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Fuzzy connectedness in terms of fuzzy grills

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Abstract. In this paper we introduce fuzzy connectedness via the notion of fuzzy grill in a fuzzy topological space. To that end we define and study fuzzy $\mathcal{G}$-separated sets, fuzzy $\mathcal{G}$-connected sets and $\mathcal{G}$-hyperconnected sets with respect to a fuzzy grill $\mathcal{G}$ and its associated topology.

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1. Introduction and preliminary results

The usual notion of set topology was generalized with the introduction of fuzzy topology by Chang [2] in 1968, based on the concept of fuzzy sets invented by Zadeh in [12]. Since then various important concepts namely compactness, connectedness in the general topology have been extended to fuzzy topology. Pu and Liu introduced the notion of fuzzy connectedness in [9]. After that different parallel notions of fuzzy connectedness were discussed by many researchers. The concept of grill in topology was introduced by Choquet [5] and fuzzy grills on fuzzy topological spaces was proposed by Azad [1]. Several applications of fuzzy grill have since then been noticed, specially in the works of Srivastava and Gupta [11] and Chattopadhyay et al. [4]. We have already introduced a new fuzzy topology $\tau_{\mathcal{G}}$, in terms of a fuzzy grill $\mathcal{G}$ in [8]. In this paper we introduce fuzzy separation of fuzzy sets, fuzzy connectedness and fuzzy hyperconnectedness via fuzzy grill $\mathcal{G}$ in usual fuzzy topological space and fuzzy $\tau_{\mathcal{G}}$-topological space.

A *fuzzy set* A in X is characterized by a membership function in the sense of Zadeh [12]. The basic fuzzy sets are the zero set, the whole set and the class of all fuzzy sets in X, to be denoted by 0_X, 1_X and I^X respectively. By a fuzzy topological space (X, τ)(henceforth abbreviated as an fts X) , we mean a non-empty set X with the fuzzy topology τ, as given by Chang [2]. For two fuzzy sets A, B in X, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$, whereas the notation AqB means that A is *quasi-coincident* [9] with B, written as AqB, if $A(x) + B(x) > 1$ for some $x \in X$. The negations of these statements are denoted by $A \not\leq B$ and $A\bar{q}B$. A is called a *q-nbd of* B [9] if BqU for some fuzzy open set U in X, with $U \leq A$; if in addition, A itself is fuzzy open then it is called an open-q-nbd of B. The collection of all open q-nbds of any fuzzy point x_α will be denoted by $\mathcal{Q}(x_\alpha)$. For a fuzzy set A in an fts X, the *fuzzy complement*, *fuzzy interior* and *fuzzy closure* of A in X are written as $1 - A$, $intA$ and clA respectively.

A non-void collection $\mathcal{G}$ of fuzzy sets in an fts (X, τ) is called a fuzzy grill on X [1] if (i) $0_X \notin \mathcal{G}$, (ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \Rightarrow B \in \mathcal{G}$ and (iii) $A, B \in I^X$ and $A \bigvee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.
In a fuzzy topological space with a fuzzy grill $\mathcal{G}$ on X, an operator $\phi : I^X \to I^X$, denoted by $\phi_{\mathcal{G}}(A)$ or simply by $\phi(A)$(where A is a fuzzy set in X) is defined [in [8]] as the union of all fuzzy points x_λ of X such that if $U \in Q(x_\alpha)$, then $A * U \in \mathcal{G}$, where $A * B$ is the Lukasiewicz conjunction on the power set I^X, given by $A * B = \max(0, A + B - 1_X)$, for $A, B \in I^X$,where $(A * B)(x) = A(x) + B(x) - 1$ if $A(x) + B(x) > 1$ and $(A * B)(x) = 0$ otherwise.
Throughout this paper, by a *fuzzy $\mathcal{G}$-space* $(X, \tau, \mathcal{G})$, we mean an fts (X, τ) endowed with a fuzzy grill $\mathcal{G}$.

Proposition 1.1 ([8]). *Let (X, τ) be an fts. Then for any fuzzy sets A and B in X, the following hold:*
(i) If $\mathcal{G}$ is any fuzzy grill on X, then $A \leq B \Rightarrow \phi_{\mathcal{G}}(A) \leq \phi_{\mathcal{G}}(B)$.
(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are any two fuzzy grills on X and $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\phi_{\mathcal{G}_1}(A) \leq \phi_{\mathcal{G}_1}(A)$.
(iii) For any fuzzy grill $\mathcal{G}$ on X and for any fuzzy set A in X, if $A \notin \mathcal{G}$, then $\phi_{\mathcal{G}}(A) = 0_X \notin \mathcal{G}$ and A is $\tau_{\mathcal{G}}$-closed.
(iv) $\phi(A \bigvee B) = \phi(A) \bigvee \phi(B)$
(v) $\phi(\phi(A)) \leq \phi(A) = cl(\phi(A)) \leq cl(A)$
(vi) $\phi(A \bigvee G) = \phi(A)$, for every $G \notin \mathcal{G}$.
(vi) $\phi(A)$ is $\tau_{\mathcal{G}}$-closed.

Definition 1.2 ([8]). In an fts (X, τ), corresponding to a fuzzy grill $\mathcal{G}$, there exists a unique fuzzy topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \in I^X / \psi(1_X - U) = 1_X - U\}$, where for any $A \in I^X$, $\psi(A) = A \bigvee \phi(A) = \tau_{\mathcal{G}}\text{-}cl(A)$.

