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Fixed point theorem for 2-fuzzy n-b metric spaces

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ABSTRACT. The object of this paper is to define 2-fuzzy n-b metric space and to establish the common fixed point theorem using two self-mappings satisfying a contractive condition in the 2-fuzzy n-b metric space.

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1. Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [5]. After that many authors have expansively developed the theory of fuzzy sets and applications George and Veeramani modified the concept of fuzzy metric space which introduced by Kramosil and Michalek [2]. R. M. Somasundaram and Thangaraj Beaula [4] has coined 2-fuzzy sets and developed 2-fuzzy 2-normed linear space. Especially, Kailash Namdeo, S. S Rajput and Rajesh Shrivastava [3] have introduced the concept of fixed point theorem for fuzzy 2- metric spaces in different ways. Recently, Zaheer K. Ansari, Rajesh Shrivastava, Gunjan Ansari and ArunGarg [1] have also studied the fixed point theorems in fuzzy 2-metric and fuzzy 3- metric spaces. In this paper we have defined the new concept of 2-fuzzy n-b-metric space. Convergent and Cauchy sequences are defined related to this space. Some of the fixed point theorems using altering function are proved for weakly compatible self mappings.

2. Preliminaries

Definition 2.1. An altering distance function (or) control function is a function $\psi : [0, \infty] \to [0, \infty]$ such that the following axioms hold:
i) ψ is monotonic increasing and continuous.
ii) $\psi(t)$=0 if and only if $t = 0$.

Definition 2.2. A function $\varphi : R \to R^+$ is said to satisfy the condition $*$ if the following axioms hold:
i) $\varphi(t) = 0$ if and only if $t = 0$.
ii) $\varphi(t)$ is increasing and $\varphi(t) \to \infty$ as $t \to \infty$.
iii) φ is left continuous in $(0, \infty)$.
iv) φ is continuous at 0.

Definition 2.3. The 3-tuple $(X, M, *)$ is called fuzzy metric space if X is an arbitrary set, M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:
i) $M(x, y, 0) = 0$.
ii) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$.
iii) $M(x, y, t) = M(y, x, t)$.
iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.
v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous $\forall x, y, z \in X$ and $t, s > 0$.
Then M is called a fuzzy metric on X and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 2.4. The 3-tuple $(X, M, *)$ is called fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t - norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$
i) $M(x, y, z, 0) = 0$.
ii) $M(x, y, z, t) = 1, t > 0$ and when atleast two of the three points are equal.
iii) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ (Symmetry about three variables).
iv) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$.
(This is corresponds to tetrahedron inequality in 2-metric space)
v) $M(x, y, z, \cdot) : [0, 1) \to (0, 1]$ is left continuous.

Definition 2.5. Let X be a set and let $s \geq 1$ be given real number. A function $d : X \times X \to R^+$ is said to be a b - metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:
i) $d(x, y) = 0$ if and only if $x = y$.
ii) $d(x, y) = d(y, x)$.
iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b- metric space with parameter s.

There exists more examples in the literature [1, 3, 5] showing that the class of b - metric spaces, since a b - metrics in effectively larger than that of metric spaces, since a b - metric is a metric when $s = 1$ in the above condition 3.

Example 2.6. Let $X = \{\ 0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 1$
$d(0, 1) = d(1, 2) = d(0, 1) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$
Then $d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Example 2.7. Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. It is obviously a b-metric on X but d is not a metric on X.

Example 2.8. Let $X = l_p(\mathbb{R})$ with $0 < p < 1$

where $l_p(\mathbb{R}) = \{\ x = \{\ x_n\}\ \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$.

Then $d(x,y) = (\sum\limits_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ is a b-metric on X with $s = 2^{\frac{1}{p}}$ as by elemenary calculation we obtain that $d(x,y) \leq 2^{\frac{1}{p}}[d(x,y) + d(y,z)]$.

Definition 2.9 ([6]). Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a φ - weak contraction if $d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y))$ for all $x,y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and non decreasing function with $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.10. A 2- fuzzy set on X is a fuzzy set on $F(X)$.

3. 2-Fuzzy n-b metric space

Definition 3.1. Let X be an arbitrary set, $*$ be the continuous t-norm and $F(X)$ be the set of all fuzzy sets on X. Let s be a real number, a fuzzy set M on $[F(X)]^{n+1} \times [0,\infty)$ is said to be a 2-fuzzy n - b metric if and only if for all $f', f'', f_1, f_2, ..., f_{n-1} \in F(X)$ the following conditions are satisfied.

i) $(2FM^{nb} - 1)M(f', f'', f_1, ..., f_{n-1}, 0) = 0$.

ii) $(2FM^{nb} - 2)M(f', f'', f_1, f_2, ..., f_{n-1}, t) = 1$ for all $t > 0$ if and only if atleast $'n'$ elements of $\{\ f', f'', f_1, ..., f_{n-1}\}$ are linearly dependent.

iii) $(2FM^{nb} - 3)M(f', f'', f_1, ..., f_{n-1}, t) = M(f'', f', f_1, ..., f_{n-1}, t)$.
$= M(f', f_1, f''..., f_{n-1}, t) = ...$ (Symmetry about 'n' variables)

iv) $(2FM^{nb} - 4)M(f', f'', f_1, ..., f_{n-1}, t_1 + t_2 +, ... + t_{n+1})$
$\geq s[M(f', f'', ..., f_{n-2}, g, t_1) * M(f', f'', ..., g, f_{n-1}, t_2) * ....$
$* M(g, f'', f_1, ..., f_{n-1}, t_{n+1})]$.

The pair $(F(X), M)$ is called a 2- fuzzy n - b metric space with parameter s. A 2-fuzzy n - b metric is a n - metric whenever $s = 1$.

Example 3.2. Let X be a non-empty set, define a metric $D : X^2 \to [0,\infty)$ as $D(x,y) = (x+y)^2$

For $s = 2$,

$$\begin{aligned} D(x,y) &= (x+y)^2 \\ &\leq (x+z+z+y)^2 \\ &= (x+z)^2 + (z+y)^2 + 2(x+z)(z+y) \\ &\leq 2[(x+z)^2 + (z+y)^2] \\ &= 2[D(x,z) + D(z,y)] \end{aligned}$$

Then (X,d) is a b-metric space. Define $M(x,y,t) = \frac{t}{t+D(x,y)}$ is a fuzzy b-metric space.

Example 3.3. Let $F(X) = \{\ f | f : X \to [0,1]\}$

Define $D : [F(X)]^{n+1} \to \mathbb{R}^+$ as

$D(f', f'', f_1, ..., f_{n-1}) = \sqrt[3]{\sup_{x\in X}[|f'(x)| + |f''(x)| + |f_1(x)| + ... + |f_{n-1}|]^3}$

is a n-b metric with constant $s = \sqrt[3]{4}$.

