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# Generated intuitionistic fuzzy subgroup, $(t, k)$ equivalence relation and congruence

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Abstract. In this paper, a new construction of intuitionistic fuzzy subgroup generated by an intuitionistic fuzzy set is provided. This is followed by a construction of intuitionistic fuzzy equivalence relation and intuitionistic fuzzy congruence generated by an intuitionistic fuzzy relation. For any intuitionistic fuzzy set $A$ of a group $G$, it is proved that the subgroup of $G$ generated by the strong $(p, q)$-cut of $A$ is contained in the strong $(p, q)$-cut of the intuitionistic fuzzy subgroup generated by $A$ and the equality holds if $p + q \geq 1$.

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## 1. Introduction

The concept of fuzzy set was introduced by Zadeh [15] in 1965 as a generalization of ordinary set. The theory of fuzzy sets has made considerable progress. It has exerted a tremendous influence on modern science and technology such as mathematics, natural sciences, management sciences, control theory, sociology, economics etc. Fuzzy subgroups were first defined by Rosenfeld [12] in 1971. Since then numerous concepts have been studied like fuzzy topological groups [7], normal fuzzy subgroups [11] and fuzzy cosets [11] in various contexts. The classical concept of binary relation was also extented into fuzzy setting and fuzzy relations were defined by zadeh [15]. Various researchers have tried to further generalize the notion of fuzzy sets in the form of rough sets, vague sets etc among which intuitionistic fuzzy sets introduced by Atanassov [3] is most interesting concept. The aim of this paper is to study some aspects of intuitionistic fuzzy sets in the realm of group theory. In 1986, Atanassov

[3] initiated the concept of intuitionistic fuzzy set (IFS) to further generalization of fuzzy set in the sense that a fuzzy set is represented by a mapping which defines the degree of membership of an element in the given set whereas an intuitionistic fuzzy set consists of two mappings: one which defines the degree of membership and the other which defines the degree of non membership of the element in the set and there is still scope for some degree of non decisiveness. The concept has been applied to various algebraic structures. In 1989, Biswas [4] defined the intuitionistic fuzzy subgroups and their properties. In 1996, intuitionistic fuzzy relations were introduced by Bustince and Burillo [5]. They also studied the intuitionistic fuzzy equivalence relations. Recently, Thomas and Nair [14] slightly altered the definition of intuitionistic fuzzy equivalence relations to make it more general. In 2005, Hur et.al [8] introduced the notion of intuitionistic fuzzy congruences. The notion of algebraic structures and fuzzy algebraic structures generated by given sets and fuzzy sets respectively are important studies of all algebras. Ajmal and Thomas [2] gave the construction of the smallest fuzzy equivalence relation containing a fuzzy relation in 1995. In 1999, Sultana and Ajmal [13] modified the construction of the smallest fuzzy subgroup containing a fuzzy set which was given by Kumar [10]. The purpose of this paper is to extend these notions to intuitionistic fuzzy setting and provide specific construction of generated intuitionistic fuzzy subgroups, intuitionistic fuzzy $(t, k)$ equivalence relations [14] and intuitionistic fuzzy $(t, k)$ congruences.

## 2. Preliminaries

First, we list some basic definitions and results which are needed in this development.

**Definition 2.1** ([3]). An intuitionistic fuzzy set (IFS) $A$ in a non empty set $X$ is an object of the form:

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where the values of the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\nu_A(x)$) of each $x \in X$ in $A$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Every fuzzy set $A$ in a non empty set $X$ is clearly an intuitionistic fuzzy set having the form :

$$A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}.$$

Throughout this paper, an intutionistic fuzzy set $A$ in a set $X$ will be written as $(\mu_A, \nu_A)$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ are the membership and non membership functions of $A$.

The set of all intuitionistic fuzzy sets in a set $X$ will be denoted by IFS($X$).

**Definition 2.2** ([3]). Let $X$ be a non empty set and $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFSs in $X$. Then

(1) $A \subseteq B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(3) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(4) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

**Definition 2.3** ([6]). Let $\{A_i\}_{i\in J}$ be a family of intuitionistic fuzzy sets in $X$, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\underset{i\in J}{\cap} A_i = (\underset{i\in J}{\wedge}\mu_{A_i}, \underset{i\in J}{\vee}\nu_{A_i})$.

(2) $\underset{i\in J}{\cup} A_i = (\underset{i\in J}{\vee}\mu_{A_i}, \underset{i\in J}{\wedge}\nu_{A_i})$.

**Definition 2.4** ([9]). Let $A$ be an intuitionistic fuzzy set in a set $X$ and $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Then

(1) The set $A_{\alpha,\beta} = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is called $(\alpha, \beta)$-level subset of $A$.