Definition 1.3 ([8]). Let $\mathcal{G}$ be a fuzzy grill on an fts (X, τ). Then τ is said to be suitable for the fuzzy grill $\mathcal{G}$, if for every fuzzy set A in X : if corresponding to each fuzzy point $x_\alpha \leq A$, there exists a $U \in \mathcal{Q}(x_\alpha)$ such that $A * U \notin \mathcal{G}$, then $A \notin \mathcal{G}$.

Corollary 1.4 ([8]). *For a fuzzy $\mathcal{G}$ space $(X, \tau, \mathcal{G})$, $\tau \subseteq \tau_{\mathcal{G}}$.*

Theorem 1.5 ([6]). *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space. Then the following are equivalent:*
(i) $\phi(1_X) = 1_X$, (ii) $\tau \setminus \{0_X\} \subseteq \mathcal{G}$, (iii) $intG = 0_X$ for each $G \notin \mathcal{G}$.

2. $\mathcal{G}$-separated fuzzy sets

Pu and Liu [9] defined separation and Q-separation for two fuzzy sets in an fts. We now want to define $\mathcal{G}$-separation of fuzzy sets in a fuzzy $\mathcal{G}$-space in an analogous way and discuss some properties which we will need later in the study of fuzzy connectedness via fuzzy grill.

Definition 2.1 ([9]). Two fuzzy sets A_1 and A_2 in an fts (X, τ) are said to be separated if there exist $U_i \in \tau (i = 1, 2)$ such that $A_i \leq U_i (i = 1, 2)$ and $U_1 \wedge A_2 = 0_X = U_2 \wedge A_1$.

Definition 2.2 ([9]). Two fuzzy sets A_1 and A_2 in (X, τ) are said to be Q-separated if there exist fuzzy closed sets H_1 and H_2 such that $A_1 \leq H_1$, $A_2 \leq H_2$ and $H_1 \wedge A_2 = 0_X = H_2 \wedge A_1$.

It is obvious that A_1 and A_2 are Q-separated iff $clA_1 \wedge A_2 = 0_X = clA_2 \wedge A_1$.

Definition 2.3. In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, any two fuzzy sets A $(\neq 0_X, 1_X)$ and B $(\neq 0_X, 1_X)$ are said to be fuzzy $\mathcal{G}$-separated if $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$ or $\tau_{\mathcal{G}}\text{-}clB \wedge A = 0_X = B \wedge clA$.

Remark 2.4. According to Piu and Liu [9], fuzzy separation and fuzzy Q-separation do not imply each other. Now since $\tau_{\mathcal{G}}\text{-}clA \leq clA$ [by Proposition 1.1(v) and Definition 1.2], Q-separation implies $\mathcal{G}$-separation of fuzzy sets. But the converse is not true in general which we show by the following:

Example 2.5. Let $X = \{a, b\}$, and $\tau = \{0_X, 1_X, P\}$ be a fuzzy topology on X where $P(a) = 1$, $P(b) = 0.4$. Let $\mathcal{G} = \{G \in I^X : 0.3 \leq G(x) \leq 1, x \in X\}$ be a fuzzy grill on X. Let us take two fuzzy sets A and B such that $A(a) = 0.8$, $A(b) = 0$ and $B(a) = 0$, $B(b) = 0.3$. Since $A \notin \mathcal{G}$, A is $\tau_{\mathcal{G}}$-closed [by Proposition 1.1(iii)]. Then $\tau_{\mathcal{G}}\text{-}clA \wedge B = A \wedge B = 0_X$. Also $clB = 1 - P$ and so $A \wedge clB = A \wedge (1 - P) = 0_X$. Thus A and B are fuzzy $\mathcal{G}$-separated. However, A and B are not fuzzy Q-separated since $clA = 1_X$ and hence $clA \wedge B \neq 0_X$.

We also note that the notions of fuzzy separation and fuzzy $\mathcal{G}$-separation are independent of each other. We show this by the following examples:

Example 2.6. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, P, Q, P \vee Q\}$ such that $P(a) = 0.8$, $P(b) = 0$; $Q(a) = 0$, $Q(b) = 0.6$, be a fuzzy topology on X and $\mathcal{G}$ be a fuzzy grill on X. Let A and B be two fuzzy sets in X such that $A(a) = 0.5$, $A(b) = 0$ and $B(a) = 0$, $B(b) = 0.3$. Then A and B are fuzzy separated since P and Q are two fuzzy open sets such that $A \leq P$ and $B \leq Q$ and $P \wedge Q = 0_X$. But A and B are fuzzy $\mathcal{G}$-separated. In fact, $clB = 1 - (P \vee Q)$ where $1 - (P \vee Q)(a) = 0.2$ and $1 - (P \vee Q)(b) = 0.4$ and $A \wedge clB = A \wedge (1 - P \vee Q) \neq 0_X$; and also $clA = 1 - Q$, so that $B \wedge clA \neq 0_X$.