For this note, if $a_1, a_2, ..., a_{n+1}$ are non-negative real numbers.
then $(a_1 + a_2 + ... + a_{n+1})^3 \leq 4(a_1^3 + a_2^3 + ... + a_{n+1}^3)$ and
$\sqrt[3]{(a_1 + a_2 + ... + a_{n-1})} \leq \sqrt[3]{a_1} + \sqrt[3]{a_2} + ... + \sqrt[3]{a_n}$
Now let us define a fuzzy set
$M : [F(X)]^{n+1} \times (0, \infty) \to [0, 1]$ as
$M(f', f'', f_1, ..., f_{n-1}, t) = \frac{t}{t+D(f', f'', f_1, ..., f_{n-1}, t)}$ which is a fuzzy n-b-metric.

Definition 3.4. Let $(F(X), M, *)$ be 2- fuzzy n - b metric space. A mapping $T : F(X) \to F(X)$ is said to be a φ - fuzzy weak contraction if

$$M(Tf, Th, g_1, ..., g_{n-1}, \varphi(t)) \geq M(f, h, g_1, ..., g_{n-1}, t) - \psi(M(f, h, g_1, ..., g_{n-1}, t))$$

For all $f, h, g_1, ..., g_{n-1} \in F(X)$ where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and non decreasing function with $\varphi(t) = 0$ if and only if $t = 0$.

Definition 3.5. Let $(F(X), M, *)$ be 2- fuzzy n - b metric space. Then a sequence $\{ f_n \}_{n \in N}$ is called

i) 2-fuzzy n - b convergent if there exists $f \in F(X)$ such that

$$M(f_n, f, g_1, ..., g_{n-1}, t) \to 1 \text{ as } n \to \infty.$$

In this case we write $lim_{n\to\infty} f_n = f$.

ii) 2-fuzzy n - b Cauchy if $M(f_n, f_m, g_1, ..., g_{n-1}, t) \to 1$ as $n, m \to \infty$.

Proposition 3.6. *In a 2- fuzzy n - b metric space $(F(X), M, *)$ the following assertions hold: A n - b convergent sequence has a unique limit.*

Proof. Let $\{ f_n \}$ converges to f_1 and f_2 in $F(X)$.

Then $M(f_n - f_1, g_1, ..., g_n, t) > 1 - r$ and $M(f_n - f_2, g_1, ...g_n, t) > 1 - r$ for all $t > 0$ and choose r such that $0 < r < 1$, where $(1 - r) * (1 - r) > 1 - \varepsilon$

Now,

$$\begin{aligned}
M(f_1 - f_2, g_1, ..., g_n, t) &= M(f_1 - f_n + f_n - f_2, g_1, ..., g_n, \frac{t}{2} + \frac{t}{2}) \\
&= M(f - f_n) + (f_n - f_2, g_1, ..., g_n, \frac{t}{2} + \frac{t}{2}) \\
&\geq M(f_1 - f_n, g_1, ..., g_n, \frac{t}{2}) * M(f_n - f_2, g_1, ..., g_n, \frac{t}{2}) \\
&\geq (1 - r) * (1 - r) = \varepsilon.
\end{aligned}$$

Therefore $f_1 = f_2$, so the limits are equal. □

Main Result

Theorem 3.7. *Let $(F(X), M, *)$ be a complete 2- fuzzy n - b metric space with parameter s and, $T : F(X) \to F(X)$ be a fuzzy continuous mapping such that*

$$M(T(f), T(h), g_1, ..., g_{n-1}, qt)$$
$$\geq \frac{\alpha M(h, T(h), g_1, ..., g_{n-1}, t), M(f, T(f), g_1, ...g_{n-1}, qt)}{M(f, h, g_1, ...g_{n-1}, t)} +$$
$$\beta M(f, h, g_1, ...g_(n-1), t) \qquad \cdots\cdots (1)$$

for all $f, h, g_1, ..., g_{n-1} \in F(X), f \neq h$ where α, β are positive real constants such that $s\beta + \alpha < 1$, then T has a unique fixed point.

Proof. For an arbitrary $f_0 \in F(X)$ construct the sequence $(f_n)_{n\in N}$ such that $f_{n+1} = T(f_n)$.

$$M(f_1, f_2, g_1, ..., g_{n-1}, qt) = M(Tf_0, Tf_1, g_1, ..., g_{n-1}, qt)$$
$$\geq \tfrac{\alpha M(f_0,Tf_0,g_1,...,g_{n-1},t)M(f_1,T(f_1),g_1,...,g_{n-1},qt)}{M(f_0,f_1,g_1,...,g_{n-1},t)} + \beta M(f_0, f_1, g_1, ..., g_{n-1}, t)$$
$$= \tfrac{\alpha M(f_0,f_1,g_1,...,g_{n-1},t)M(f_1,f_2,g_1,...,g_{n-1},qt)}{M(f_0,f_1,g_1,...,g_{n-1},t)} + \beta M(f_0, f_1, g_1, ..., g_{n-1}, t)$$
$$M(f_1, f_2, g_1, ..., g_{n-1}, qt) \geq \alpha M(f_1, f_2, g_1, ..., g_{n-1}, qt) + \beta M(f_0, f_1, g_1, ..., g_{n-1}, t)$$
$$M(f_1, f_2, g_1, ..., g_{n-1}, qt)(1-\alpha) \geq \beta M(f_0, f_1, g_1, ..., g_{n-1}, t)$$
$$M(f_1, f_2, g_1, ..., g_{n-1}, t) \geq \tfrac{\beta}{1-\alpha} M(f_0, f_1, g_1, ..., g_{n-1}, \tfrac{t}{|q|}), q > 0$$

where $\frac{\beta}{1-\alpha} = k < 1$. Similarly,

$$M(f_2, f_3, g_1, ..., g_{n-1}, qt) = M(Tf_1, Tf_2, g_1, ..., g_{n-1}, qt)$$
$$\geq \tfrac{\alpha M(f_1,Tf_1,g_1,...,g_{n-1},t)M(f_2,T(f_2),g_1,...,g_{n-1},qt)}{M(f_1,f_2,g_1,...,g_{n-1},t)} + \beta M(f_1, f_2, g_1, ..., g_{n-1}, t)$$
$$= \tfrac{\alpha M(f_1,f_2,g_1,...,g_{n-1},t)M(f_2,f_3,g_1,..,g_{n-1},qt)}{M(f_1,f_2,g_1,...,g_{n-1},t)} + \beta M(f_1, f_2, g_1, ..., g_{n-1}, t)$$
$$M(f_2, f_3, g_1, ..., g_{n-1}, qt) \geq \alpha M(f_2, f_3, g_1, ..., g_{n-1}, qt) + \beta M(f_1, f_2, g_1, ..., g_{n-1}, t)$$
$$M(f_2, f_3, g_1, ..., g_{n-1}, qt)(1-\alpha) \geq \beta M(f_1, f_2, g_1, ..., g_{n-1}, t)$$
$$M(f_2, f_3, g_1, ...g_(n-1), t) \geq \tfrac{\beta}{1-\alpha} M(f_1, f_2, g_1, ..., g_{n-1}, \tfrac{t}{q}), q > 0$$
$$= \left(\tfrac{\beta}{1-\alpha}\right)^2 M(f_0, f_1, g_1, ..., g_{n-1}, \tfrac{t}{q^2}), q > 0t)(1-\alpha)$$
$$\geq \beta M(f_1, f_2, g_1, ..., g_{n-1}, t)$$