(2) The set $A^{>}_{\alpha,\beta} = \{x \in X : \mu_A(x) > \alpha, \nu_A(x) < \beta\}$ is called $(\alpha, \beta)$-strong level subset of $A$.

**Definition 2.5** ([14]). Let $A$ be an intuitionistic fuzzy set in a set $X$ and $\alpha, \beta \in [0, 1]$. Then The sets $U(\mu_A, \alpha) = \{x \in X : \mu_A(x) \geq \alpha\}$ and $L(\nu_A, \beta) = \{x \in X : \nu_A(x) \leq \beta\}$ are called the upper and lower level subsets of $A$, respectively.

Clearly $A_{\alpha,\beta} = U(\mu_A, \alpha) \cap L(\nu_A, \beta)$ and $A^{>}_{\alpha,\beta} \subseteq A_{\alpha,\beta}$.

One can easily verify that for any $\alpha, \beta \in [0, 1]$, the $(\alpha, \beta)$-level subset $A_{\alpha,\beta} = A_{\alpha,\delta}$ for some $\delta \in [0, 1]$ such that $\gamma + \delta \leq 1$. A similar statement holds for strong level subsets.

**Definition 2.6** ([4]). An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in a group $G$ is said to be an intuitionistic fuzzy subgroup of $G$ if

(1) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$.

(2) $\mu_A(x^{-1}) = \mu_A(x)$.

(3) $\nu_A(xy) \leq max\{\nu_A(x), \nu_A(y)\}$.

(4) $\nu_A(x^{-1}) = \nu_A(x)$, for all $x, y \in G$.

Equivalently, an IFS $A$ is said to be intuitionistic subgroup of $G$ if both $\mu_A$ and $1 - \nu_A$ are fuzzy subgroups of $G$, where $1 - \nu_A$ is defined as $(1 - \nu_A)(x) = 1 - \nu_A(x)$ for all $x \in G$.

If $A$ is an intuitionistic fuzzy subgroup of a group $G$, then

$$\sup_{x\in G}\{\mu_A(x)\} = \mu_A(e), \text{ and}$$

$$\inf_{x\in G}\{\nu_A(x)\} = \nu_A(e),$$

where $e$ is the identity of $G$.

The set of all intuitionistic fuzzy subgroups of a group $G$ is denoted by IFSG$(G)$.

**Definition 2.7** ([5]). An intuitionistic fuzzy relation $R$ on a set $X$ is defined as the intuitionistic fuzzy subset of $X \times X$. That is, $R = (\mu_R, \nu_R)$, where $\mu_R : X \times X \to [0, 1]$ and $\nu_R : X \times X \to [0, 1]$ are the functions satisfying $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for all $x, y \in X$.

The family of all Intuitionistic fuzzy relations on $X$ will be denoted by IFR$(X)$.

**Definition 2.8** ([5]). Let $Q$ and $R$ be two intuitionistic fuzzy relations on a set $X$. Then their composition $QoR = (\mu_{QoR}, \nu_{QoR})$ is defined as: for any $x, y \in X$,

$$\mu_{QoR}(x, y) = \sup_{z\in X}\{min\{\mu_Q(x, z), \mu_R(z, y)\}\};$$

$$\nu_{QoR}(x,y) = \inf_{z\in X}\{max\{\nu_Q(x,z), \nu_R(z,y)\}\}.$$

**Definition 2.9** ([5]). An intuitionistic fuzzy relation $R = (\mu_R, \nu_R)$ on a set $X$ is called an intuitionistic fuzzy equivalence relation (in short, IFER) on $X$ if it satisfies the following conditions:

(1) $R$ is an intuitionistic fuzzy reflexive, i.e., $\mu_R(x,x) = 1$ and $\nu_R(x,x) = 0$ for all $x \in X$.
(2) $R$ is an intuitionistic fuzzy symmetric, i.e., $\mu_R(x,y) = \mu_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)$ for all $x, y \in X$.
(3) $R$ is an intuitionistic fuzzy transitive, i.e., $RoR \subseteq R$.

Above definition was given by Burstince and Burillo in 1996. Recently in 2012, Thomas and Nair [14] slightly altered the definition of intutionistic fuzzy equivalence relation as follows:

**Definition 2.10** ([14]). For $t, k \in [0,1]$, an intuitionististic fuzzy relation $R$ on a set $X$ is called $(t,k)$ reflexive if $\mu_R(x,x) = t$ and $\nu_R(x,x) = k$, for all $x \in X$.