Example 2.7. Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, U, V, U \wedge V\}$, where $U(a) = 1$, $U(b) = 0.6$; $V(a) = 0.4$, $V(b) = 1$, be a fuzzy topology on X and $\mathcal{G}$ be a fuzzy grill on X. Let G and H be two fuzzy sets in X such that $G(a) = 0$, $G(b) = 0.2$ and $H(a) = 0.3$, $H(b) = 0$. Now $clH = 1 - V$. Thus $G \wedge clH = G \wedge (1 - V) = 0_X$. Also $clG = 1 - U$ and $clG \wedge H = 0_X \Rightarrow \tau_{\mathcal{G}}\text{-}clG \wedge H = 0_X$. Thus G and H are fuzzy $\mathcal{G}$-separated. But we have fuzzy open sets U and V such that $G \leq U$ and $H \leq V$ and $U \wedge V \neq 0_X$ which implies that G and H are not fuzzy separated.

However fuzzy $\mathcal{G}$-separated sets may be made fuzzy Q-separated if some condition is assumed. For this, we recall the following:

Definition 2.8 ([7]). Any fuzzy set A in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be $\tau_{\mathcal{G}}$-dense-in-itself if $A \leq \phi(A)$.

Theorem 2.9. *Any two fuzzy $\mathcal{G}$-separated sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ are fuzzy Q-separated if one of them is $\tau_{\mathcal{G}}$-dense-in-itself, whose $\tau_{\mathcal{G}}$-closure has a void intersection with the other.*

Proof. Let A and B be any two fuzzy $\mathcal{G}$-separated sets in $(X, \tau, \mathcal{G})$. Then $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$ or $\tau_{\mathcal{G}}\text{-}clB \wedge A = 0_X = B \wedge clA$.
Without any loss of generality, let $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$. Now, A is $\tau_{\mathcal{G}}$-dense-in-itself $\Rightarrow A \le \phi(A) \Rightarrow clA \le cl\phi(A) \le clA$ [by Proposition1.1(v)] $\Rightarrow \phi(A) = clA$. Thus $\tau_{\mathcal{G}}\text{-}clA = A \vee \phi(A) = \phi(A) = clA$ and so $clA \wedge B = 0_X = A \wedge clB$ and consequently A and B are fuzzy Q-separated. □

Theorem 2.10. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, if A and B are two fuzzy $\mathcal{G}$-separated sets then so are $\phi(A)$ and $\phi(B)$.*

Proof. Let A and B be any two fuzzy $\mathcal{G}$-separated sets in $(X, \tau, \mathcal{G})$. Then either $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$ or $\tau_{\mathcal{G}}\text{-}clB \wedge A = 0_X = B \wedge clA$.

Case-I : $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$.
Then $\phi(A) \wedge B = 0_X = A \wedge \phi(B)$[since $\phi(A) \le cl(A)$] $\Rightarrow cl(\phi(A)) \wedge B = 0_X = A \wedge cl(\phi(B))$[by Proposition1.1(v)] $\Rightarrow \tau_{\mathcal{G}}\text{-}cl(\phi(A)) \wedge B = 0_X = A \wedge cl(\phi(B))$[since $\tau_{\mathcal{G}}\text{-}cl(\phi(A)) \le cl(\phi(A))$] $\Rightarrow \phi(A)$ and $\phi(B)$ are fuzzy $\mathcal{G}$-separated.

Case-II: $\tau_{\mathcal{G}}\text{-}clB \wedge A = 0_X = B \wedge clA$.
Interchanging A and B in case-I we obtain the same conclusion. □

Theorem 2.11. *For any two fuzzy $\mathcal{G}$-separated sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, $A\overline{q}\phi(B)$ and $B\overline{q}\phi(A)$.*

Proof. Since A and B are fuzzy $\mathcal{G}$-separated sets then $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X = A \wedge clB$ or $\tau_{\mathcal{G}}\text{-}clB \wedge A = 0_X = B \wedge clA$. Now $\tau_{\mathcal{G}}\text{-}clA \wedge B = 0_X \Rightarrow \tau_{\mathcal{G}}\text{-}clA\overline{q}B \Rightarrow B\overline{q}\phi(A)$[as $\tau_{\mathcal{G}}\text{-}clA = A \vee \phi(A)$].
Also, $A \wedge clB = 0_X \Rightarrow A \wedge \tau_{\mathcal{G}}\text{-}clB = 0_X \Rightarrow A\overline{q}\phi(B)$.
Similarly we will get the same result if the other possibility occurs. □

3. $\mathcal{G}$-connected fuzzy sets

We will now discuss about an important concept of fuzzy topology, namely fuzzy connectedness for two fuzzy $\mathcal{G}$-spaces $(X, \tau, \mathcal{G})$ and $(X, \tau_{\mathcal{G}}, \mathcal{G})$. For this we require to recall the following definition and lemma given by Pu and Liu [9]

Definition 3.1. A fuzzy set D in (X, τ) is called disconnected if there exist two non-null fuzzy sets A and B such that A and B are Q-separated and $D = A \vee B$. A fuzzy set is called connected iff it is not disconnected.

Lemma 3.2. *A fuzzy set D in an fts X is disconnected iff there are fuzzy closed sets A and B such that $A \wedge D \neq 0_X$, $B \wedge D \neq 0_X$, $A \wedge B = 0_X$ and $D \le A \vee B$.*

Remark 3.3. From Definition 3.1and Lemma 3.2, we can state that a fuzzy topological space (X, τ) is disconnected iff there exists non-null τ-closed fuzzy sets A, B in X such that $A \wedge B = 0_X$ and $A \vee B = 1_X$.

We now want to relate the connectedness defined in terms of the grill-oriented topology $\tau_{\mathcal{G}}$ with the original topology τ on the same underlying set X.