where $\frac{\beta}{1-\alpha} = k < 1$. Inductively,

$$M(f_n, f_{n+1}, g_1, ..., g_{n-1}, qt) = M(Tf_{n-1}, Tf_n, g_1, ..., g_{n-1}, t)$$
$$\geq \tfrac{\alpha M(f_{n-1},Tf_{n-1},g_1,...,g_{n-1},t)M(f_n,Tf_n,g_1,...,g_{n-1},t)}{M(f_{n-1},f_n,g_1,...,g_{n-1},t)} + \beta M(f_{n-1}, f_n, g_1, ..., g_{n-1}, t)$$
$$\geq \tfrac{\beta}{1-\alpha} M(f_{n-1}, f_n, g_1, ..., g_{n-1}, qt)$$

Inductively,

$$M(f_n, f_{n+1}, g_1, ..., g_{n-1}, qt) \geq kM(f_0, f_1, g_1, ..., g_{n-1}, \frac{t}{q^n})$$

For every positive integer p and k in N we have

$$M(f_k, f_{K+1}, g_1, ..., g_{n-1}, qt) \to 1$$

The above sequence is Cauchy in complete 2- fuzzy n - b metric space $(F(X), M, *)$ so there exists a $f \in F(X)$ such that $\lim_{n\to\infty} f_n = f$

By the continuity of T and M
We have

$$\begin{aligned} Tf &= T(\lim_{n\to\infty} f_n) \\ &= \lim_{n\to\infty} Tf_n \\ &= \lim_{n\to\infty} f_(n+1) \\ &= \lim_{n\to\infty} f_n \\ &= f \\ \text{Therefore } Tf &= f \end{aligned}$$

And this proves that f is a fixed point.
If there exists a another point $g \neq f$ in $F(X)$ such that $Tg = g$ then

$$\begin{aligned} M(g, f, g_1, ..., g_{n-1}, qt) &= M(Tg, Tf, g_1, ..., g_{n-1}, t) \\ &\geq \frac{\alpha M(g, Tg, g_1, ..., g_{n-1}, t)M(f, Tf, g_1, ..., g_{n-1}, qt)}{M(g, f, g_1, ..., g_{n-1}, t)} \\ &\quad +\beta M(g, f, g_1, ..., g_{n-1}, t) \\ &= \beta M(g, f, g_1, ..., g_{n-1}, qt) \\ &\geq M(g, f, g_1, ..., g_{n-1}, qt) \end{aligned}$$

which implies
$M(g, f_{n+1}, g_1, ..., g_{n-1}, qt) \geq M(g, f_{n+1}, g_1, ..., g_{n-1}, qt)$
and hence $f = g$
Hence the fixed point is unique. □

Theorem 3.8. *Let $(F(X), M, *)$ be a complete 2-fuzzy n - b metric space and $S, T : F(X) \to F(X)$ be two self-mappings satisfying*
i) $TF(X) \subseteq SF(X)$
ii) The functions $\psi, \alpha : [0, 1] \to [0, 1]$ are continuous, monotonically increasing with $\psi(0) = 0 = \alpha(0)$ and $t - \frac{1}{s}(\alpha(t) - \psi(t)) < 0$ also $\frac{1}{s^n}(\alpha - \psi)^n(a_n) \to 1$ when $a_n \to 1$ as $n \to \infty$
iii) $\frac{1}{s}M(Sf, Sh, g_1, ..., g_{n-1}, \varphi(t)) > 0$ for all $t > 0$ where the function φ satisfies the definition
iv) $sM(Tf, Th, g_1, ..., g_{n-1}, \varphi(ct)) \geq M(f, h, g_1, ..., g_{n-1}, t)$
$-\psi(M(f, h, g_1, ..., g_{n-1}, t))$
Also the contraction with above conditions

$$\frac{1}{M(Tf,Th,g_1,..,g_{n-1},\varphi(ct))} \geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Sf,Sh,g_1,...,g_{n-1},\psi(t))}\right) - \psi\left(\frac{1}{M(Sf,Sh,g_1,..,g_{n-1},\varphi(t))}\right)\right] \quad (1)$$

holds for all $f, h \in F(X), t > 0, 0 < c < 1$.
If $S(F(X))$ is a complete subspace of $F(X)$ and the mappings (S, T) are weakly compatible, then S and T have a unique common fixed point.

Proof. Let f_0 be an element in $F(X)$. Define two sequences (h_n) and (f_n) such that $h_n = Tf_n = Sf_{n+1}$, we claim that $\{ h_n\}$ is a Cauchy sequence.

For some n, assume that

$$\frac{1}{M(Tf_{n-1},Tf_n,g_1,...,g_{n-1},\varphi(ct))} \geq \frac{1}{M(Tf_n,Tf_{n+1},g_1,...,g_{n-1},\varphi(ct))}$$

is true. Then from condition (1)

$$\frac{s}{M(Tf_n,Tf_{n+1},g_1,...,g_{n-1},\varphi(ct))} \geq \alpha\left(\frac{1}{M(Sf_n,Sf_{n+1},g_1,...,g_{n-1},\varphi(t))}\right) - \psi\left(\frac{1}{M(Sf_n,Sf_{n+1},g_1,...,g_{n-1},\varphi(t))}\right).$$

Then using the above assumption contraction becomes,

$$\frac{1}{M(Tf_{n-1},Tf_n,g_1,...,g_{n-1},\varphi(ct))} \geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Tf_{n-1},Tf_n,g_1,...,g_{n-1},\varphi(t))}\right) - \psi\left(\frac{1}{M(Tf_{n-1},Tf_n,g_1,...,g_{n-1},\varphi(t))}\right)\right] \quad (2).$$

Given $t - \frac{1}{s}(\alpha(t) - \psi(t)) < 0$ is a contrary to our assumption, because above inequality (1) yields $st - \alpha(t) + \psi(t) \geq 0$

$$\frac{1}{M(Tf_n,Tf_{n+1},g_1,...,g_{n-1},\varphi(ct))} \geq \frac{1}{M(Tf_{n-1},Tf_n,g_1,...,g_{n-1},\varphi(ct))} \quad (3)$$

Again assume $\{\ h_n\} \ \neq \{\ h_{n+1}\}$ for every n.

