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On pairwise fuzzy σ-Baire spaces

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Abstract. In this paper the concepts of pairwise fuzzy σ-nowhere dense sets and pairwise fuzzy σ-Baire spaces are studied. Several characterizations of pairwise fuzzy σ-Baire spaces are also investigated.

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Keywords: Pairwise fuzzy open set, Pairwise fuzzy G_δ-set, Pairwise fuzzy F_σ-set, Pairwise fuzzy σ-nowhere dense set, Pairwise fuzzy σ-first category set, Pairwise fuzzy Baire space, Pairwise fuzzy σ-Baire space, Pairwise fuzzy σ-second category space.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh in his classical paper [17] in the year 1965 . This concept provides a natural foundation for treating mathematically the fuzzy phenomena, which exist pervasively in our real world and for building new branches of fuzzy mathematics. Thereafter, the paper of C. L. Chang [6] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

In 1989, A. Kandil [8] introduced the concept of fuzzy bitopological spaces as an extension of fuzzy topological space and as a generalization of bitopological spaces. The concepts of Baire bitopological spaces have been studied extensively in classical topology in [1], [2], [7] and [5]. The purpose of this paper is to introduce the concepts of pairwise fuzzy σ-nowhere nowhere dense sets, pairwise fuzzy σ-Baire bitopological space and pairwise fuzzy σ-second category space and study several characterizations of pairwise fuzzy σ-Baire bitopological spaces. Inter-relations between pairwise fuzzy σ-Baire spaces, pairwise fuzzy Baire spaces, pairwise fuzzy σ-second category spaces are also investigated in this paper. Several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to CHANG (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are fuzzy topologies on the non-empty set X.

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define :

(i) $\lambda \vee \mu$: X $\to$ [0,1] as follows : $(\lambda \vee \mu)(x) = \max\ \{\lambda(\text{x}), \mu(\text{x})\}$.
(ii) $\lambda \wedge \mu$: X $\to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = \min\ \{\lambda(\text{x}), \mu(\text{x})\}$.
(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

More generally, for a family of $\{\lambda_i / i \in I\}$ of fuzzy sets in X, $\vee_i \lambda_i$ and $\wedge_i \lambda_i$ are defined as $\vee_i \lambda_i = sup_i\{\lambda_i(x)\ /\ x \in X\}$ and $\wedge_i \lambda_i = inf_i\{\lambda_i(x)\ /\ x \in X\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define the interior and the closure of λ respectively as follows :

(i) $Int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$,
(ii) $Cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([3])**.** *For a fuzzy set λ in a fuzzy topological space X,*

(i) $1 - Int(\lambda) = Cl(1 - \lambda)$,
(ii) $1 - Cl(\lambda) = Int(1 - \lambda)$.

Definition 2.4 ([9])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.5 ([4])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ where $1 - \lambda_i \in T$ for $i \in I$.

Definition 2.6 ([4])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ where $\lambda_i \in T$ for $i \in I$.

Definition 2.7 ([9])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy nowhere dense set if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $intcl(\lambda) = 0$.

Definition 2.8 ([12])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ-nowhere dense set if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.9 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i$ $(i = 1, 2)$. The complement of pairwise fuzzy open set in (X, T_1, T_2) is called a pairwise fuzzy closed set.

Definition 2.10 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$, where (λ_i)'s are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.11 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty} \lambda_i$, where (λ_i)'s are pairwise fuzzy closed sets in (X, T_1, T_2).

Lemma 2.12 ([3]). *For a family of* $\{\lambda_\alpha\}$ *of fuzzy sets of a fuzzy topological space* (X,T), $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. *In case* $\mathscr{A}$ *is a finite set,* $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. *Also* $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$ *in* (X,T).

Definition 2.13 ([10]). A fuzzy set λ in a fuzzy bitopological space (X,T_1,T_2) is called a pairwise fuzzy dense set if $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = 1$.