Theorem 3.4. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space such that $\tau \setminus \{0_X\} \subseteq \mathcal{G}$. Then $(X, \tau, \mathcal{G})$ is fuzzy connected iff $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is fuzzy connected.*

Proof. Since $\tau \subseteq \tau_{\mathcal{G}}$, if $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is fuzzy connected then $(X, \tau, \mathcal{G})$ is also so. Conversely, suppose $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is fuzzy disconnected. Then there exist non-null $\tau_{\mathcal{G}}$-closed fuzzy sets A and B in $(X, \tau, \mathcal{G})$ such that $A \wedge B = 0_X$ and $A \vee B = 1_X$ ...(1)
Then $\phi(1_X) = \phi(A \vee B) \Rightarrow 1_X = \phi(A) \vee \phi(B))$ [Using Theorem 1.5, and Proposition 1.1(iv)] ...(2)
Now, A is $\tau_{\mathcal{G}}$-closed $\Rightarrow A = \tau_{\mathcal{G}}\text{-}clA = A \vee \phi(A) \Rightarrow \phi(A) \leq A$. Similarly $\phi(B) \leq B$. Thus $A \wedge B = 0_X \Rightarrow \phi(A) \wedge \phi(B)) = 0_X$.
Also $\phi(A)$ and $\phi(B))$ are non-null. Indeed, if $\phi(A)(say) = 0_X$, then $\phi(B)) = 1_X$[by(2)] $\Rightarrow 1_X = \phi(B)) \leq B \leq 1_X \Rightarrow B = 1_X \Rightarrow A \wedge B \neq 0_X$, contradicting (1). Thus $\phi(A)$ and $\phi(B)$ are non-null τ-closed sets [by Proposition 1.1(v)] in X such that $\phi(A) \wedge \phi(B)) = 0_X$ and $\phi(A) \vee \phi(B)) = 1_X$. Thus $(X, \tau, \mathcal{G})$ is fuzzy disconnected. □

Next we discuss about another type of fuzzy connectedness namely fuzzy hyperconnectedness of a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$.

Definition 3.5 ([3])**.** An fts (X, τ) is said to be fuzzy hyperconnected if every fuzzy open set in X is fuzzy dense in X. i.e., $clU = 1_X$, for each $U \in \tau$.

Theorem 3.6. *For any fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following are equivalent:*
(i) X is fuzzy hyperconnected and $U \leq \phi(U)$, for each $U \in \tau$.
(ii) for each non-null fuzzy open set U in X, $\phi(U) = 1_X$.

Proof. $i) \Rightarrow ii)$: X is fuzzy hyperconnected and $U(\neq 0_X) \in \tau \Rightarrow clU = 1_X$. Then $1_X = clU \leq cl\phi(U)$ [by (i)] $= \phi(U)$[by Proposition 1.1(v)] $= 1_X \Rightarrow \phi(U) = 1_X$.

$ii) \Rightarrow i)$: Let $U(\neq 0_X)$ be any fuzzy open set in X such that $\phi(U) = 1_X$. Then obviously $U \leq \phi(U)$. Now for each fuzzy point y_β in X, $y_\beta \leq 1_X = \phi(U) \leq clU$[by Proposition 1.1(v)]. Thus $clU = 1_X$ and hence X is fuzzy hyperconnected. □

Now we define fuzzy connectedness and fuzzy hyperconnectedness in terms of fuzzy grills and we want to show their relationships with the corresponding ones with respect to the original topology.

Definition 3.7. A fuzzy set D in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be a fuzzy $\mathcal{G}$-connected set if D cannot be expressed as the union of two fuzzy $\mathcal{G}$-separated sets. A fuzzy set D is called fuzzy $\mathcal{G}$-disconnected iff it is not fuzzy $\mathcal{G}$-connected.

In particular, a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be a fuzzy $\mathcal{G}$-disconnected if there exist two fuzzy sets A and B $(\neq 0_X, 1_X)$ such that A and B are fuzzy $\mathcal{G}$-separated and $1_X = A \vee B$. $(X, \tau, \mathcal{G})$ is called fuzzy $\mathcal{G}$-connected space iff it is not fuzzy $\mathcal{G}$-connected.

Definition 3.8. A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be fuzzy $\mathcal{G}$-hyperconnected if for each $U \in \tau_{\mathcal{G}}$, $\tau_{\mathcal{G}}\text{-}clU = 1_X$.

Theorem 3.9. *For any fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following hold:*
i) Every fuzzy $\mathcal{G}$-connected space is fuzzy connected.
ii) Every fuzzy $\mathcal{G}$-hyperconnected space is fuzzy hyperconnected.