**Definition 2.11** ([14]). An intuitionistic fuzzy relation $R$ on a set $X$ is called an intuitionistic fuzzy $(t,k)$ equivalence relation if

(1) $R$ is an intuitionistic fuzzy $(t,k)$ reflexive.
(2) $R$ is an intuitionistic fuzzy symmetric.
(3) $R$ is an intuitionistic fuzzy transitive.

Definition 2.9 becomes a particular case of Definition 2.10 by taking $t = 1$ and $k = 0$.

If $R$ is an intuitionistic fuzzy $(t,k)$ equivalence relation on a set $X$, then

$$\sup_{x,y\in X}\{\mu_R(x,y)\} = t$$

and

$$\inf_{x,y\in X}\{\nu_R(x,y)\} = k.$$

In 2005, Hur et. al. [8] defined intuitionistic fuzzy congruence on a group $G$ as follows:

**Definition 2.12** ([8]). An intuitionistic fuzzy equivalence relation $R = (\mu_R, \nu_R)$ on a group $G$ is said to be an intuitionistic fuzzy congruence on $G$ if

$$\mu_R(ac, bd) \geq min\{\mu_R(a,b), \mu_R(c,d)\}$$

and

$$\nu_R(ac, bd) \leq max\{\nu_R(a,b), \nu_R(c,d)\}$$

for all $a, b, c, d \in G$.

The same definition holds for an intuitionistic fuzzy $(t,k)$ equivalence relation to be an intuitionistic fuzzy $(t,k)$ congruence.

We shall use the notation IF to denote all Intuitionistic fuzzy concepts.

## 3. Constructions

The proof of the following proposition being trivial is omitted.

**Proposition 3.1.** *The intersection of an arbitrary family of intuitionistic fuzzy subgroups of a group $G$ is an intuitionistic fuzzy subgroup of $G$.*

The above proposition is instrumental in ensuring the existence of the least intuitionisic fuzzy subgroup containing a given intuitionistic fuzzy set of a group $G$.

**Definition 3.2.** The intuitionistic fuzzy subgroup generated by the intuitionistic fuzzy set $A$ of a group $G$, denoted by $< A >$ is defined as the least IF subgroup of $G$ containing $A$, that is,

$$< A >= \cap\{A_i : A \subseteq A_i, A_i \in IFSG(G)\}.$$

Before providing a specific and new construction of an IF subgroup of a group $G$, generated by an IF set $A$, we recall the construction of a fuzzy subgroup generated by a fuzzy set given by Sultana and Ajmal [13].

**Theorem 3.3** ([13]). *Let $\mu$ be a fuzzy set in a group $G$. Then the fuzzy set $\mu^*$ in $G$ defined by*

$$\mu^*(x) = \sup_{\alpha \leq t}\{\alpha : x \in< \mu_\alpha >\}, \text{ where } t = \sup_{x\in G}\{\mu(x)\}$$

*is the fuzzy subgroup of $G$ generated by $\mu$, that is, $\mu^* =< \mu >$.*

**Theorem 3.4.** *Let $A$ be an intuitionistic fuzzy set in a group $G$. Define an intuitionistic fuzzy set $A^* = (\mu_{A^*}, \nu_{A^*})$ in $G$ as:*

$$\mu_{A^*}(x) = \sup_{\alpha\leq t}\{\alpha : x \in< U(\mu_A, \alpha) >\}$$

*and*

$$\nu_{A^*}(x) = \inf_{\beta\geq k}\{\beta : x \in< L(\nu_A, \beta) >\},$$

*where $t = \sup_{x\in G}\{\mu_A(x)\}, k = \inf_{x\in G}\{\nu_A(x)\}$ and $< U(\mu_A, \alpha) >, < L(\nu_A, \beta) >$ denote the smallest subgroups of $G$ containing $U(\mu_A, \alpha)$ and $L(\nu_A, \beta)$ respectively. Then $A^*$ is an intuitionistic fuzzy subgroup generated by IF set $A$ in $G$, that is, $A^* =< A >$.*

*Proof.* Let $\{A_i\}_{i\in J}$ be the family of intuitionistic fuzzy subgroups of $G$ containing $A$, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

$$< A >= \bigcap_{i\in J} A_i = \cap(\bigwedge_{i\in J}\mu_{A_i}, \bigvee_{i\in J}\nu_{A_i}).$$