By virtue of the properties of φ, we can find a $t > 0$ such that $sM(Sf_1, Sf_2, g_1, .., g_{n-1}, \varphi(t)) > 0$ Therefore using condition (1) we get

$$\frac{1}{M(h_0,h_1,g_1,...,g_{n-1},\varphi(ct))} = \frac{1}{M(Tf_0,Tf_1,g_1,...,g_{n-1},\varphi(ct))} \text{ (since } h_n = Tf_n)$$
$$\geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(t))}\right) - \psi\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(t))}\right)\right]$$

On using (3) we get

$$\frac{1}{M(Tf_1,Tf_2,g_1,...g_{n-1},\varphi(ct))} \geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(t))}\right) - \psi\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(t))}\right)\right]$$

Since $\frac{1}{s}M(Sf_1, Sf_2, g_1, ...g_{n-1}, \varphi(t)) > 0$ implies

$\frac{1}{s}M(Sf_1, Sf_2, g_1, ...g_{n-1}, \varphi(\frac{t}{c})) > 0$

By applying in (1) we get

$$\frac{1}{M(h_0,h_1,g_1,...g_{n-1},\varphi(t))} = \frac{1}{M(Tf_0,Tf_1,g_1,...g_{n-1},\varphi(t))}$$
$$\geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(\frac{t}{c}))}\right) - \psi(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\psi(\frac{t}{c}))}\right)\right]$$

Again by using (3),

$$\frac{1}{M(Tf_1,Tf_2,g_1,...g_{n-1},\varphi(t))} \geq \frac{1}{s}\left[\alpha\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(\frac{t}{c}))}\right) - \psi\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\psi(\frac{t}{c}))}\right)\right]$$

Repeating the process n times, we obtain

$$\frac{1}{M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(t))} \geq \frac{1}{s^n}(\alpha-\psi)^n\left(\frac{1}{M(Sf_1,Sf_2,g_1,...g_{n-1},\varphi(\frac{t}{c^n}))}\right)$$

Since $\frac{1}{s}M(Sf_2,Sh_3,g_1,...g_{n-1},\varphi(ct)) > 0$ (by condition (iii))

Then following the above process we get,

$$\frac{1}{M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(t))} \geq \frac{1}{s^n}(\alpha-\psi)^n\left(\frac{1}{M(Sf_2,Sf_3,g_1,...g_{n-1},\varphi(\frac{ct}{c^n}))}\right)$$

Continuing this process r times, We get,

$$\frac{1}{M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(c^rt))} \geq \frac{1}{s^{n-r+1}(\alpha-\psi)^{n-r+1}}\left(\frac{1}{M(Sf_{r+1},Sf_{n+2},g_1,...g_{n-1},\varphi((\frac{c^rt}{c^{n-r+1}}))}\right)$$

Take $h_n = Sf_{n+1}$ then

$$\frac{1}{M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(c^rt))} \geq \frac{1}{s^{n-r+1}}(\alpha-\psi)^{n-r+1}\left(\frac{1}{M(h_r,h_{r+1},g_1,...g_{n-1},\varphi(\frac{c^rt}{c^{n-r+1})}}\right)$$

Since
$\frac{1}{s^n}(\alpha-\psi)^n(a_n) \to 1$ when $a_n \to 1$ as $n \to \infty$,

hence for all $r > 0$.

$$\frac{1}{M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(c^rt))} \geq 1$$

Therefore as $n \to \infty$

$$M(h_{n-1},h_n,g_1,...g_{n-1},\varphi(c^rt)) \to 1 as n \to \infty$$

Choose $\varphi(c^rt) < \varepsilon$ then it follows that $M(h_{n-1},h_n,g_1,...g_{n-1},\varepsilon) \to 1$ as $n \to \infty$

By triangle inequality,

$$M(h_n,h_{n+p},g_1,...g_{n-1},\varepsilon) \geq M\left(h_n,h_{n+1},g_1,...g_{n-1},\frac{\varepsilon}{p}\right) * ... * M\left(h_{n+p},h_{n+p+1},g_1,...g_{n-1},\frac{\varepsilon}{p}\right)$$

And so
$M(h_n,h_{n+p},g_1,...g_{n-1},\varepsilon) \to 1$ as $n \to \infty$
which implies $\{h_n\}$ is a Cauchy sequence and it converges to $h \in F(X)$ such that $h_n \to h$ as $n \to \infty$

Let $h_n = Tf_n = Sf_{n+1} \to h$

Our aim is to show that $Th = h$

Since

$M(Th, h, g_1, ...g_{n-1}, \varepsilon) \geq M(Th, h_n, g_1, ...g_{n-1}, \frac{\varepsilon}{2}) * M(h_n, h, g_1, ...g_{n-1}, \frac{\varepsilon}{2})$

By the property of φ, there exists a $t_1 > 0$ such that $\varphi(t_1) < \frac{\varepsilon}{2}$ as $h_n \to h$ as $n \to \infty$, there exists $m \in N$ such that for all $n > m$,

$M(h_n, h, g_1, ...g_{n-1}, \varphi(t_1)) > 0$, then for $n > m$

$$\begin{aligned}\frac{1}{M(Th, h_n, g_1, ...g_{n-1}, \frac{\varepsilon}{2})} &= \frac{1}{M(Th, Tf_n, g_1, ...g_{n-1}, \varphi(t_1))}\\ &\geq \frac{1}{s}\Big[\alpha\Big(\frac{1}{M(Sh, Sh_{n+1}, g_1, ...g_{n-1}, \varphi(\frac{t_1}{c}))}\Big)\\ &\qquad - \psi\Big(\frac{1}{M(Sh, Sf_{n+1}, g_1, ...g_{n-1}, \varphi(\frac{t_1}{c}))}\Big)\Big]\end{aligned}$$

Again on applying (1) we get

$$\begin{aligned}\Big(\frac{1}{M(Th, Tf_{n+1}, g_1, ...g_{n-1}, \varphi(t_1))}\Big) &\geq \frac{1}{s}\Big[\alpha\Big(\frac{1}{M(Sh, Sf_{n+1}, g_1, ...g_{n-1}, \varphi(\frac{t_1}{c}))}\Big)\\ &\qquad -\psi\Big(\frac{1}{M(Sh, Sf_{n+1}, g_1, ...g_{n-1}, \varphi(\frac{t_1}{c}))}\Big)\Big]\end{aligned}$$

Proceeding the limit as $n \to \infty$ we obtain

$$M(Th, h_n, g_1, ...g_{n-1}, \frac{t_1}{2}) \to 1 \text{ as } n \to \infty$$

As $n \to \infty, h_n \to h$ and $M(Th, h, g_1, ...g_{n-1}, \varepsilon) = 1$, for every $\varepsilon > 0$ gives $Th = h$. Thus $Sh = Th = h$ which implies that h is a common fixed point of S and T.