Proof. i) Follows from Remark 2.4.
ii) Let $(X, \tau, \mathcal{G})$ be fuzzy $\mathcal{G}$-hyperconnected. Then for each $U \in \tau_{\mathcal{G}}$, $\tau_{\mathcal{G}}$-$clU = 1_X$. Since $\tau \subseteq \tau_{\mathcal{G}}$, $1_X = \tau_{\mathcal{G}}$-$clU \leq clU$ for all $U \in \tau_{\mathcal{G}} \Rightarrow cl(U) = 1_X, \forall U \in \tau \Rightarrow (X, \tau, \mathcal{G})$ is fuzzy hyperconnected. □

However, the converses of the results in the above theorem are not true in general, as we show by the following two examples:

Example 3.10. Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A\}$, where $A(a) = 1$, $A(b) = 0$, be a fuzzy topology on X and $\mathcal{G} = \{G \in I^X / 0 < G(x) \leq 1, x \in X\}$ be a fuzzy grill on X. Let us consider a fuzzy set B in X such that $B(a) = 0$, $B(b) = 1$. Then A and B are non-null fuzzy sets such that $1_X = A \vee B$ and $A \wedge B = 0_X$. To show that A and B are fuzzy $\mathcal{G}$-separated, we see that A , $B \notin \mathcal{G}$. So by Proposition 1.1(iii), A and B are $\tau_{\mathcal{G}}$-closed. Thus $\tau_{\mathcal{G}}$-$clA \wedge B = A \wedge B = 0_X = A \wedge (1 - A)$[since $B = 1 - A = clB$] $= A \wedge clB$ and consequently X is fuzzy $\mathcal{G}$-disconnected.
But $(X, \tau, \mathcal{G})$ is fuzzy connected since if $1_X = P \vee Q$, where P and Q are two non-null τ-closed fuzzy sets in X and $P \wedge Q = 0_X$, then each of clP and clQ is either $(1 - A)$ or 1_X but $(1 - A) \wedge 1_X \neq 0_X$. In any case $P \wedge Q \neq 0_X$ contradicting our hypothesis.

Example 3.11. Consider the fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, where $X = \{a, b\}$, $\tau = \{0_X, 1_X, P\}$ with $P(a) = 0.4$ and $P(b) = 0.6$; and $\mathcal{G} = \{G \in I^X / 0.6 < G(x) \leq 1, x \in X\}$. We have $clP = 1_X$ and hence X is fuzzy hyperconnected. Also X is fuzzy $\mathcal{G}$-connected since 1_X cannot be expressed as a disjoint union of two fuzzy $\mathcal{G}$-separated sets. Indeed, if possible, $1_X = A \vee B$, where A, $B (\neq 0_X, 1_X)$ are two fuzzy sets such that $\tau_{\mathcal{G}}$-$clA \wedge B = 0_X = A \wedge clB$ or $\tau_{\mathcal{G}}$-$clB \wedge A = 0_X = B \wedge clA$. But clB is either $1 - P$ or 1_X. But there cannot exist any non-null set A such that either $A \wedge (1 - P) = 0_X$ or $A \wedge 1_X = 0_X$. Similarly $B \wedge clA = 0_X$ is also not possible.
But X is not fuzzy $\mathcal{G}$-hyperconnected. In fact, for $P \in \tau \subseteq \tau_{\mathcal{G}}$, we claim that $P = \tau_{\mathcal{G}}$-clP. Indeed, for any fuzzy point x_λ in X and for any $U \in \mathcal{Q}(x_\lambda)$, $P + U - 1 \notin \mathcal{G}$, so that $\phi(P) = 0_X$. Thus $P = P \vee \phi(P) = \tau_{\mathcal{G}}$-$clP \neq 1_X \Rightarrow X$ is not fuzzy $\mathcal{G}$-hyperconnected.

Lemma 3.12. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, if A and B are fuzzy $\mathcal{G}$-separated sets in X, and A_1 and B_1 are two non-null fuzzy sets in X such that $A_1 \leq A$ and $B_1 \leq B$, then A_1 and B_1 are also fuzzy $\mathcal{G}$-separated.*

Proof. Clear from definition of fuzzy $\mathcal{G}$-separated sets. □

Theorem 3.13. *A non-null fuzzy set C in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is fuzzy $\mathcal{G}$-connected iff for every pair of fuzzy $\mathcal{G}$-separated sets A and B in X with $C \leq A \vee B$, exactly one of the possibilities (a) and (b) holds:*

(a) $C \leq A$ and $C \wedge B = 0_X$.

(b) $C \leq B$ and $C \wedge A = 0_X$.

Proof. First let C be fuzzy $\mathcal{G}$-connected. As $C \leq A \vee B$, both of $C \wedge A = 0_X$ and $C \wedge B = 0_X$ cannot hold simultaneously. Also if $C \wedge A \neq 0_X$ and $C \wedge B \neq 0_X$ hold then by Lemma 3.12, $C \wedge A$ and $C \wedge B$ are also fuzzy $\mathcal{G}$-separated sets and $C = (C \wedge A) \vee (C \wedge B)$ which contradicts the fact that C is fuzzy $\mathcal{G}$-connected. Thus exactly one of the possibilities ($C \wedge A \neq 0_X$ but $C \wedge B = 0_X$) and ($C \wedge A = 0_X$ and $C \wedge B \neq 0_X$) holds. Now if $C \wedge A = 0_X$ then $C \leq B$ and if $C \wedge B = 0_X$ then $C \leq A$. Conversely let the given condition hold. If possible, let C be not fuzzy $\mathcal{G}$-connected. Then there exist two fuzzy $\mathcal{G}$-separated sets A and B in X such that $C = A \vee B$. By hypothesis, either $C \wedge A = 0_X$ or $C \wedge B = 0_X$. i.e., either $A = 0_X$ or $B = 0_X$, none of which is true. Thus C is fuzzy $\mathcal{G}$-connected. □

Theorem 3.14. *For any two fuzzy sets A and B in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, if A is fuzzy $\mathcal{G}$-connected and $A \leq B \leq \tau_{\mathcal{G}}\text{-}clA$, then B is also fuzzy $\mathcal{G}$-connected.*