Definition 2.14 ([13]). A fuzzy set λ in a fuzzy bitopological space (X,T_1,T_2) is called a pairwise fuzzy nowhere dense set if $int_{T_1}cl_{T_2}(\lambda) = int_{T_2}cl_{T_1}(\lambda) = 0$.

3. Pairwise fuzzy σ-nowhere dense sets

Definition 3.1 ([16]). A fuzzy set λ in a fuzzy bitopological space (X,T_1,T_2) is called a pairwise fuzzy σ-nowhere dense set if λ is a pairwise fuzzy F_σ-set in (X,T_1,T_2) such that $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$.

Example 3.2. Let $X = \{a,b,c\}$. The fuzzy sets α, β, δ and μ are defined on X as follows :

$$\alpha : X \to [0,1] \text{ is defined as } \alpha(a) = 0.2;\ \ \alpha(b) = 0.7;\ \ \alpha(c) = 0.5.$$
$$\beta : X \to [0,1] \text{ is defined as } \beta(a) = 0.7;\ \ \beta(b) = 0.5;\ \ \beta(c) = 0.3.$$
$$\delta : X \to [0,1] \text{ is defined as } \delta(a) = 0.5;\ \ \delta(b) = 0.7;\ \ \delta(c) = 0.5.$$
$$\mu : X \to [0,1] \text{ is defined as } \mu(a) = 0.5;\ \ \mu(b) = 0.2;\ \ \mu(c) = 0.8.$$

Clearly $T_1 = \{0,\beta,\mu,\delta,\beta\vee\mu,\beta\vee\delta,\mu\vee\delta,\beta\wedge\mu,\beta\wedge\delta,\mu\wedge\delta,\beta\vee[\mu\wedge\delta],\mu\vee[\beta\wedge\delta],\delta\wedge[\beta\vee\mu],\beta\vee\mu\vee\delta,1\}$ and $T_2 = \{0,\alpha,\beta,\mu,\alpha\vee\beta,\alpha\vee\mu,\beta\vee\mu,\alpha\wedge\beta,\alpha\wedge\mu,\beta\wedge\mu,\alpha\vee[\beta\wedge\mu],\beta\vee[\alpha\wedge\mu],\mu\vee[\alpha\wedge\beta],\alpha\wedge[\beta\vee\mu],\beta\wedge[\alpha\vee\mu],\mu\wedge[\alpha\vee\beta],\beta\vee\mu\vee\delta,1\}$ are fuzzy topologies on X. The fuzzy sets $\beta,\mu,\delta,\alpha\vee\mu,\beta\vee\mu,\beta\vee\delta,\beta\wedge\mu,\beta\wedge\delta,\mu\wedge\delta,\beta\vee[\mu\wedge\delta],\mu\vee[\alpha\wedge\beta],\beta\vee\mu\vee\delta,1$ are pairwise fuzzy open sets in (X,T_1,T_2). The fuzzy sets $1-[\beta\wedge\mu] = (1-\beta)\vee(1-\mu)\vee(1-[\beta\wedge\delta])\vee(1-[\mu\wedge\delta])\vee(1-[\beta\vee(\mu\wedge\delta)])$ and $\delta\wedge(\beta\vee\mu) = (1-\delta)\vee(1-[\alpha\vee\mu])\vee(1-[\beta\vee\mu])\vee(1-[\beta\vee\delta])\vee(1-[\mu\vee(\alpha\wedge\beta)])\vee(1-[\beta\vee\mu\vee\delta])$ are pairwise fuzzy F_σ-sets in (X,T_1,T_2). Also $int_{T_1}int_{T_2}(1-[\beta\wedge\mu]) = \delta \neq 0$ and $int_{T_2}int_{T_1}(1-[\beta\wedge\mu]) = \alpha\vee[\beta\wedge\mu] \neq 0$ and $int_{T_1}int_{T_2}(\delta\wedge[\beta\vee\mu]) = \beta\wedge\delta \neq 0$ and $int_{T_2}int_{T_1}(\delta\wedge[\beta\vee\mu]) = \beta\wedge[\alpha\vee\mu] \neq 0$. Hence $1-[\beta\wedge\mu]$ and $\delta\wedge[\beta\vee\mu]$ are not pairwise fuzzy σ-nowhere dense sets in (X,T_1,T_2).