To verify that the family $\{\mu_\alpha : \mu_\alpha$ is a fuzzy subgroup of G containing $\mu_A\}$ is equal to $\{\mu_{A_i}\}_{i\in J}$, note that $\{\mu_{A_i}\}_{i\in J} \subseteq \{\mu_\alpha : \mu_\alpha$ is a fuzzy subgroup of G containing $\mu_A\}$. Otherway, let $\mu_\alpha$ be any fuzzy subgroup of $G$ containing $\mu_A$. Then $(\mu_\alpha, 1-\mu_\alpha)$ is an intuitionistic fuzzy subgroup of $G$ containing $A$. So $(\mu_\alpha, 1 - \mu_\alpha) = A_i$ for some $i \in J$. Hence $\mu_\alpha = \mu_{A_i}$ for some $i \in J$. Therefore, the above equality holds.
Thus,

$$\begin{aligned} \bigcap_{i\in J}\mu_{A_i} &= \cap\mu_\alpha \\ &= \text{smallest fuzzy subgroup of G containing } \mu_A \\ &= < \mu_A > . \end{aligned}$$

By Theorem 3.3,

$$< \mu_A > (x) = \sup_{\alpha \leq t}\{\alpha : x \in < U(\mu_A, \alpha) >\} = \mu_{A^*}(x).$$

Therefore,

$$\bigcap_{i\in J} \mu_{A_i} = \mu_{A^*}.$$

Now,

$$\begin{aligned} 1 - \bigcup_{i\in J} \nu_{A_i} &= \bigcap_{i\in J}(1 - \nu_{A_i}) \\ &= \text{smallest fuzzy subgroup of G containing } 1 - \nu_A \\ &= < 1 - \nu_A > . \end{aligned}$$

Again by Theorem 3.3,

$$< 1 - \nu_A > (x) = \sup_{\alpha \leq t'}\{\alpha : x \in < U(1 - \nu_A, \alpha) >\},$$

where $t^{'} = \sup_{x\in G}\{(1 - \nu_A)(x)\} = 1 - \inf_{x\in G}\{\nu_A(x)\} = 1 - k.$

So

$$< 1 - \nu_A > (x) = \sup_{\alpha \leq 1-k}\{\alpha : x \in < U(1 - \nu_A, \alpha) >\}.$$

Thus,

$$\begin{aligned} (\bigcup_{i\in J}(\nu_{A_i}))(x) &= 1 - (1 - \bigcup_{i\in J} \nu_{A_i})(x) \\ &= 1 - < 1 - \nu_A > (x) \\ &= 1 - \sup_{\alpha \leq 1-k}\{\alpha : x \in < U(1 - \nu_A, \alpha) >\} \\ &= \inf_{1-\alpha \geq k}\{1 - \alpha : x \in < L(\nu_A, 1 - \alpha) >\} \\ &= \inf_{\beta \geq k}\{\beta : x \in < L(\nu_A, \beta) >\} \\ &= \nu_{A^*}(x). \end{aligned}$$

Therefore, $< A >= A^*$. □

Another construction of fuzzy subgroup generated by fuzzy set is given by Ajmal and Jain [1] as follows:

**Theorem 3.5** ([1]). *Let $\mu$ be a fuzzy set in a group $G$. Then the fuzzy set $S(\mu)$ in $G$ defined by*

$$S(\mu)(x) = \sup_{\alpha \in Im\mu}\{\alpha : x \in < \mu_\alpha >\}$$

*is the fuzzy subgroup generated by $\mu$ in $G$. That is $S(\mu) =< \mu >$.*

Following theorem is now easy to prove:

**Theorem 3.6.** *Let $A$ be an intuitionistic fuzzy set in a group $G$. Define an intuitionistic fuzzy set $A^* = (\mu_{A^*}, \nu_{A^*})$ in $G$ as:*

$$\mu_{A^*}(x) = \sup_{\alpha \in Im\mu_A}\{\alpha : x \in < U(\mu_A, \alpha) >\}$$

*and*

$$\nu_{A^*}(x) = \inf_{\beta \in Im\nu_A}\{\beta : x \in < L(\nu_A, \beta) >\}.$$

*Then $A^*$ is an intuitionistic fuzzy subgroup generated by an IF set $A$ in $G$, that is, $A^* =< A >$.*

Construction of $< A >$ can also be given in terms of level subsets and strong level subsets of $A$ as follows:

**Theorem 3.7.** *Let $A$ be an intuitionistic fuzzy set in a group $G$. Let $t = \sup_{x\in G}\{\mu_A(x)\}$ and $k = \inf_{x\in G}\{\nu_A(x)\}$. If $A^{'} = (\mu_{A'}, \nu_{A'})$ is defined by*

$$\mu_{A'}(x) = \sup_{\substack{\alpha\le t\\ \beta\ge k}}\{\alpha : x \in< A_{\alpha,\beta} >\};$$

$$\nu_{A'}(x) = \inf_{\substack{\alpha\le t\\ \beta\ge k}}\{\beta : x \in< A_{\alpha,\beta} >\}.$$