Finally let us prove the uniqueness of h.

Let h, h' be two fixed points of S and T. by the properties of φ there exists $k > 0$ such that $M(h, h', g_1, ...g_{n-1}, \varphi(k)) > 0$ then again by applying (1) we obtain the following equation

$$\begin{aligned}\frac{1}{M(h, h', g_1, ...g_{n-1}, \varphi(k))} &= \frac{1}{((Th, Th', g_1, ...g_{n-1}, \varphi(ck)))}\\ &\geq \frac{1}{s}\Big[\alpha\Big(\frac{1}{M(Sh, Sh', g_1, ...g_{n-1}, \varphi(k))}\Big)\\ &\qquad - \psi\Big(\frac{1}{M(Sh, Sh', g_1, ...g_{n-1}, \varphi(k))}\Big)\Big]\end{aligned}$$

On replacing k by $\frac{k}{c}$ we get,

$$\begin{aligned}\frac{1}{M(h, h', g_1, ...g_{n-1}, \varphi(k))} &\geq \frac{1}{s}\Big[\alpha\big(\frac{1}{M(Sh, Sh', g_1, ...g_{n-1}, \varphi(\frac{S}{c}))}\Big)\\ &\qquad - \psi\Big(\frac{1}{M(Sh, Sh', g_1, ...g_{n-1}, \varphi(\frac{S}{c}))}\Big)\Big]\end{aligned}$$

Repeating the procedure $'n'$ times

$$\frac{1}{M(h,h',g_1,...g_{n-1},\varphi(S))} \geq \frac{1}{s^n}(\alpha-\psi)^n\left(\frac{1}{M(Sh,Sh',g_1,...g_{n-1},\varphi(\frac{S}{c}))}\right)$$

and so $M(h,h',g_1,...g_{n-1},\varphi(S)) \to 1$ as $n \to \infty$ since $\frac{1}{s^n}(\alpha-\psi)^n(a_n) \to 1$ when $a_n \to 1$ as $n \to \infty$

which establishes the uniqueness of fixed point. □

Theorem 3.9. *Let $(F(X), M, *)$ be a complete 2-fuzzy n-metric space and let S and T be continuous mappings of $F(X)$ in $F(X)$ then S and T have common fixed point in F(X) if there exists continuous mapping A of $F(X)$ into $S(F(X))\cap T(F(X))$ which commute weakly with S and T and*

$$\begin{aligned} &M(Af',Af'',g_1,...g_{n-1},qt)\\ &\quad\geq smin\{\ M(Tf'',Af'',g_1,...g_{n-1},t), M(Sf',Af',g_1,...g_{n-1},t),\\ &\qquad M(Sf',Tf'',g_1,...g_{n-1},t), \frac{M(Sf',Tf'',g_1,...g_{n-1},t))}{M(Af',Tf'',g_1,...g_{n-1},t)}\} \end{aligned} \tag{1}$$

For all $f',f'',...f_{n+1},g_1,....,g_{n-1} \in F(X), t>0$ and $0<q<1$ and

$\lim_{n\to\infty} M(f',f'',...f_{n+1},g_1,...g_{n-1},t) = 1$ for all $f',f'',...f_{n+1},g_1,....g_{n-1}$ in F(X).

Then S, T and A have a unique common fixed point.

Proof. We define a sequence $\{\ f'_n\}$ such that

$Af'_{2n} = Sf'_{2n-1}$ and $Af'_{2n-1} = Tf'_{2n}, n = 1,2,..$

We shall prove that $\{\ Af'_n\}$ is a Cauchy sequence. For this suppose $f' = f_{2n}$ and $f'' = f_{2n+1}$, we write

$$\begin{aligned} &M(Af'_{2n},Af'_{2n+1},g_1,...,g_{n-1},qt)\\ &\quad\geq s\ min\{\ M(Tf'_{2n+1},Af'_{2n+1},g_1,...,g_{n-1},t), M(S'f_{2n},Af'_{2n},g_1,...,g_{n-1},t),\\ &\qquad M(Sf'_{2n},Tf'_{2n+1},g_1,...,g_{n-1},t), \frac{M(Sf'_{2n},Tf'_{2n+1},g_1,...,g_{n-1},t)}{M(Af'_{2n},Af'_{2n+1},g_1,...,g_{n-1},t)}\} \end{aligned}$$

$$\begin{aligned} &M(Af'_{2n},Af'_{2n+1},g_1,\dots g_{n-1},qt)\\ &\quad\geq s\ min\{\ M(Af'_{2n},Af'_{2n+1},\ g_1,\dots g_{n-1},t), M(Af'_{2n+1},Af'_{2n},\ g_1,\dots g_{n-1},t),\\ &\qquad MAf'_{2n+1},Af'_{2n},\ g_1,\dots g_{n-1},t), \frac{M(Af'_{2n+1},Af'_{2n},\ g_1,\dots g_{n-1},t}{M(Af'_{2n},Af_{2n},\ g_1,\dots g_{n-1},t)}\} \end{aligned}$$

$= s\ min\{\ M(Af'_{2n},Af'_{2n+1},\ g_1,\dots g_{n-1},t), M(Af'_{2n+1},Af'_{2n},\ g_1,\dots g_{n-1},t),$
$M(Af'_{2n+1},Af'_{2n},\ g_1,\dots g_{n-1},t),1\}$
$= s\ min\left\{M\left(Af'_{2n-1},Af'_{2n},\ g_1,\dots g_{n-1},\frac{t}{q}\right), M\left(Af'_{2n},Af'_{2n-1},\ g_1,\dots g_{n-1},\frac{t}{q}\right)\right\}$

Therefore, $M(Af'_{2n},Af'_{2n+1},g_1,\dots g_{n-1},qt) \geq sM\left(Af'_{2n-1},Af'_{2n},\ g_1,\dots g_{n-1},\frac{t}{q}\right)$

By induction

$$M(Af'_{2k},Af'_{2m+1},g_1,\dots g_{n-1},qt) \geq sM\left(Af'_{2m},Af'_{2k-1},\ g_1,\dots g_{n-1},\frac{t}{q}\right) \tag{2}$$