Proof. Suppose that B is not fuzzy $\mathcal{G}$-connected. Then there exist non-null fuzzy $\mathcal{G}$-separated sets G and H such that $B = G \vee H$. Then either $\tau_{\mathcal{G}}\text{-}clG \wedge H = 0_X = G \wedge clH$ or $\tau_{\mathcal{G}}\text{-}clH \wedge G = 0_X = H \wedge clG$ holds. Without any loss of generality we take $\tau_{\mathcal{G}}\text{-}clG \wedge H = 0_X = G \wedge clH$. Now $A \leq B \leq G \vee H$ and A is fuzzy $\mathcal{G}$-connected. Then by Theorem 3.13, either ($A \leq G$ and $H \wedge A = 0_X$) or ($A \leq H$ and $A \wedge G = 0_X$) but not both.

<u>Case-I.</u> Suppose $A \leq G$ and $H \wedge A = 0_X$. Then $\tau_{\mathcal{G}}\text{-}clA \leq \tau_{\mathcal{G}}\text{-}clG$. So, $\tau_{\mathcal{G}}\text{-}clA \wedge H = 0_X = G \wedge clH$. Also by hypothesis, $H \leq B \leq \tau_{\mathcal{G}}\text{-}clA \Rightarrow H = \tau_{\mathcal{G}}\text{-}clA \wedge H = 0_X$ which contradicts that H is non-null.

<u>Case-II.</u> $A \leq H$ and $A \wedge G = 0_X$. By similar arguments as in Case-I, we again arrive at a contradiction.

Thus B is fuzzy $\mathcal{G}$-connected. □

Definition 3.15 ([6]). A fuzzy set A in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be a fuzzy $\mathcal{G}$-open set if $A \leq int\phi(A)$.

Corollary 3.16. *If A is a fuzzy $\mathcal{G}$-connected set in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, then*
(a) $\tau_{\mathcal{G}}\text{-}clA$ is fuzzy $\mathcal{G}$-connected.
(b) $\phi(A)$ is fuzzy $\mathcal{G}$-connected if A is fuzzy $\mathcal{G}$-open.

Proof. (a) It is clear from Theorem 3.14.

(b) If A is a fuzzy $\mathcal{G}$-open set in X, then $A \leq int\phi(A) \leq \phi(A) \leq \tau_{\mathcal{G}}\text{-}clA$. So by Theorem 3.14, $\phi(A)$ is fuzzy $\mathcal{G}$-connected. □

Theorem 3.17. *The union of any aggregate of fuzzy $\mathcal{G}$-connected sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, no two of which are fuzzy $\mathcal{G}$-separated, is a fuzzy $\mathcal{G}$-connected set.*

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy $\mathcal{G}$-connected sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, no two of which are fuzzy $\mathcal{G}$-separated in X and let $G = \bigvee\{G_\alpha : \alpha \in \Lambda\}$. If possible, let G be not fuzzy $\mathcal{G}$-connected in X. Then there exist two non null fuzzy sets A and B which are fuzzy $\mathcal{G}$-separated in X and $G = A \vee B$. Now for each $\alpha \in \Lambda$, G_α is a fuzzy $\mathcal{G}$-connected set and also $G_\alpha \leq A \vee B$. Then by Theorem 3.13, either $G_\alpha \leq A$ and $G_\alpha \wedge B = 0_X$ or else $G_\alpha \leq B$ and $G_\alpha \wedge A = 0_X$. If possible, let for some $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$, $G_\alpha \leq A$ and $G_\beta \leq B$. Then G_α and G_β, being non null subsets of fuzzy $\mathcal{G}$-separated sets, are also fuzzy $\mathcal{G}$-separated [by Lemma 3.12] which is not the case. Thus either $G_\alpha \leq A$ and $G_\alpha \wedge B = 0_X$ for each $\alpha \in \Lambda$

or else $G_\alpha \leq B$ and $G_\alpha \wedge A = 0_X$ for each $\alpha \in \Lambda$. In the first case $B = 0_X$ [since $B \leq G$] and in the second case $A = 0_X$, none of which is true. Thus G is fuzzy $\mathcal{G}$-connected. □

Corollary 3.18. *Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy $\mathcal{G}$-connected sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ such that $G_\alpha \wedge G_\beta \neq 0_X$ for any $\alpha, \beta \in \Lambda$. Then $\bigvee_{\alpha \in \Lambda} G_\alpha$ is fuzzy $\mathcal{G}$-connected.*

Proof. Follows from Theorem 3.17. □

Corollary 3.19. *Let $\{G_\alpha : \alpha \in \Lambda\}$ be a non-null family of fuzzy $\mathcal{G}$-connected sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ with $\bigwedge_{\alpha \in \Lambda} G_\alpha \neq 0_X$, then $\bigvee_{\alpha \in \Lambda} G_\alpha$ is fuzzy $\mathcal{G}$-connected.*

Proof. Follows from Theorem 3.17. □

Theorem 3.20. *Let $\mathcal{A}$ be a family of fuzzy $\mathcal{G}$-connected sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ such that there is a non null member A_0 of $\mathcal{A}$ with the property that A_0 and A are not fuzzy $\mathcal{G}$-separated for each $A \in \mathcal{A}$. Then $\bigvee\{A : A \in \mathcal{A}\}$ is fuzzy $\mathcal{G}$-connected.*