*Then $A^{'}$ is the smallest intuitionistic fuzzy subgroup of $G$ containing $A$. That is, $< A >= A^{'}$.*

*Proof.* First we show that $A^{'}$ is an IFS. Clearly $0 \le \mu_{A'}(x) \le 1$ and $0 \le \nu_{A'}(x) \le 1$ for all $x \in G$. Suppose $\mu_{A'}(x) + \nu_{A'}(x) > 1$ for some $x \in G$. Then $\exists\ \alpha^{'} \le t, \beta^{'} \ge k$ such that $x \in< A_{\alpha',\beta'} >$ and $\alpha^{'} + \nu_{A'}(x) > 1$. This implies $\alpha^{'} + \beta > 1$ for every $\beta$ such that $x \in< A_{\alpha',\beta>}$. This is a contradiction as $x \in< A_{\alpha',\beta'} >=< A_{\alpha',\gamma} >$, where $\gamma \ge k$ such that $\alpha^{'} + \gamma \le 1$. Hence $\mu_{A'}(x) + \nu_{A'}(x) \le 1$, for all $x \in G$. In order to show that $A^{'}$ is an intuitionistic fuzzy subgroup of $G$, firstly we establish that

$$\mu_{A'}(xy) \ge min\{\mu_{A'}(x), \mu_{A'}(y)\}$$

and

$$\nu_{A'}(xy) \le max\{\nu_{A'}(x), \nu_{A'}(y)\},$$

for all $x, y \in G$. Suppose

$$\mu_{A'}(xy) < min\{\mu_{A'}(x), \mu_{A'}(y)\}$$

for some $x, y \in G$. Then,

$$\mu_{A'}(xy) < \mu_{A'}(x) = \sup_{\substack{\alpha\le t\\ \beta\ge k}}\{\alpha : x \in< A_{\alpha,\beta} >\}$$

and

$$\mu_{A'}(xy) < \mu_{A'}(y) = \sup_{\substack{\alpha\le t\\ \beta\ge k}}\{\alpha : y \in< A_{\alpha,\beta} >\}.$$

This implies there exists $\alpha_1, \alpha_2 \le t$ and some $\beta_1, \beta_2 \ge k$ such that

$$x \in< A_{\alpha_1,\beta_1} >\subseteq< A_{\alpha_1,\beta} >;\ y \in< A_{\alpha_2,\beta_2} >\subseteq< A_{\alpha_2,\beta} >,$$

where $\beta = \beta_1 \vee \beta_2$ and $\mu_{A'}(xy) < \alpha_1$, $\mu_{A'}(xy) < \alpha_2$. Now without any loss of generality assume that $\alpha_1 \le \alpha_2$, then

$$< A_{\alpha_2,\beta} >\subseteq< A_{\alpha_1,\beta} > .$$

Therefore,

$$x, y \in< A_{\alpha_1,\beta} > \text{ implies } xy \in< A_{\alpha_1,\beta} >$$

and hence $\mu_{A'}(xy) \geq \alpha_1$ which leads to a contradiction. Thus

$$\mu_{A'}(xy) \geq min\{\mu_{A'}(x), \mu_{A'}(y)\}.$$

Using similar technique, one can easily prove

$$\nu_{A'}(xy) \leq max\{\nu_{A'}(x), \nu_{A'}(y)\}.$$

Further $x \in< A_{\alpha,\beta} >$ if and only if $x^{-1} \in< A_{\alpha,\beta} >$, so for all $x \in G$

$$\mu_{A'}(x) = \mu_{A'}(x^{-1})$$

and

$$\nu_{A'}(x) = \nu_{A'}(x^{-1}).$$

Moreover $A \subseteq A^{'}$, as for each $x \in G$,

$$\mu_A(x) = \underset{\substack{\alpha \leq t \\ \beta \geq k}}{sup}\{\alpha : x \in A_{\alpha,\beta}\} \leq \underset{\substack{\alpha \leq t \\ \beta \geq k}}{sup}\{\alpha : x \in< A_{\alpha,\beta} >\} = \mu_{A'}(x)$$

and

$$\nu_A(x) = \underset{\substack{\alpha \leq t \\ \beta \geq k}}{inf}\{\beta : x \in A_{\alpha,\beta}\} = \underset{\substack{\alpha \leq t \\ \beta \geq k}}{inf}\{\beta : x \in< A_{\alpha,\beta} >\} = \nu_{A'}(x).$$