For every k and m in N,
Further if $2m+1 > 2k$ then,
$M\left(Af'_{2k}, Af'_{2m+1}, g_1, \dots g_{n-1}, qt\right) \geq sM\left(Af'_{2k-1}, Af'_{2m},\ g_1, \dots g_{n-1}, \frac{t}{q}\right)$
$\geq s^{2k} M\left(Af_0, Af'_{2m+1-2k},\ g_1, \dots g_{n-1}, \frac{t}{q^{2k}}\right)$ (3)
If 2k > 2m+1 then,

$$M\left(Af'_{2k}, Af'_{2m+1}, g_1, \dots g_{n-1}, qt\right) \geq sM\left(Af'_{2k-1}, Af'_{2m},\ g_1, \dots g_{n-1}, \frac{t}{q}\right)$$
$$\geq s^{2m+1} M\left(Af'_{2k-(2m+1)}, Af_0,\ g_1, \dots g_{n-1}, \frac{t}{q^{2m+1}}\right) \qquad (4)$$

By simple induction with (3) and (4) , we have
$M\left(Af_n, Af_{n+p}, g_1, \dots g_{n-1}, qt\right) \geq s^n M\left(Af_0, Af_p,\ g_1, \dots g_{n-1}, \frac{t}{q^n}\right)$
For n = 2k, p = 2m + 1 (or) n = 2k + 1, p = 2m + 1 and by (2FM^n – 3)
$M\left(Af_n, Af_{n+p}, g_1, \dots g_{n-1}, qt\right)$
$\geq s^n(s)\left[M\left(Af_0, Af_1,\ g_1, \dots g_{n-1}, \frac{t}{2q^n}\right) * M\left(Af_1, Af_p,\ g_1, \dots g_{n-1}, \frac{t}{q^n}\right)\right]$ (5)
where $0 \leq s \leq 1$
For every positive integer p and n in N we have
$M\left(Af_0, Af_p,\ g_1, \dots g_{n-1}, \frac{t}{q^n}\right) \to 1$ as $n \to \infty$
Thus $\{Af_n\}$ is a Cauchy sequence.

Since the space F(X) is complete there exists $f_{n+1} \in F(X)$ such that,
$\lim_{n\to\infty} Af_n = \lim_{n\to\infty} Sf_{2n-1} = \lim_{n\to\infty} Tf_{2n} = h$

It follows that $Ah = Sh = Th$ and therefore,

$$\begin{aligned} &M\left(Ah, AAh, g_1, \dots g_{n-1}, qt\right) \\ &\geq s\ min\{\ M\left(TAh, AAh, g_1, \dots g_{n-1}, t\right), M\left(Sh, Ah, g_1, \dots g_{n-1}, t\right), \\ &\qquad M\left(Sh, TAh, g_1, \dots g_{n-1}, t\right), \frac{M\left(Sh, TAh, g_1, \dots g_{n-1}, t\right)}{M\left(Ah, TAh, g_1, \dots g_{n-1}, t\right)}\} \end{aligned}$$

$$\begin{aligned} M\left(Ah, A^2h, g_1, \dots g_{n-1}, qt\right) &\geq s\ M\left(Sh, TAh, g_1, \dots g_{n-1}, t\right) \\ &\geq s\ M\left(Sh, ATh, g_1, \dots g_{n-1}, t\right) \\ &\geq sM\left(Ah, A^2h, g_1, \dots g_{n-1}, t\right) \\ &\geq s^n M\left(Ah, A^2h, g_1, \dots g_{n-1}, \frac{t}{q^n}\right) \end{aligned}$$

Since, $\lim_{n\to\infty} M\left(Ah, A^2h, g_1, \dots g_{n-1}, \frac{t}{q^n}\right) = 1$
$\Rightarrow Ah = A^2h$
Thus h is common fixed point of A, S and T.
For uniqueness let $k(k \neq h)$ be another common fixed point of S, T and A

By (1) we write,

$$M\left(Ah, Ak, g_1, \dots g_{n-1}, qt\right) \geq s\ min\{\ M\left(Tk, Ak, g_1, \dots g_{n-1}, t\right), \\ M\left(Sh, Ah, g_1, \dots g_{n-1}, t\right), M\left(Sh, Tk, g_1, \dots g_{n-1}, t\right), \\ \frac{M\left(Sh, Tk, g_1, \dots g_{n-1}, t\right)}{M\left(Ah, Tk, g_1, \dots g_{n-1}, t\right)}\}$$

$$M\left(Ah, Ak, g_1, \dots g_{n-1}, qt\right) \geq s\ min\left\{M\left(h, k, g_1, \dots g_{n-1}, t\right)\right\}$$

This implies that,
$M\left(h, k, g_1, \dots g_{n-1}, qt\right) \geq s\min\left\{M\left(h, k, g_1, \dots g_{n-1}, t\right)\right\}$ hence $h = k$ and this completes the proof. ☐

Theorem 3.10. *Let $(F(X),\ M_1,\ *)$ and $(F(Y),\ M_2,\ *)$ be two complete 2- fuzzy n - b metric spaces. Let A and B be mappings from F(X) to F(Y) and S and T be mappings from F(Y) to F(X) satisfying the following inequalities:*

$$M_1\left(f, f', g_1, \dots g_{n-1}, t\right) M_1\left(SAf, TBf', g_1, \dots g_{n-1}, t\right) \\ \geq s\ \{\ min\{M_1(f', f, g_1, \dots g_{n-1}, t), M_1(f', TBf', g_1, \dots g_{n-1}, t), \\ M_1(f', SAf', g_1, \dots g_{n-1}, t),\ M_1(f, SAf, g_1, \dots g_{n-1}, t), \\ M_1(f', TBf', g_1, \dots g_{n-1}, t),\ M_2(h, BTh, g_1, \dots g_{n-1}, t), \\ M_1(f, f', g_1, \dots g_{n-1}, t), M_1(TBf, TBf', g_1, \dots g_{n-1}, t)\}\ \cdots\cdots \quad (1)$$

$$M_2\left(h, h', g_1, \dots g_{n-1}, t\right), M_2\left(BSh, ATh', g_1, \dots g_{n-1}, qt\right) \\ \geq min\{M_2\left(h, h', g_1, \dots g_{n-1}, t\right), M_2\left(h', ATh', g_1, \dots g_{n-1}, t\right), \\ M_2\left(h', ATh', g_1, \dots g_{n-1}, t\right), M_2\left(h', BSh', g_1, \dots g_{n-1}, t\right), \\ M_2\left(h', BSh', g_1, \dots g_{n-1}, t\right),\ M_1\left(Sh', Th', g_1, \dots g_{n-1}, t\right), \\ M_2\left(h, h', g_1, \dots g_{n-1}, t\right),\ M_2\left(BSh, h, g_1, \dots g_{n-1}, t\right)\}\ \cdots\cdots \quad (2)$$

For all f and f' in F(X) and h and h' in F(Y) and $0 < q < 1$.
If one of the mapping A, B, S and T is continuous, then SA and TB have a common fixed point g in F(X) and BS and AT have a common fixed point k in F(Y).
Further, $Ag = Bg = k$ and $Sk = Tk = g$.