The fuzzy set $1-(\mu\vee[\alpha\wedge\beta]) = (1-[\alpha\vee\mu])\vee(1-[\beta\vee\mu])\vee(1-[\beta\vee\mu\vee\delta])$ is a pairwise fuzzy F_σ-set in (X,T_1,T_2) such that $int_{T_1}int_{T_2}\big(1-(\mu\vee[\alpha\wedge\beta])\big) = 0$ and $int_{T_2}int_{T_1}\big(1-(\mu\vee[\alpha\wedge\beta])\big) = 0$ and hence $1-(\mu\vee[\alpha\wedge\beta])$ is a pairwise fuzzy σ-nowhere dense set in (X,T_1,T_2)

Proposition 3.3. *If* λ *is a pairwise fuzzy dense set and pairwise fuzzy* G_δ*-set in a fuzzy bitopological space* (X,T_1,T_2)*, then* $1-\lambda$ *is a pairwise fuzzy* σ*-nowhere dense set in* (X,T_1,T_2).

Proof. Let λ be a pairwise fuzzy dense set and pairwise fuzzy G_δ-set in a fuzzy bitopological space (X,T_1,T_2). Then, $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = 1$. This implies that $1-cl_{T_1}cl_{T_2}(\lambda) = 0$ and $1-cl_{T_2}cl_{T_1}(\lambda) = 0$ and hence we have $int_{T_1}int_{T_2}(1-\lambda) = 0$ and $int_{T_2}int_{T_1}(1-\lambda) = 0$. Also, since λ is a pairwise fuzzy G_δ-set, $1-\lambda$ is a pairwise fuzzy F_σ-set in (X,T_1,T_2). Hence $1-\lambda$ is a pairwise fuzzy F_σ-set in (X,T_1,T_2) such that $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$. Therefore $1-\lambda$ is a pairwise fuzzy σ-nowhere dense set in (X,T_1,T_2). □

Remark 3.4. The complement of a pairwise fuzzy σ-nowhere dense set in a pairwise fuzzy bitopological space need not be a pairwise fuzzy σ-nowhere dense set. For, consider the following example :

Example 3.5. Let $X = \{a, b, c\}$. The fuzzy sets α, β, δ and μ are defined on X as follows :

$$\begin{aligned} \alpha : X \to [0,1] \text{ is defined as } \alpha(a) = 0.6;\quad \alpha(b) = 0.9;\quad \alpha(c) = 0.8. \\ \beta : X \to [0,1] \text{ is defined as } \beta(a) = 0.7;\quad \beta(b) = 0.8;\quad \beta(c) = 0.9. \\ \delta : X \to [0,1] \text{ is defined as } \delta(a) = 0.8;\quad \delta(b) = 0.6;\quad \delta(c) = 0.7. \\ \mu : X \to [0,1] \text{ is defined as } \mu(a) = 0.7;\quad \mu(b) = 0.5;\quad \mu(c) = 0.9. \end{aligned}$$