Proof. By use of Theorem 3.17, $A \vee A_0$ is fuzzy $\mathcal{G}$-connected, for all $A \in \mathcal{A} \Rightarrow \bigvee\{A \vee A_0 : A \in \mathcal{A}\}$ i.e., $\bigvee\{A : A \in \mathcal{A}\}$ is fuzzy $\mathcal{G}$-connected. □

However intersection of even two fuzzy $\mathcal{G}$-connected sets may not be fuzzy $\mathcal{G}$-connected as shown by the following:

Example 3.21. Consider the fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, where $X = \{a, b\}$, $\tau = \{0_X, 1_X, U\}$ with $U(a) = 1$ and $U(b) = 0.5$; and $\mathcal{G} = \{G \in I^X / 0.6 \leq G(x) \leq 1, x \in X\}$. Consider two fuzzy sets A and B in X where $A(a) = 0.2$ and $A(b) = 1$; $B(a) = 1$ and $B(b) = 0.3$. We show that A and B are fuzzy $\mathcal{G}$-connected in X. Indeed, if $A = G \vee H$ for any two non-null fuzzy sets G and H in X, then we claim that $G \wedge clH \neq 0_X \neq clG \wedge H$. In fact, clH is either 1_X or $1 - U$. If $clH = 1_X$ then $G \wedge clH \neq 0_X$. Otherwise $clH = 1 - U$, then $clH(b) = (1 - U)(b) = 0.5$. But $A(b) = 1$, so $G(b) = 1$[since $A(b) = G(b) \vee H(b)$ and $H(b) \leq clH(b) = 0.5$]. In this case $G(b) \wedge clH(b) \neq 0$ and hence $G \wedge clH \neq 0_X$. Again G is non-null, so clG is either 1_X or $1 - U$. By similar argument we get $clG \wedge H \neq 0_X$. Thus G and H are not fuzzy $\mathcal{G}$-separated and hence A is fuzzy $\mathcal{G}$-connected.
To show B to be fuzzy $\mathcal{G}$-connected, let us assume to the contrary that $B = Y \vee Z$, where Y and Z are two non-null fuzzy $\mathcal{G}$-separated sets in X. Then $clY = 1 - U$ [since if $clY = 1_X$ then $clY \wedge Z \neq 0_X$]. Thus $clY(a) = (1-U)(a) = 0$. But $B(a) = 1 \Rightarrow Z(a) = 1$ and $B(b) = 0.3 \Rightarrow \max(Y(b), Z(b)) = 0.3$.
Now if $Y(b) \neq 0 \neq Z(b)$, then $clY(b) \wedge Z(b) \neq 0 \Rightarrow clY \wedge Z \neq 0_X$.
If $Y(b) = 0$, then $Y = 0_X$ which contradicts that Y is non-null.
If $Z(b) = 0$, then since $Z(a) = 1$, $clZ \neq 1 - U$ and hence $clZ = 1_X$ so that $clZ \wedge Y \neq 0_X$.
Thus in each case we arrive at a contradiction. Thus Y and Z are not fuzzy $\mathcal{G}$-separated and hence B is also fuzzy $\mathcal{G}$-connected.
But we claim that $A \wedge B$ is not fuzzy $\mathcal{G}$-connected. Here $(A \wedge B)(a) = 0.2$;

$(A \wedge B)(b) = 0.3$. We write $A \wedge B = P \vee Q$, where $P(a) = 0.2$ and $P(b) = 0$; $Q(a) = 0$ and $Q(b) = 0.3$. Then P and Q are non-null fuzzy sets in X. We want to show that $\tau_{\mathcal{G}}\text{-}clP \wedge Q = 0_X = P \wedge clQ$. We see that $\phi(P) = 0_X$ since for any fuzzy point x_λ in X and for any $U_x \in \mathcal{Q}(x_\lambda)$, $P + U_x - 1 \notin \mathcal{G} \Rightarrow x_\lambda \not\leq \phi(P)$. Thus $\tau_{\mathcal{G}}\text{-}clP \wedge Q = P \wedge Q = 0_X$. Again $clQ = 1 - U$. So $P \wedge clQ = P \wedge (1 - U) = 0_X$.

Theorem 3.22. *A non-null fuzzy set A in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is fuzzy $\mathcal{G}$-connected iff for any two fuzzy points x_α and y_β in A, where $x_\alpha \neq y_\beta$, there is a fuzzy $\mathcal{G}$-connected set B such that $B \leq A$ and $x_\alpha, y_\beta \leq B$.*

Proof. The condition is necessary because in that case we can take $B = A$.
Conversely, suppose that A is not fuzzy $\mathcal{G}$-connected. Then there exist two fuzzy $\mathcal{G}$-separated sets P and Q in X such that $A = P \vee Q$. As P and Q are non-null, let us choose any two fuzzy points x_α and y_β such that $x_\alpha \leq P$ and $y_\beta \leq Q$. Then $x_\alpha, y_\beta \leq A$ and hence by hypothesis there exists a fuzzy $\mathcal{G}$-connected set B such that $B \leq A$ and $x_\alpha, y_\beta \leq B$. Then $B \wedge P$ and $B \wedge Q$ are two non-null fuzzy sets in X and by Lemma 3.12, they are fuzzy $\mathcal{G}$-separated. Also $B = (B \wedge P) \vee (B \wedge Q)$, which contradicts the fact that B is fuzzy $\mathcal{G}$-connected. □

Next we want to discuss briefly about the behavior of fuzzy $\mathcal{G}$-separated sets and fuzzy $\mathcal{G}$-connected sets under fuzzy continuous function. For this we first recall that a function $f : X \to Y$ (where X, Y are fts's) is called fuzzy continuous [2], if $f^{-1}(V)$ is fuzzy open in X, for each fuzzy open set V in X. Some of the well known characterizations of fuzzy continuity are given below.