Proof. Let f be an arbitrary point in F(X). we define sequence $\{f_n\}$ in F(X) and $\{h_n\}$ in F(Y) such that $Af_{2n} = h_{2n+1}, Bf_{2n-1} = h_{2n}$, $Th_{2n} = f_{2n}$ and $Sh_{2n-1} = f_{2n-1}$for n = 1, 2,,. . .
Applying inequality (1) we have,

$$M_1\left(f_{2n}, f_{2n-1}, g_1, \dots g_{n-1}, t\right) M_1\left(f_{2n+1}, f_{2n}, g_1, \dots g_{n-1}, t\right) \\ \geq s\ minM_1\left(f_{2n-1}, f_{2n}, g_1, \dots g_{n-1}, t\right)\ M_1\left(f_{2n-1}, f_{2n}, g_1, \dots g_{n-1}, t\right) \\ M_1\left(f_{2n-1}, f_{2n}, g_1, \dots g_{n-1}, t\right) M_1\left(f_{2n}, f_{2n+1}, g_1, \dots g_{n-1}, t\right) \\ M_1\left(f_{2n-1}, f_{2n}, g_1, \dots g_{n-1}, t\right) M_2\left(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t\right) \\ M_1\left(f_{2n}, f_{2n-1}, g_1, \dots g_{n-1}, t\right) M_1\left(f_{2n}, f_{2n}, g_1, \dots g_{n-1}, t\right),$$

i.e. $M_1\left(f_{2n+1}, f_{2n}, g_1, \dots g_{n-1}, qt\right) \geq s\ min\{M_1\left(f_{2n}, f_{2n-1}, g_1, \dots g_{n-1}, t\right), M_2\left(h_{2n+1}, h_{2n}, g_1, \dots g_{n-1}, t\right), M_1\left(f_{2n+1}, f_{2n}, g_1, \dots g_{n-1}, t\right), 1\}$
which implies that,

$$M_1\left(f_{2n+1}, f_{2n}, g_1, \dots g_{n-1}, qt\right) \geq s\ \min\{M_1\left(f_{2n}, f_{2n-1}, g_1, \dots g_{n-1}, t\right), \\ M_2\left(h_{2n+1}, h_{2n}, g_1, \dots g_{n-1}, t\right)\}\ \cdots\cdots \quad (3)$$

Applying inequality (2), we have

$$M_2\left(h_{2n}, h_{2n-1}, g_1, \dots g_{n-1}, t\right) M_2\left(h_{2n+1}, h_{2n}, g_1, \dots g_{n-1}, qt\right)$$

$$= M_2\left(h_{2n}, h_{2n-1}, g_1, \ldots g_{n-1}, t\right) M_2\left(BSh_{2n}, ATh_{2n-1}, g_1, \ldots g_{n-1}, qt\right),$$
$$\geq s\ \min\{M_2\left(h_{2n}, h_{2n-1}, g_1, \ldots g_{n-1}, t\right), M_2\left(h_{2n-1}, h_{2n}, g_1, \ldots g_{n-1}, t\right),$$
$$M_2\left(h_{2n-1}, h_{2n}, g_1, \ldots g_{n-1}, t\right), M_2\left(h_{2n-1}, h_{2n}, g_1, \ldots g_{n-1}, t\right),$$
$$M_2\left(h_{2n-1}, h_{2n}, g_1, \ldots g_{n-1}, t\right), M_1\left(f_{2n}, f_{2n-1}, g_1, \ldots g_{n-1}, t\right),$$
$$M_2\left(h_{2n-1}, h_{2n}, g_1, \ldots g_{n-1}, t\right), M_2\left(h_{2n}, h_{2n}, g_1, \ldots g_{n-1}, t\right)\}.$$

This implies that, $M_2\left(h_{2n+1}, h_{2n}, g_1, \ldots g_{n-1}, qt\right)$

$$\geq s\ \min\{M_2\left(h_{2n}, h_{2n-1}, g_1, \ldots g_{n-1}, t\right), M_1\left(f_{2n}, f_{2n-1}, g_1, \ldots g_{n-1}, t\right)\}\ (4)$$

In general we have,

$$M_1\left(f_n, f_{n+1}, g_1, \ldots g_{n-1}, qt\right) \geq s\ \min\{M_1\left(f_{n-1}, f_n, g_1, \ldots g_{n-1}, t\right), M_2\left(h_n, h_{n+1}, g_1, \ldots g_{n-1}, t\right)\}$$

and

$$M_2\left(h_n, h_{n+1}, g_1, \ldots g_{n-1}, t\right) \geq s\ \min\{M_2\left(h_{n-1}, h_n, g_1, \ldots g_{n-1}, t\right), M_1\left(f_{n-1}, f_n, g_1, \ldots g_{n-1}, t\right)$$

i.e.

$$M_1\left(f_n, f_{n+1}, g_1, \ldots g_{n-1}, t\right) \geq s\ \min\{M_1\left(f_{n-1}, f_n, g_1, \ldots g_{n-1}, \tfrac{t}{q}\right), M_2\left(h_n, h_{n+1}, g_1, \ldots g_{n-1}, \tfrac{t}{q}\right) \cdots\cdots (5)$$

and

$$M_2\left(h_n, h_{n+1}, g_1, \ldots g_{n-1}, t\right) \geq s\ \min\{M_2\left(h_{n-1}, h_n, g_1, \ldots g_{n-1}, \tfrac{t}{q}\right), M_1\left(f_{n-1}, f_n, g_1, \ldots g_{n-1}, \tfrac{t}{q}\right)\ \cdots\cdots\ (6)$$

repeated use of (5) and (6) give

$$M_1\left(f_n, f_{n+1}, g_1, \ldots g_{n-1}, t\right) \geq s^n \min\{M_1\left(f_0, f_1, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right), M_2\left(h_1, h_2, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right) \to 1$$

as $n \to \infty$

and

$$M_2\left(h_n, h_{n+1}, g_1, \ldots g_{n-1}, t\right) \geq s^n \min\{M_2\left(h_0, h_1, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right), M_1\left(f_0, f_1, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right) \to 1$$

as $n \to \infty$

In general,

$$M_1\left(f_n, f_{n+m}, g_1, \ldots g_{n-1}, t\right) \geq s^n \min\{M_1\left(f_0, f_m, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right), M_2\left(h_1, h_{m+1}, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right) \to 1$$

as $n \to \infty$

and

$$M_2\left(h_n, h_{n+m}, g_1, \ldots g_{n-1}, t\right) \geq s^n \min\{M_2\left(h_0, h_m, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right), M_1\left(f_0, f_m, g_1, \ldots g_{n-1}, \tfrac{t}{q^n}\right) \to 1$$

as $n \to \infty$

For n = 1, 2,…, since q < 1, it follows that $\{f_n\}$ and $\{h_n\}$ are Cauchy sequences in F(X) and F(Y) with limits g in F(X) and k in F(Y) respectively.
i.e. $\lim_{n\to\infty} f_n = g$ and $\lim_{n\to\infty} h_n = k$.