Clearly $T_1 = \{0, \alpha, \beta, \delta, \alpha\vee\beta, \alpha\vee\delta, \beta\vee\delta, \alpha\wedge\beta, \alpha\wedge\delta, \beta\wedge\delta, \alpha\vee[\beta\wedge\delta], \delta\vee[\alpha\wedge\beta], \beta\wedge[\alpha\vee\delta], \delta\wedge[\alpha\vee\beta, \alpha\vee\beta\vee\delta, 1\}$ and $T_2 = \{0, \alpha, \mu, \delta, \alpha\vee\mu, \alpha\vee\delta, \mu\vee\delta, \alpha\wedge\mu, \alpha\wedge\delta, \mu\wedge\delta, \alpha\vee[\mu\wedge\delta], \mu\vee[\alpha\wedge\delta], \delta\vee[\alpha\wedge\mu], \alpha\wedge[\mu\vee\delta], \delta\wedge[\alpha\vee\mu], \alpha\vee\mu\vee\delta, \alpha\wedge\mu\wedge\delta, 1\}$ are fuzzy topologies on X. The fuzzy sets $\alpha, \delta, \alpha\vee\delta, \alpha\wedge\delta, \alpha\vee\beta, \mu\wedge\delta, \alpha\vee[\mu\wedge\delta], \alpha\vee\beta\vee\delta, 1$ are pairwise fuzzy open sets in (X, T_1, T_2). The fuzzy sets $(1-\alpha), (1-\delta), (1-[\alpha\vee\delta]), (1-[\alpha\wedge\delta]), (1-[\alpha\vee\beta]), (1-[\mu\wedge\delta]), \big(1-(\alpha\vee[\mu\wedge\delta])\big), (1-[\alpha\vee\beta\vee\delta]), 1$ are pairwise fuzzy closed sets in (X, T_1, T_2). Now the fuzzy sets $\lambda = (1-\alpha)\vee(1-\delta)\vee(1-[\alpha\vee\delta])\vee(1-[\alpha\wedge\delta])$ and $\eta = (1-[\alpha\vee\beta])\vee(1-[\mu\wedge\delta])\vee\big(1-(\alpha\vee[\mu\wedge\delta]\big)\vee(1-[\alpha\vee\beta\vee\delta])$ are pairwise fuzzy F_σ-sets in (X, T_1, T_2). Also $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$ and $int_{T_1}int_{T_2}(\eta) = int_{T_2}int_{T_1}(\eta) = 0$. Hence λ and η are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). But $1-\lambda = \alpha\wedge\delta$ is a fuzzy set with $int_{T_1}int_{T_2}(1-\lambda) = int_{T_1}int_{T_2}(\alpha\wedge\delta) = int_{T_1}(\alpha\wedge\delta) = \alpha\wedge\delta \neq 0$. $int_{T_2}int_{T_1}(1-\lambda) \neq 0$. Hence $1-\lambda$ is not a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2).

Theorem 3.6 ([16])**.** *If λ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2) such that $\mu \leq 1-\lambda$, where μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2), then μ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2).*

Theorem 3.7 ([13])**.** *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1-\lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proposition 3.8. *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2) such that $\mu \leq \lambda$, where μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2), then μ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2) such that $\mu \leq \lambda$, where μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Since λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2), by theorem 3.7, $1-\lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Now $\mu \leq \lambda$, implies that $\mu \leq 1-(1-\lambda)$, where μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2) and $1-\lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). By theorem 3.6, μ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). □

Definition 3.9. Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy σ-first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Any other fuzzy set in (X, T_1, T_2) is said to be a pairwise fuzzy σ-second category set in (X, T_1, T_2).

Definition 3.10. If λ is a pairwise fuzzy σ-first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a pairwise fuzzy σ-residual set in (X, T_1, T_2).

Definition 3.11. A fuzzy bitopological space (X, T_1, T_2) is called pairwise fuzzy σ-first category space if the fuzzy set 1_X is a pairwise fuzzy σ-first category set in (X, T_1, T_2). That is., $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Otherwise, (X, T_1, T_2) will be called a pairwise fuzzy σ-second category space.

4. Pairwise fuzzy σ-Baire spaces

Definition 4.1 ([16])**.** A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-Baire space if $int_{T_i}\Big(\vee_{k=1}^{\infty}(\lambda_k)\Big) = 0,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2).

