) is a smooth bts. The fuzzy set $a_{0.7}$ is not a 1-(τ_1, τ_2)-gfc set on X because $a_{0.7} \leq a_{0.7}$, $\tau_1(a_{0.7}) \geq s$, $0 < s \leq 1$, $C_{\tau_2}(a_{0.7}, s) = \bar{1} \nleq a_{0.7}$.
Since $a_{0.7} \vee b_s$ is a 1-(τ_1, τ_2)-gfc set for $s \in I_0$, then $(1,2)$-$\mathcal{GC}(a_0.7, 1) = \bigwedge_{s \in I_0} (a_{0.7} \vee b_s) = a_{0.7} \vee \bigwedge_{s \in I_0} b_s = a_{0.7}$.

Theorem 4.5. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:*

(1) *(i,j)-$\mathcal{GC}$ (resp., (i,j)-$\mathcal{GI}$) is a fuzzy closure (resp., interior) operator.*

(2) *define $\tau_{(i,j)\text{-}\mathcal{GC}} : I^X \longrightarrow I$ as*

$$\tau_{(i,j)\text{-}\mathcal{GC}}(\lambda) = \bigvee \{ r \in I \mid (i,j)\text{-}\mathcal{GC}(\bar{1} - \lambda, r) = \bar{1} - \lambda \}.$$

Then, $\tau_{(i,j)\text{-}\mathcal{GC}}$ is a smooth topology on X such that $\tau_j \leq \tau_{(i,j)\text{-}\mathcal{GC}}$.

Proof. We have proven that (i,j)-$\mathcal{GC}$ is a fuzzy closure operator and in a similar way can prove that (i,j)-$\mathcal{GI}$ is a fuzzy interior operator. To prove (1), we need to satisfy conditions $(C1) - (C5)$ in Definition 2.4.

$(C1)$ Since $\bar{0}$ is an r-τ_j-closed fuzzy set in X, then from Proposition 3.4(1), $\bar{0}$ is an r-(τ_i, τ_j)-gfc in X and, from Proposition 4.3(3), we have (i,j)-$\mathcal{GC}(\bar{0}, r) = \bar{0}$.

$(C2)$ Follows immediately from Definition 4.1.

$(C3)$ Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, then from Proposition 4.3(2),

$$(i,j)\text{-}\mathcal{GC}(\lambda, r) \leq (i,j)\text{-}\mathcal{GC}(\lambda \vee \mu, r) \text{ and } (i,j)\text{-}\mathcal{GC}(\mu, r) \leq (i,j)\text{-}\mathcal{GC}(\lambda \vee \mu, r).$$

This implies that (i,j)-$\mathcal{GC}(\lambda, r) \vee$ (i,j)-$\mathcal{GC}(\mu, r) \leq (i,j)$-$\mathcal{GC}(\lambda \vee \mu, r)$.

Suppose (i,j)-$\mathcal{GC}(\lambda \vee \mu, r) \nleq (i,j)$-$\mathcal{GC}(\lambda, r) \vee$ (i,j)-$\mathcal{GC}(\mu, r)$. Consequently, $x \in X$ and $t \in (0,1)$ exist such that

$$(4.1) \qquad (i,j)\text{-}\mathcal{GC}(\lambda, r)(x) \vee (i,j)\text{-}\mathcal{GC}(\mu, r)(x) < t < (i,j)\text{-}\mathcal{GC}(\lambda \vee \mu, r)(x).$$

Since (i,j)-$\mathcal{GC}(\lambda, r)(x) < t$ and (i,j)-$\mathcal{GC}(\mu, r)(x) < t$, there exist r-(τ_i, τ_j)-gfc sets ρ_1, ρ_2 with $\lambda \leq \rho_1$ and $\mu \leq \rho_2$ such that

$$\rho_1(x) < t, \rho_2(x) < t.$$

Since $\lambda \vee \mu \leq \rho_1 \vee \rho_2$ and $\rho_1 \vee \rho_2$ is an r-(τ_i, τ_j)-gfc from Proposition 3.8(1), we have (i,j)-$\mathcal{GC}(\lambda \vee \mu, r)(x) \leq (\rho_1 \vee \rho_2)(x) < t$. This, however, contradicts (4.1). Hence, (i,j)-$\mathcal{GC}(\lambda, r) \vee$ (i,j)-$\mathcal{GC}(\mu, r) = (i,j)$-$\mathcal{GC}(\lambda \vee \mu, r)$.