Result 3.23 ([10]). *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*
a) f is fuzzy continuous.
b) For any fuzzy set A in X, $f(clA) \leq clf(A)$.
b) For any fuzzy set B in Y, $cl(f^{-1}(B)) \leq f^{-1}(clB)$.

Theorem 3.24. *Let $(X, \tau, \mathcal{G}_1)$ and $(Y, \sigma, \mathcal{G}_2)$ be two fuzzy $\mathcal{G}$-spaces and $f : X \to Y$ be a fuzzy continuous function. If C and D be any two $\tau_{\mathcal{G}_2}$-dense-in-itself, $\mathcal{G}$-separated sets in Y, then $f^{-1}(C)$ and $f^{-1}(D)$ are fuzzy $\mathcal{G}$-separated sets in X.*

Proof. If possible, let $f^{-1}(C)$ and $f^{-1}(D)$ be not fuzzy $\mathcal{G}$-separated sets in X. Then $(\tau_{\mathcal{G}_1}\text{-}cl(f^{-1}(C)) \wedge f^{-1}(D) \neq 0_X$ or $f^{-1}(C) \wedge clf^{-1}(D) \neq 0_X)$ and $(cl(f^{-1}(C)) \wedge f^{-1}(D) \neq 0_X$ or $f^{-1}(C) \wedge \tau_{\mathcal{G}_1}\text{-}clf^{-1}(D) \neq 0_X)$.

<u>Case-I</u>: $\tau_{\mathcal{G}_1}\text{-}cl(f^{-1}(C)) \wedge f^{-1}(D) \neq 0_X$.
In this case $cl(f^{-1}(C)) \wedge f^{-1}(D) \neq 0_X$[since $\tau_{\mathcal{G}_1}\text{-}clA \leq clA$ for $A \in I^X$] $\Rightarrow f^{-1}(clC) \wedge f^{-1}(D) \neq 0_X$[by Result 3.23(c)] $\Rightarrow f[f^{-1}(clC) \wedge f^{-1}(D)] \neq 0_Y \Rightarrow f(f^{-1}(clC)) \wedge f(f^{-1}(D)) \neq 0_Y \Rightarrow clC \wedge D \neq 0_Y$. From this we also get $\tau_{\mathcal{G}_2}\text{-}clC \wedge D \neq 0_Y$, because $C \leq \phi(C)$ and Proposition 1.1(v) imply that $clC \leq cl\phi(C) = \phi(C) = C \vee \phi(C) = \tau_{\mathcal{G}_2}\text{-}clC \leq clC$. Thus C and D are not fuzzy $\mathcal{G}$-separated sets in Y, a contradiction.

<u>Case-II</u>: If $f^{-1}(C) \wedge clf^{-1}(D) \neq 0_X$, then $f^{-1}(C) \wedge f^{-1}(clD) \neq 0_X \Rightarrow f[f^{-1}(C) \wedge f^{-1}(clD)] \neq 0_Y \Rightarrow f(f^{-1}(C)) \wedge f(f^{-1}(clD)) \neq 0_Y \Rightarrow C \wedge clD \neq 0_Y \Rightarrow C \wedge \tau_{\mathcal{G}_2}\text{-}clD \neq 0_Y$ (since $clD = \tau_{\mathcal{G}_2}\text{-}clD$, as above) which contradicts again the fact that C

and D are fuzzy $\mathcal{G}$-separated sets in Y.
Thus $f^{-1}(C)$ and $f^{-1}(D)$ are fuzzy $\mathcal{G}$-separated sets in X. □

Theorem 3.25. *Let $(X, \tau, \mathcal{G}_1)$ and $(Y, \sigma, \mathcal{G}_2)$ be two fuzzy $\mathcal{G}$-spaces such that each fuzzy set in Y is $\tau_{\mathcal{G}_2}$-dense-in-itself. Let $f : X \to Y$ be a fuzzy continuous and one-to-one. If A is a fuzzy $\mathcal{G}$-connected set in X then $f(A)$ is fuzzy $\mathcal{G}$-connected in Y.*

Proof. If possible, let $f(A)$ be not fuzzy $\mathcal{G}$-connected in Y. Then $f(A) = B \vee C$, where B and C are two non-null fuzzy $\mathcal{G}$-separated sets in Y. Since each fuzzy set in Y is $\tau_{\mathcal{G}_2}$-dense-in-itself, by Theorem 3.24, $f^{-1}(B)$ and $f^{-1}(C)$ are also fuzzy $\mathcal{G}$-separated sets in X. Since f is one-to-one, $A = f^{-1}f(A) = f^{-1}(B \vee C) = f^{-1}(B) \vee f^{-1}(C)$ which shows that A is not fuzzy $\mathcal{G}$-connected in X and contradicts our hypothesis. Hence $f(A)$ is fuzzy $\mathcal{G}$-connected in Y. □

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