If A is continuous then $\lim_{n\to\infty} Af_{2n} = \lim_{n\to\infty} h_{2n+1}$ i.e. Ag = k
Applying inequality (1), we have

$$M_1(g, f_{2n-1}, g_1, \dots g_{n-1}, t)\, M_1(SAg, TBf_{2n-1}, g_1, \dots g_{n-1}, qt)$$
$$\geq s \min\{M_1(f_{2n-1}, g, g_1, \dots g_{n-1}, t), M_1(f_{2n-1}, TBf_{2n-1}, g_1, \dots g_{n-1}, t),$$
$$M_1(f_{2n-1}, SAf_{2n-1}, g_1, \dots g_{n-1}, t), M_1(g, SAg, g_1, \dots g_{n-1}, t),$$
$$M_1(f_{2n-1}, TBf_{2n-1}, g_1, \dots g_{n-1}, t), M_2(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t),$$
$$M_1(g, f_{2n-1}, g_1, \dots g_{n-1}, t),\ M_1(f_{2n}, f_{2n}, g_1, \dots g_{n-1}, t)\}$$

Taking limit $n \to \infty$, we have

$$M_1(SAg, g, g_1, \dots g_{n-1}, qt) \geq s \min\{1,\ M_1(g, SAg, g_1, \dots g_{n-1}, t)\}$$

This implies that,

$$M_1(SAg, g, g_1, \dots g_{n-1}, qt) \geq s\ M_1(g, SAg, g_1, \dots g_{n-1}, t),$$

a contradiction
Therefore $SAg = g$. Hence $SAg = g = Sk$
Again by (2), we have

$$M_2(k, h_{2n}, g_1, \dots g_{n-1}, t), M_2(BSk, h_{2n+1}, g_1, \dots g_{n-1}, qt)$$
$$\geq s \min\{ M_2(k, h_{2n}, g_1, \dots g_{n-1}, t), M_2(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t),$$
$$M_2(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t),\ M_2(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t),$$
$$M_1(Sk, f_{2n}, g_1, \dots g_{n-1}, t),\ M_2(k, h_{2n}, g_1, \dots g_{n-1}, t),$$
$$M_2(BSk, k, g_1, \dots g_{n-1}, t)\}$$

Taking limit $n \to \infty$, we have

$$M_2(BSk, k, g_1, \dots g_{n-1}, qt) \geq s \min\{M_2(BSk, k, g_1, \dots g_{n-1}, t), 1\}$$

This implies that,

$$M_2(BSk, k, g_1, \dots g_{n-1}, qt) \geq M_2(BSk, k, g_1, \dots g_{n-1}, t),$$

a contradiction. Therefore, BSk = k, BSk = k = Bg and Ag = Bg = k.
Again applying inequality (1), we have

$$M_1(g, g, g_1, \dots g_{n-1}, t), M_1(SAg, TBg, g_1, \dots g_{n-1}, qt)$$
$$\geq s \min\{M_1(g, g, g_1, \dots g_{n-1}, t),\ M_1(g, TBg, g_1, \dots g_{n-1}, t),$$
$$M_1(g, SAg, g_1, \dots g_{n-1}, t),\ M_1(g, SAg, g_1, \dots g_{n-1}, t),$$
$$M_1(g, TBg, g_1, \dots g_{n-1}, t),\ M_2(h_{2n}, h_{2n+1}, g_1, \dots g_{n-1}, t),$$
$$M_1(g, g, g_1, \dots g_{n-1}, t),\ M_1(TBg, TBg, g_1, \dots g_{n-1}, t)\}$$

i.e. $M_1(g, TBg, g_1, \dots g_{n-1}, qt) \geq s \min\{M_1(g, TBg, g_1, \dots g_{n-1}, t), 1\}$
this implies

$$M_1(g, TBg, g_1, \dots g_{n-1}, qt) \geq M_1(g, TBg, g_1, \dots g_{n-1}, t),$$

a contradiction. Therefore, TBg = g. Hence TBg = Tk = g and Sk = g= Tk.
Again by using (2) we have

$$M_2(k, k, g_1, \dots g_{n-1}, t)\, M_2(BSk, ATk, g_1, \dots g_{n-1}, qt)$$
$$\geq s \min\{M_2(k, k, g_1, \dots g_{n-1}, t),\ M_2(k, ATk, g_1, \dots g_{n-1}, t),$$
$$M_2(k, ATk, g_1, \dots g_{n-1}, t),\ M_2(k, BSk, g_1, \dots g_{n-1}, t),$$
$$M_2(k, BSk, g_1, \dots g_{n-1}, t),\ M_1(Sk, Tk, g_1, \dots g_{n-1}, t),$$

$$M_2\left(k,k,g_1,\dots g_{n-1},t\right),\quad M_2\left(BSk,k,g_1,\dots g_{n-1},t\right)\}$$

$$\text{i.e. } M_2\left(k,ATk,g_1,\dots g_{n-1},qt\right)\geq s\ \min\{M_2\left(k,ATk,g_1,\dots g_{n-1},t\right),\ M_2\left(k,ATk,g_1,\dots g_{n-1},t\right)\}$$

this implies,
$M_2\left(k,ATk,g_1,\dots g_{n-1},qt\right)\geq M_2\left(k,ATk,g_1,\dots g_{n-1},t\right)$a contradiction
Therefore, ATk = k, BSk = ATk = k
Ag = Bg = g and Tk = Sk = g
Hence g is a common fixed point of SA and TB and k is a common fixed point of BS and AT.
This completes the proof. □

References

[1] Z. K. Ansari, R. Shrivastava, G. Ansari and A. Garg, Some fixed point theorems in fuzzy 2-metric and fuzzy 3-metric spaces, Int. J. Contemp. Math. Sci. 6(46) (2011) 2291–2301.

[2] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika (Prague) 11(5) (1975) 336–344.

[3] Kailash Namdeo, S. S. Rajput and Rajesh Shrivastavai, Fixed point theorem for fuzzy 2-metric spaces, Int. J. Theoretical & Applied Sciences 2(2) (2010) 16–18.

[4] R. M. Somasundaram and Thangaraj Beaula, Some aspects of 2-fuzzy 2-normed linear space, Bull. Malays. Math. Sci. Soc. 32(2) (2009) 211–221.

[5] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[6] Q. Zhang and Y. Song, Fixed point theory for generalised ϕ-weak contractions, Appl. Math. Lett. 22(1) (2009) 75–78.

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