$(C4)$ Let $r \leq s$, $r, s \in I_0$. Suppose (i,j)-$\mathcal{GC}(\lambda, r) \nleq (i,j)$-$\mathcal{GC}(\lambda, s)$. Consequently, $x \in X$ and $t \in (0,1)$ exist such that

$$(4.2) \qquad (i,j)\text{-}\mathcal{GC}(\lambda, s)(x) < t < (i,j)\text{-}\mathcal{GC}(\lambda, r)(x).$$

Since (i,j)-$\mathcal{GC}(\lambda, s)(x) < t$, there is an s-(τ_i, τ_j)-gfc set ρ with $\lambda \leq \rho$ such that $\rho(x) < t$. This yields $C_{\tau_j}(\rho, s_1) \leq \mu$, whenever $\rho \leq \mu$ and $\tau_i(\mu) \geq s_1$, for $0 < s_1 \leq s$. Since $r \leq s$, then $C_{\tau_j}(\rho, r_1) \leq \mu$ whenever $\rho \leq \mu$ and $\tau_i(\mu) \geq r_1$, for $0 < r_1 \leq r \leq s_1 \leq s$. This implies ρ is an r-(τ_i, τ_j)-gfc. From Definition 4.1, we have

(i,j)-$\mathcal{GC}(\lambda,r)(x) \le \rho(x) < t$. This contradicts (4.2). Hence, (i,j)-$\mathcal{GC}(\lambda,r) \le (i,j)$-$\mathcal{GC}(\lambda,s)$.

(C5) Let ρ be any r-(τ_i,τ_j)-gfc containing λ. Then, from Definition 4.1, we have (i,j)-$\mathcal{GC}(\lambda,r) \le \rho$. From proposition 4.3(2), we obtain (i,j)-$\mathcal{GC}(\ (i,j)$-$\mathcal{GC}(\lambda,r),r) \le (i,j)$-$\mathcal{GC}(\rho,r) = \rho$. This mean that (i,j)-$\mathcal{GC}(\ (i,j)$-$\mathcal{GC}(\lambda,r),r)$ is contained in every r-(τ_i,τ_j)-gfc set containing λ. Hence, (i,j)-$\mathcal{GC}(\ (i,j)$-$\mathcal{GC}(\lambda,r),r) \le (i,j)$-$\mathcal{GC}(\lambda,r)$. However, (i,j)-$\mathcal{GC}(\lambda,r) \le (i,j)$-$\mathcal{GC}(\ (i,j)$-$\mathcal{GC}(\lambda,r),r)$. Therefore, (i,j)-$\mathcal{GC}(\ (i,j)$-$\mathcal{GC}(\lambda,r),r) = (i,j)$-$\mathcal{GC}(\lambda,r)$. Thus (i,j)-$\mathcal{GC}$ is a fuzzy closure operator.

To prove (2), using (1) and Definition 2.4, we get $\tau_{(i,j)\text{-}\mathcal{GC}}$, which is a smooth topology. By Proposition 4.2, we have (i,j)-$\mathcal{GC}(\lambda,r) \le C_{\tau_j}(\lambda,r)$. This means that $C_{\tau_j}(\bar{1}-\lambda,r) = \bar{1}-\lambda$ and implies (i,j)-$\mathcal{GC}(\bar{1}-\lambda,r) = \bar{1}-\lambda$. Thus, $\tau_j(\lambda) \le \tau_{(i,j)\text{-}\mathcal{GC}}(\lambda)\ \forall \lambda \in I^X$. □

Proposition 4.6. *Let (X,τ_1,τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:*

(1) *If $\tau_1 \le \tau_2$, then $(1,2)$-$\mathcal{GC}(\lambda,r) \le (2,1)$-$\mathcal{GC}(\lambda,r)$.*
(2) *If λ is an r-(τ_i,τ_j)-gfc, then λ is an r-$\tau_{(i,j)\text{-}\mathcal{GC}}$-closed fuzzy set.*
(3) *If $\tau_1 \le \tau_2$, then $\tau_{(2,1)\text{-}\mathcal{GC}} \le \tau_{(1,2)\text{-}\mathcal{GC}}$.*