Finally to complete the proof we show that $A^{'}$ is the smallest intuitionistic fuzzy subgroup containing $A$. For this let $C$ be any intuitionistic fuzzy subgroup of $G$ containing $A$. Let $x \in G$ and $\alpha \leq t, \beta \geq k$ be such that $x \in< A_{\alpha,\beta} >$. Then $x = x_1x_2...x_n$; $x_i \in A_{\alpha,\beta}$ or $x_i^{-1} \in A_{\alpha,\beta}$. Now

$$\begin{array}{rcccl} \mu_C(x) & = & \mu_C(x_1x_2...x_n) & \geq & min\{\mu_C(x_1), \mu_C(x_2), ..., \mu_C(x_n)\} \\ & & & \geq & min\{\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)\} \\ & & & \geq & \alpha. \end{array}$$

Thus

$$\mu_C(x) \geq \alpha \text{ for all } \alpha \leq t : x \in< A_{\alpha,\beta} > .$$

This implies

$$\mu_C(x) \geq \underset{\substack{\alpha \leq t \\ \beta \geq k}}{sup}\{\alpha : x \in< A_{\alpha,\beta} >\} = \mu_{A'}(x).$$

Similarly

$$\nu_C(x) \leq \nu_{A'}(x) \text{ for all } x \in G.$$

Hence $A^{'} \subseteq C$. □

**Theorem 3.8.** *Let $A$ be an intuitionistic fuzzy set in a group $G$. Let $t = \underset{x \in G}{sup}\{\mu_A(x)\}$ and $k = \underset{x \in G}{inf}\{\nu_A(x)\}$. Define intuitionistic fuzzy set $A^{''}$ in $G$ as*

$$\mu_{A''}(x) = \underset{\substack{\alpha < t \\ \beta > k}}{sup}\{\alpha : x \in< A^{>}_{\alpha,\beta} >\};$$

$$\nu_{A''}(x) = \underset{\substack{\alpha < t \\ \beta > k}}{inf}\{\beta : x \in< A^{>}_{\alpha,\beta} >\}.$$

*Then $< A >= A^{''}$.*

*Proof.* Straightforward. □

**Remark 3.9.** In the above Theorem 3.7, the intuitionistic fuzzy subgroup $A^{'} = (\mu_{A'}, \nu_{A'})$ generated by an IFS $A$ in $G$ is given by

$$\mu_{A'}(x) = \sup_{\substack{\alpha\leq t\\ \beta\geq k}}\{\alpha : x \in< A_{\alpha,\beta} >\}; \nu_{A'}(x) = \inf_{\substack{\alpha\leq t\\ \beta\geq k}}\{\beta : x \in< A_{\alpha,\beta} >\}.$$

If we take $B = (\mu_B, \nu_B)$ as

$$\mu_B(x) = \sup_{\substack{\alpha\in[0,1]\\ \beta\in[0,1]}}\{\alpha : x \in< A_{\alpha,\beta} >\}; \nu_B(x) = \inf_{\substack{\alpha\in[0,1]\\ \beta\in[0,1]}}\{\beta : x \in< A_{\alpha,\beta} >\},$$

then $B$ is an intuitionistic fuzzy subgroup of $G$ but not the least, as for $\alpha > t$, $A_{\alpha,\beta} = \varnothing$ implies $< A_{\alpha,\beta} >= \{e\}$. Similarly for $\beta < k$, $< A_{\alpha,\beta} >= \{e\}$. Therefore we always have $\mu_B(e) = 1$ and $\nu_B(e) = 0$, but this may not be true in $< A >$. For example, let $A = (\mu_A, \nu_A)$, where $\mu_A(x) = 1/2$ and $\mu_A(x) = 1/2$ for all $x \in G$. Then $A$ is an intuitionistic fuzzy subgroup of $G$. So $< A >= A$ and here $\mu_{<A>}(e) = 1/2$; $\nu_{<A>}(e) = 1/2$.

**Theorem 3.10.** *Let $A$ be an intuitionistic fuzzy set in a group $G$. Let $t = \sup_{x\in G}\{\mu_A(x)\}$ and $k = \inf_{x\in G}\{\nu_A(x)\}$. Then the following hold:*

(1) $< A^{>}_{p,q} >\subseteq< A >^{>}_{p,q}$, $(p, q) \in [0, t[\times]k, 1]$ *and equality holds if* $p + q \geq 1$.
(2) $< A_{p,q} >\subseteq< A >_{p,q}$, $(p, q) \in [0, t] \times [k, 1]$ *and equality holds if* $p+q \geq 1$ *and* $\mu_A$ *has sup property.*