Example 4.2. Let $X = \{a, b, c\}$. The fuzzy sets α, β, δ and μ are defined on X as follows :

$$\alpha : X \to [0,1] \text{ is defined as } \alpha(a) = 0.2;\ \ \alpha(b) = 0.4;\ \ \alpha(c) = 0.7.$$
$$\beta : X \to [0,1] \text{ is defined as } \beta(a) = 0.3;\ \ \beta(b) = 0.2;\ \ \beta(c) = 0.6.$$
$$\delta : X \to [0,1] \text{ is defined as } \delta(a) = 0.1;\ \ \delta(b) = 0.3;\ \ \delta(c) = 0.5.$$
$$\mu : X \to [0,1] \text{ is defined as } \mu(a) = 0.4;\ \ \mu(b) = 0.3;\ \ \mu(c) = 0.5.$$

Clearly $T_1 = \{0, \alpha, \beta, \delta, \alpha \vee \beta, \beta \vee \delta, \alpha \wedge \beta, \alpha \wedge \delta, \beta \wedge \delta, \alpha \wedge [\beta \vee \delta], 1\}$ and $T_2 = \{0, \alpha, \beta, \mu, \alpha\vee\beta, \alpha\vee\mu, \beta\vee\mu, \alpha\wedge\beta, \alpha\wedge\mu, \beta\wedge\mu, \beta\vee[\alpha\wedge\mu], \alpha\wedge[\beta\vee\mu], \mu\wedge[\alpha\vee\beta], \alpha\wedge\beta\wedge\mu, 1\}$ are fuzzy topologies on X. The fuzzy sets $\alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, \beta \vee \delta, \alpha \wedge [\beta \vee \mu], 1$ are pairwise fuzzy open sets in (X, T_1, T_2). Now the fuzzy sets $1 - \beta$ and $1 - (\alpha \wedge \beta)$ are pairwise fuzzy F_σ-sets in (X, T_1, T_2). Also $int_{T_1} int_{T_2}(1-\beta) = int_{T_2} int_{T_1}(1-\beta) = 0$ and $int_{T_1} int_{T_2}(1-(\alpha\wedge\beta)) = int_{T_2} int_{T_1}(1-(\alpha\wedge\beta)) = 0$. Hence $1-\beta$ and $1-(\alpha\wedge\beta)$ are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2) and $int_{T_i}[1-\beta]\vee[1-(\alpha\wedge\beta)] = 0,\ (i = 1, 2)$ implies that (X, T_1, T_2) is a pairwise fuzzy σ-Baire space.

Proposition 4.3. *Let (X, T_1, T_2) be a fuzzy bitopological space. The following are equivalent for a fuzzy bitopological space (X, T_1, T_2) :*

(1.) *(X, T_1, T_2) is a pairwise fuzzy σ-Baire space.*
(2.) *$Int_{T_i}(\lambda) = 0,\ (i = 1, 2)$ for every pairwise fuzzy σ-first category set λ in (X, T_1, T_2).*
(3.) *$Cl_{T_i}(\mu) = 1,\ (i = 1, 2)$ for every pairwise fuzzy σ-residual set μ in (X, T_1, T_2).*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$: Let λ be a pairwise fuzzy σ-first category set in a fuzzy bitopological space (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now $int_{T_i}(\lambda) = int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big),\ (i = 1, 2)$. Since (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Hence $int_{T_i}(\lambda) = 0,\ (i = 1, 2)$ where λ is a pairwise fuzzy σ-first category set in (X, T_1, T_2).

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Let μ be a pairwise fuzzy σ-residual set in (X, T_1, T_2). Then $1 - \mu$ is a σ-first category set in (X, T_1, T_2). By hypothesis, $int_{T_i}(1 - \mu) = 0,\ (i = 1, 2)$. Then $1 - cl_{T_i}(\mu) = 0$. Hence $cl_{T_i}(\mu) = 1,\ (i = 1, 2)$ for a pairwise fuzzy σ-residual set μ in (X, T_1, T_2).