Proof. To show (1), suppose $(1,2)$-$\mathcal{GC}(\lambda,r) \not\le (2,1)$-$\mathcal{GC}(\lambda,r)$. There exists $x \in X$ and $t \in (0,1)$ such that

$$(2,1)\text{-}\mathcal{GC}(\lambda,r)(x) < t < (1,2)\text{-}\mathcal{GC}(\lambda,r)(x). \tag{4.3}$$

Since $(2,1)$-$\mathcal{GC}(\lambda,r)(x) < t$, there exists an r-(τ_2,τ_1)-gfc set ρ such that $\lambda \le \rho$ and $\rho(x) < t$. From Proposition 3.11, ρ is an r-(τ_1,τ_2)-gfc, which implies $(1,2)$-$\mathcal{GC}(\lambda,r)(x) < \rho(x) < t$. This contradicts (4.3).

The proof of (2) follows from Proposition 4.3(3). Finally (3), follows directly from (1). □

The following example shows that the converse of Proposition 4.6(2) is not true.

Example 4.7. Let $X = \{a,b,c\}$. We define smooth topologies $\tau_1,\tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0},\bar{1}, \\ \frac{1}{2} & \text{if } \lambda = a_1, \\ \frac{1}{4} & \text{if } \lambda = b_1 \vee c_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0},\bar{1}, \\ \frac{1}{2} & \text{if } \lambda = b_1, \\ \frac{1}{2} & \text{if } \lambda = c_1, \\ \frac{1}{2} & \text{if } \lambda = b_1 \vee c_1, \\ \frac{1}{4} & \text{if } \lambda = a_1 \vee c_1, \\ 0 & \textit{otherwise}. \end{cases}$$

Then (X,τ_1,τ_2) is a smooth bts. The fuzzy set $\lambda = b_1$ is a $\frac{1}{2}$-closed fuzzy set in $\tau_{(2,1)\text{-}\mathcal{GC}}$ but not a $\frac{1}{2}$-(τ_2,τ_1)-gfc.

5. (i,j)-GF-continuous and (i,j)-GF-irresolute mappings

In this section we introduce the concepts of (i,j)-generalized fuzzy continuous (resp., irresolute) mappings in smooth bts and study the relationship between them. We also investigate some of their properties and also, we introduce the definition of (i,j)-$T_{\frac{1}{2}}$ space and the strongly fuzzy pairwise $T_{\frac{1}{2}}$ space in smooth bts (X,τ_1,τ_2).

Throughout this section consider (X, τ_1, τ_2), (Y, σ_1, σ_2) and $(Z, \varphi_1, \varphi_2)$ as smooth bts's. For a mapping f from (X, τ_1, τ_2) into (Y, σ_1, σ_2), we shall denote the fuzzy continuous (resp., closed, open) mapping from (X, τ_j) into (Y, σ_j), $j \in \{1, 2\}$ by j-fuzzy continuous (resp., closed, open) mapping. Firstly, we state the definition of (i, j)-generalized fuzzy continuous (resp., irresolute) mappings.

Definition 5.1. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called:

1. (1) (i, j)-generalized fuzzy continuous ((i, j)-GF-continuous, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc in X for each $\mu \in I^Y$ with $\sigma_j(\bar{1} - \mu) \geq r$.
2. (2) (i, j)-generalized fuzzy irresolute ((i, j)-GF-irresolute, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc in X for each r-(σ_i, σ_j)-gfc $\mu \in I^Y$.

The following theorem gives an equivalent definition of (i, j)-GF-continuous mapping.

Theorem 5.2. *A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-GF-continuous if and only if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfo in X for each $\mu \in I^Y$ with $\sigma_j(\mu) \geq r$.*

Proof. This follows directly from Definition 3.1(2) and Definition 5.1(1). □

The relationship between the concepts of fuzzy continuous, FP-continuous and (i, j)-GF-continuous will be introduced in the following theorem and its corollary.

Theorem 5.3. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is a j-fuzzy continuous, then f is (i, j)-GF-continuous.*

Proof. Let $\mu \in I^Y$, such that $\sigma_j(\bar{1} - \mu) \geq r$. Since f is a j-fuzzy continuous, then $\tau_j(f^{-1}(\bar{1} - \mu)) \geq r$. Consequently, $f^{-1}(\mu)$ is an r-τ_j-closed fuzzy set in X. From Proposition 3.4(1), we have that $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc. Hence, f is (i, j)-GF-continuous. □

The proof of the next corollary follows directly from Theorem 5.3.