*Proof.* $(i)$. As $A \subseteq< A >$, so $A^{>}_{p,q} \subseteq< A >^{>}_{p,q}$, for any $(p, q) \in [0, t[\times]k, 1]$. This implies $< A^{>}_{p,q} >\subseteq< A >^{>}_{p,q}$. Otherway, let $(p, q) \in [0, t[\times]k, 1]$ be such that $p+q \geq 1$ and let $x \in< A >^{>}_{p,q}$. Then

$$\mu_{<A>}(x) > p.$$

By Theorem 3.6,

$$\sup_{\alpha\in Im\mu_A}\{\alpha : x \in< U(\mu_A, \alpha) >\} > p.$$

Thus,

$$\exists\, \alpha^{'} \in Im\mu_A \text{ such that } \alpha^{'} > p \text{ and } x \in< U(\mu_A, \alpha^{'}) > .$$

Now $x \in< U(\mu_A, \alpha^{'}) >$ implies $x = x_1x_2...x_n$; $x_i$ or $x_i^{-1} \in U(\mu_A, \alpha^{'})$. If $x_i \in U(\mu_A, \alpha^{'})$, then $\mu_A(x_i) \geq \alpha^{'} > p \geq 1 - q$ and so $\nu_A(x_i) \leq 1 - \mu_A(x_i) < q$. This gives $x_i \in A^{>}_{p,q}$. Similary if $x_i^{-1} \in U(\mu_A, \alpha')$, then $x_i^{-1} \in A^{>}_{p,q}$. So $x \in< A^{>}_{p,q} >$. Hence $< A >^{>}_{p,q}\subseteq< A^{>}_{p,q} >$. Thus result follows.

$(ii)$. Similar to $(i)$. □

The next example illustrates that $< A^{>}_{p,q} >$ need not be equal to $< A >^{>}_{p,q}$ if $p + q < 1$.

**Example 3.11.** Let $G = S_3 = \{I, (12), (13), (23), (123), (132)\}$ be the symmetric group. Define an intuitionistic fuzzy set $A$ on $G$ as:

$\mu_A(I) = 0.1, \mu_A(12) = 0.2, \mu_A(13) = 0.3, \mu_A(23) = 0.4, \mu_A(123) = 0.5, \mu_A(132) = 0.6$

and

$\nu_A(I) = 0.6, \nu_A(12) = 0.5, \nu_A(13) = 0.4, \nu_A(23) = 0.3, \nu_A(123) = 0.3, \nu_A(132) = 0.4.$

Then by simple calculations, one can verify

$$< U(\mu_A, 0.1) >=< U(\mu_A, 0.2) >=< U(\mu_A, 0.3) >=< U(\mu_A, 0.4) >= S_3;$$

$$< U(\mu_A, 0.5) >=< U(\mu_A, 0.6) >= \{I, (123), (132)\}$$

and

$$< L(\nu_A, 0.3) >=< L(\nu_A, 0.4) >=< L(\nu_A, 0.5) >=< L(\nu_A, 0.4) >= S_3.$$

Therefore the intuitionistic fuzzy subgroup generated by $A$ is given by

$$\mu_{<A>}(I) = \mu_{<A>}(123) = \mu_{<A>}(132) = 0.6$$

$$\mu_{<A>}(12) = \mu_{<A>}(13) = \mu_{<A>}(23) = 0.4$$

and

$$\nu_{<A>}(x) = 0.3, \text{ for all } x \in G.$$

If we take $p = 1/2$ and $q = 1/3$, then $A^{>}_{p,q} = \varnothing$ implies $< A^{>}_{p,q} >= \{I\}$. On the other hand, $< A >^{>}_{p,q} = \{I, (123), (132)\}$. So $< A^{>}_{p,q} > \neq < A >^{>}_{p,q}$.

The next two theorems provide the constructions of generated intuitionistic fuzzy $(t, k)$ equivalence relation and congruence.

It can easily be verified that if $\{R_\alpha : \alpha \in \Omega\}$ is a family of intuitionistic fuzzy $(t, k)$ equivalence relations on $X$, then $\underset{\alpha\in\Omega}{\cap} R_\alpha$ is an intuitionistic fuzzy $(t, k)$ equivalence relation on $X$.

An intuitionistic fuzzy $(t, k)$ equivalence relation on a set $X$ generated by an intuitionistic fuzzy relation $R$ on $X$ is defined as the least intuitionistic fuzzy $(t, k)$ equivalence relation on $X$ containing $R$ and is denoted by $< R >$. Similarly, an intuitionistic fuzzy $(t, k)$ congruence on a group $G$ generated by an intuitionistic fuzzy relation $R$ on $G$ is defined as the least intuitionistic fuzzy $(t, k)$ congruence on $G$ containing $R$ and is also denoted by $< R >$.