(3) $\Rightarrow$ **(1)** : Let λ be a pairwise fuzzy σ-first category set in a fuzzy bitopological space (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now λ is a pairwise fuzzy σ-first category set in (X, T_1, T_2) implies that $1 - \lambda$ is a pairwise fuzzy σ-residual set in (X, T_1, T_2). By hypothesis, $cl_{T_i}(1 - \lambda) = 1$, $(i = 1, 2)$. Then $1 - int_{T_i}(\lambda) = 1$. Hence we have $int_{T_i}(\lambda) = 0$, $(i = 1, 2)$. That is., $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$, $(i = 1, 2)$ where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy σ-Baire space. □

Proposition 4.4. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, then $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).*

Proof. Let (λ_k)'s be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). By proposition 3.3, $(1-\lambda_k)$'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Then the fuzzy sets $\lambda = \vee_{k=1}^{\infty}(1 - \lambda_k)$ is a pairwise fuzzy σ-first category set in (X, T_1, T_2). Now $int_{T_i}(\lambda) = int_{T_i}\big(\vee_{k=1}^{\infty}(1 - \lambda_k)\big) = int_{T_i}\big(1 - \wedge_{k=1}^{\infty}(\lambda_k)\big) = 1 - cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big)$. Since (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, by proposition 4.3, we have $int_{T_i}(\lambda) = 0$. Then $1 - cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 0$. This implies that $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). □

Proposition 4.5. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, then (X, T_1, T_2) is a pairwise fuzzy σ-second category space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy σ-Baire bitopological space. Then we have $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$, $(i = 1, 2)$ where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now we claim that $\vee_{k=1}^{\infty}(\lambda_k) \neq 1_X$. Suppose that $\vee_{k=1}^{\infty}(\lambda_k) = 1_X$. Then $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = int_{T_i}(1_X) = 1_X$, $(i = 1, 2)$ which implies that 0 = 1, a contradiction. Hence we must have $\vee_{k=1}^{\infty}(\lambda_k) \neq 1_X$. Therefore the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-second category space. □

Definition 4.6 ([14])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy strongly irresolvable space* if $cl_{T_1}int_{T_2}(\lambda) = 1 = cl_{T_2}int_{T_1}(\lambda)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).

Proposition 4.7. *If λ is a pairwise fuzzy σ-nowhere dense set in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then λ is a pairwise fuzzy nowhere dense set and pairwise fuzzy F_σ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). Then λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Since $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$. Then $1 - int_{T_1}int_{T_2}(\lambda) = 1$ and $1 - int_{T_2}int_{T_1}(\lambda) = 1$ implies that $cl_{T_1}cl_{T_2}(1-\lambda) = 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) = 1$. Hence $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set $1 - \lambda$ in (X, T_1, T_2), we have $cl_{T_1}int_{T_2}(1 - \lambda) = 1$ and $cl_{T_2}int_{T_1}(1 - \lambda) = 1$. Then $1 - int_{T_1}cl_{T_2}(\lambda) = 1$ and $1 - int_{T_2}cl_{T_1}(\lambda) = 1$ implies that $int_{T_1}cl_{T_2}(\lambda) = 0$ and $int_{T_2}cl_{T_1}(\lambda) = 0$. Therefore λ is a pairwise fuzzy nowhere dense set and pairwise fuzzy F_σ-set in (X, T_1, T_2). □

Proposition 4.8. *If the fuzzy bitopological space* (X, T_1, T_2) *is a pairwise fuzzy* σ*-first category space, then* (X, T_1, T_2) *is not a pairwise fuzzy* σ*-Baire space.*

Proof. Let the fuzzy bitopological space (X, T_1, T_2) be a pairwise fuzzy σ-first category space. Then $\vee_{k=1}^{\infty}(\lambda_k) = 1_X$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = int_{T_i}(1_X) = 1 \neq 0$. Hence we have $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) \neq 0$ where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is not a pairwise fuzzy σ-Baire space. □

5. Conclusions

In this paper, several characterizations of pairwise fuzzy σ-Baire bitopological spaces are studied. Inter-relations between pairwise fuzzy σ-Baire spaces, pairwise fuzzy Baire spaces, pairwise fuzzy σ-second category spaces are also investigated.

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