Corollary 5.4. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is FP-continuous, then f is (i, j)-GF-continuous.*

The converse of above Theorem 5.3 is not true as seen from the following example.

Example 5.5. Let $X = \{a, b, c\}$ and $Y = \{p, q\}$. Define fuzzy sets $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{0.5} \vee b_{0.3} \vee c_{0.7}, \quad \lambda_2 = a_{0.5} \vee b_{0.4} \vee c_{0.8}, \quad \mu_1 = p_{0.7} \vee q_{0.4}, \quad \mu_2 = p_{0.9} \vee q_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & otherwise; \end{cases} \quad and \quad \sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p,\ f(b) = p,\ f(c) = q$. Then, f is $(1,2)$-GF-continuous but not 2-fuzzy continuous, as μ_2 is a $\frac{1}{2}$-σ_2-open fuzzy set in Y. However, $f^{-1}(\mu_2) = a_{0.9} \vee b_{0.9} \vee c_{0.2}$ is not a $\frac{1}{2}$-τ_2-open fuzzy set in X.

Thus we have the following implication and none of them is reversible.

$$FP\text{-}continuous \Longrightarrow j\text{-}fuzzy\ continuous \Longrightarrow (i,j)\text{-}GF\text{-}continuous$$

Theorem 5.6. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a mapping. If f is (i,j)-GF-irresolute, then f is (i,j)-GF-continuous.*

Proof. This follows directly from Proposition 3.4(1) and Definition 5.1(2). □

Then converse of above theorem is not true as seen from the following example.

Example 5.7. Let $X = \{a, b, c\}$ and $Y = \{p, q\}$. Define fuzzy sets $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$\lambda_1 = a_{0.5} \vee b_{0.3} \vee c_{0.7},\ \ \lambda_2 = a_{0.5} \vee b_{0.4} \vee c_{0.8},\ \ \mu_1 = p_{0.9} \vee q_{0.6},\ \ \mu_2 = p_{0.1} \vee q_{0.8}$.

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \mu = \mu_1, \\ 0 & otherwise; \end{cases} \quad and \quad \sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p,\ f(b) = q,\ f(c) = q$. Then, f is $(1,2)$-GF-continuous but not $(1,2)$-GF-irresolute, as $\mu = p_{0.3} \vee q_{0.2}$ is a $\frac{1}{4}$-(σ_1, σ_2)-gfc set in Y. However, $f^{-1}(\mu) = a_{0.3} \vee b_{0.2}$ is not a $\frac{1}{4}$-(τ_1, τ_2)-gfc set in X.

The following theorem provides the conditions to establish (i,j)-GF-irresolute from (i,j)-GF-continuous.

Theorem 5.8. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is bijective, i-fuzzy open and (i,j)-GF-continuous, then f is (i,j)-GF-irresolute.*

Proof. Let μ be an r-(σ_i, σ_j)-gfc of Y. Let $f^{-1}(\mu) \leq \nu$, where $\tau_i(\nu) \geq s$ for $0 < s \leq r$. Clearly, $\mu \leq f(\nu)$ as f is a i-fuzzy open. Then, $f(\nu)$ is an r-σ_i-open fuzzy set. As μ is an r-(σ_i, σ_j)-gfc in Y, then $C_{\sigma_j}(\mu, s) \leq f(\nu)$ implies $f^{-1}(C_{\sigma_j}(\mu, s)) \leq \nu$. Since f is (i,j)-GF-continuous, then $f^{-1}(C_{\sigma_j}(\mu, s))$ is r-(τ_i, τ_j)-gfc in X and given $f^{-1}(C_{\sigma_j}(\mu, s)) \leq \nu$, we have $C_{\tau_j}(f^{-1}(C_{\sigma_j}(\mu, s)), s) \leq \nu$, which implies $C_{\tau_j}(f^{-1}(\mu), s) \leq \nu$. Therefore, $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc in X. Hence f is (i,j)-GF-irresolute. □

Theorem 5.9. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a mapping. Consider the following statements:*

(1) f *is* (i, j)-GF*-continuous.*
(2) $f((i, j)\text{-}\mathcal{GC}(\lambda, r)) \leq C_{\sigma_j}(f(\lambda), r)$, *for each* $\lambda \in I^X$, $r \in I_0$.
(3) $(i, j)\text{-}\mathcal{GC}(f^{-1}(\mu), r) \leq f^{-1}(C_{\sigma_j}(\mu, r))$, *for each* $\mu \in I^Y$.

Then (1)⇒(2)⇒(3).

Proof. (1) ⇒ (2) Let $\lambda \in I^X$. Since $f(\lambda) \in I^Y$, then $f(\lambda) \leq C_{\sigma_j}(f(\lambda), r)$. Then, $\lambda \leq f^{-1}(C_{\sigma_j}(f(\lambda), r))$. Since f is (i, j)-GF-continuous, then $f^{-1}(C_{\sigma_j}(f(\lambda), r))$ is an r-(τ_i, τ_j)-gfc in X. Hence, $(i, j)\text{-}\mathcal{GC}(\lambda, r) \leq f^{-1}(C_{\sigma_j}(f(\lambda), r))$ implies

$$f((i, j)\text{-}\mathcal{GC}(\lambda, r)) \leq f(f^{-1}(C_{\sigma_j}(f(\lambda), r))).$$

Thus, $f((i, j) - \mathcal{GC}(\lambda, r)) \leq C_{\sigma_j}(f(\lambda), r)$.

(2) ⇒ (3) Letting $\lambda = f^{-1}(\mu)$ and applying (2), we arrive at $f((i, j)\text{-}\mathcal{GC}(f^{-1}(\mu), r)) \leq C_{\sigma_j}(f(f^{-1}(\mu)), r) \leq C_{\sigma_j}(\mu, r)$. Consequently, $f((i, j)\text{-}\mathcal{GC}(f^{-1}(\mu), r)) \leq C_{\sigma_j}(\mu, r)$ implies $f^{-1}\ (f((i, j)\text{-}\mathcal{GC}(f^{-1}(\mu), r))) \leq f^{-1}(C_{\sigma_j}(\mu, r))$, which yields

$$(i, j)\text{-}\mathcal{GC}(f^{-1}(\mu), r) \leq f^{-1}(C_{\sigma_j}(\mu, r)).$$

□

Next, we give an example to show that (3) does not lead to (1) in above theorem.

Example 5.10. Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{0.6} \vee b_{0.3}, \quad \lambda_2 = a_{0.7} \vee b_{0.6}, \quad \mu_1 = p_{0.4} \vee q_{0.6}, \quad \mu_2 = p_{0.4} \vee q_{0.7}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_1, \\ 0 & otherwise; \end{cases} \quad \textit{and} \quad \sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p,\ f(b) = q$. Then $(1, 2)\text{-}\mathcal{GC}(f^{-1}(\mu), \frac{1}{3}) \leq f^{-1}(C_{\sigma_2}(\mu, \frac{1}{3}))$, for each $\mu \in I^Y$, but f is not $(1, 2)$-GF-continuous since $\bar{1} - \mu_2$ is a $\frac{1}{3}$-σ_2-closed fuzzy set in Y, but $f^{-1}(\bar{1} - \mu_2)$ is not a $\frac{1}{3}$-(τ_1, τ_2)-gfc set in X.

Theorem 5.11. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a mapping. If* f *is* (i, j)-GF*-continuous, then for each* $x_t \in Pt(X)$ *and for each* r-σ_j*-open fuzzy set* ν *in* Y *such that* $f(x_t) \in \nu$, *there exists an* r-(τ_i, τ_j)*-gfo* η *in* X *such that* $x_t \in \eta$ *and* $f(\eta) \leq \nu$.

Proof. Let $x_t \in Pt(X)$, let ν be an r-σ_j-open fuzzy set in Y such that $f(x_t) \in \nu$. Since f is (i, j)-GF-continuous then, by Theorem 5.2, $f^{-1}(\nu)$ is an r-(τ_i, τ_j)-gfo in X such that $x_t \in f^{-1}(\nu)$, let $\eta = f^{-1}(\nu)$, then $f(\eta) \leq \nu$. □

Theorem 5.12. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \varphi_1, \varphi_2)$ be mappings. Then:*

(1) *If g is j-fuzzy continuous and f is (i, j)-GF-continuous, then $g \circ f$ is (i, j)-GF-continuous.*
(2) *If g is (i, j)-GF-irresolute and f is (i, j)-GF-irresolute, then $g \circ f$ is (i, j)-GF-irresolute.*
(3) *If g is (i, j)-GF-continuous and f is (i, j)-GF-irresolute, then $g \circ f$ is (i, j)-GF-continuous.*

Proof. We prove (1), and the proof of (2) and (3) are similar to (1). Let μ be an r-φ_j-closed fuzzy set of Z. Since g is a j-fuzzy continuous, then $g^{-1}(\mu)$ is an r-σ_j-closed fuzzy set of Y. When f is (i, j)-GF-continuous, then $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is an r-(τ_i, τ_j)-gfc of X. Hence, $g \circ f$ is (i, j)-GF-continuous. □

We now introduce the definition of (i, j)-$T_{\frac{1}{2}}$ space and strongly fuzzy pairwise $T_{\frac{1}{2}}$ space in a smooth bts (X, τ_1, τ_2).

Definition 5.13. A smooth bts (X, τ_1, τ_2) is said to be (i, j)-$T_{\frac{1}{2}}$ space if every r-(τ_i, τ_j)-gfc is an r-τ_j-closed fuzzy set of X.

Definition 5.14. A smooth bts (X, τ_1, τ_2) is said to be strongly fuzzy pairwise $T_{\frac{1}{2}}$ space, if it is $(1, 2)$-$T_{\frac{1}{2}}$ space and $(2, 1)$-$T_{\frac{1}{2}}$ space.

Remark 5.15. In [7] (recall that in a bts $(X, \mathcal{T}_1, \mathcal{T}_2)$, if λ is $(\mathcal{T}_i, \mathcal{T}_j)$-generalized closed then $C_{\mathcal{T}_j}(\lambda) - \lambda$ contains no non-empty $\mathcal{T}_i$-closed set), we notice that this result is not true in a smooth bts. Yet many properties of $T_{\frac{1}{2}}$ space depend on this fact (e.g. Proposition 2.13(ii), pp.21 and Theorem 2.15, pp.22). The following example explains Remark 5.15.

Example 5.16. Let $X = \{a, b\}$. Define fuzzy sets $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.5} \vee b_{0.8}, \quad \lambda_2 = a_{0.5} \vee b_{0.5}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then (X, τ_1, τ_2) is a smooth bts. The fuzzy set $\lambda = a_{0.5} \vee b_{0.2}$ is a $\frac{1}{3}$-(τ_1, τ_2)-gfc but $C_{\tau_2}(\lambda, \frac{1}{3}) - \lambda = a_{0.5} \vee b_{0.5}$ contains a τ_1-closed fuzzy set $\bar{1} - \lambda_1$.

Theorem 5.17. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-GF-irresolute and X is (i, j)-$T_{\frac{1}{2}}$ space, then f is a j-fuzzy continuous.*

Proof. Let μ be an r-σ_j-closed fuzzy set of Y. Then, from Proposition 3.4(1), we have that μ is an r-(τ_i, τ_j)-gfc of Y. Since f is (i, j)-GF-irresolute, then $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc of X, but X is (i, j)-$T_{\frac{1}{2}}$ space, which implies $f^{-1}(\mu)$ is an r-τ_j-closed fuzzy set of X. Hence, f is a j-fuzzy continuous. □

Theorem 5.18. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is i-fuzzy continuous and j-fuzzy closed mapping, then every r-(τ_i, τ_j)-gfc set $\lambda \in I^X$, $f(\lambda)$ is an r-(τ_i, τ_j)-gfc in Y.*

Proof. Let λ be an r-(τ_i, τ_j)-gfc in X, let $f(\lambda) \leq \mu$, where $\sigma_i(\mu) \geq s$, for $0 < s \leq r$. Then, $\lambda \leq f^{-1}(\mu)$ and $f^{-1}(\mu)$ is an r-τ_i-open fuzzy set of X since f is a i-fuzzy continuous. As λ is an r-(τ_i, τ_j)-gfc in X, then $C_{\tau_j}(\lambda, s) \leq f^{-1}(\mu)$ implies $f(C_{\tau_j}(\lambda, s)) \leq \mu$. Since $\lambda \leq C_{\tau_j}(\lambda, s)$, then $f(\lambda) \leq f(C_{\tau_j}(\lambda, s))$. Therefore we have, $C_{\sigma_j}(f(\lambda), s) \leq C_{\sigma_j}(f(C_{\tau_j}(\lambda, s)), s) = f(C_{\tau_j}(\lambda, s)) \leq \mu$ since f is a j-fuzzy closed. Hence, $f(\lambda)$ is an r-(τ_i, τ_j)-gfc in Y. □

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