**Lemma 3.12** ([14]). *Let $R = (\mu_R, \nu_R)$ be an intuitionistic fuzzy equivalence relation on a set $X$ and $\underset{x,y\in X}{sup}\{\mu_R(x, y)\} = t$ and $\underset{x,y\in X}{inf}\{\nu_R(x, y)\} = k$. Then each level subset $R_{\alpha,\beta}$, $(\alpha, \beta) \in [0, t] \times [k, 1]$ is an equivalence relation on $X$.*

**Theorem 3.13.** *Let $R$ be an intuitionistic fuzzy relation on a set $X$. Define an intuitionistic fuzzy relation $R^* = (\mu_{R^*}, \nu_{R^*})$ on $X$ by*

$$\mu_{R^*}(x, y) = \underset{\substack{\alpha\leq t\\ \beta\geq k}}{sup}\{\alpha : (x, y) \in < R_{\alpha,\beta} >\};$$

$$\nu_{R^*}(x, y) = \underset{\substack{\alpha\leq t\\ \beta\geq k}}{inf}\{\beta : (x, y) \in < R_{\alpha,\beta} >\},$$

*where* $t = \underset{x,y\in X}{sup}\{\mu_R(x,y)\}, k = \underset{x,y\in X}{inf}\{\nu_R(x,y)\}$ *and* $< R_{\alpha,\beta} >$ *denotes the smallest equivalence relation containing* $R_{\alpha,\beta}$ *on* $X$*. Then* $R^*$ *is an intuitionistic fuzzy* $(t,k)$ *equivalence relation on* $X$ *generated by* $R$*, that is,* $R^* =< R >$ *.*

*Proof.* First we show that $R^*$ is an intuitionistic fuzzy $(t,k)$ equivalence relation. For any $x \in X$, $(x,x) \in < R_{\alpha,\beta} >$ for all $\alpha \le t$, $\beta \ge k$. So for every $x \in X$,

$$\mu_{R^*}(x,x) = \underset{\substack{\alpha\le t\\ \beta\ge k}}{sup}\{\alpha : (x,x) \in < R_{\alpha,\beta} >\} = t$$

and

$$\nu_{R^*}(x,x) = \underset{\substack{\alpha\le t\\ \beta\ge k}}{inf}\{\beta : (x,x) \in < R_{\alpha,\beta} >\} = k.$$

Hence $R^*$ is IF $(t,k)$ reflexive. Now we establish IF symmetry, i.e.,

$$\mu_{R^*}(x,y) = \mu_{R^*}(y,x)$$

and

$$\nu_{R^*}(x,y) = \nu_{R^*}(y,x)$$

for all $x,y \in X$. Let if possible $\mu_{R^*}(x,y) < \mu_{R^*}(y,x)$ for some $x,y \in X$. Then there exists $\gamma$ such that $\mu_{R^*}(x,y) < \gamma < \mu_{R^*}(y,x)$. Now

$$\gamma < \mu_{R^*}(y,x) = \underset{\substack{\alpha\le t\\ \beta\ge k}}{sup}\{\alpha : (y,x) \in < R_{\alpha,\beta} >\}.$$

This implies there exists $\alpha \le t$ and $\beta \ge k$ such that $(y,x) \in < R_{\alpha,\beta} >$ and $\gamma < \alpha$. So $R_{\alpha,\beta} \subseteq R_{\gamma,\beta}$ and hence $< R_{\alpha,\beta} > \subseteq < R_{\gamma,\beta} >$. Thus $(y,x) \in < R_{\gamma,\beta} >$ and therefore $(x,y) \in < R_{\gamma,\beta} >$. This gives $\mu_{R^*}(x,y) \ge \gamma$, which is a contradiction. Therefore for all $x,y \in X$

$$\mu_{R^*}(x,y) = \mu_{R^*}(y,x).$$

Similarly

$$\nu_{R^*}(x,y) = \nu_{R^*}(y,x).$$

This proves that $R^*$ is IF symmetric. Next we show IF transitivity, i.e

$$\mu_{R^*}(x,y) \ge \mu_{R^*oR^*}(x,y) = \underset{z\in X}{sup}\{min\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\}\}$$

and

$$\nu_{R^*}(x,y) \le \nu_{R^*oR^*}(x,y) = \underset{z\in X}{inf}\{max\{\nu_{R^*}(x,z), \nu_{R^*}(z,y)\}\}$$

for all $x,y \in X$. For this we prove

$$\mu_{R^*}(x,y) \ge min\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\}$$

for all $x,y,z \in X$. Let if possible

$$\mu_{R^*}(x,y) < min\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\}$$

for some $x,y,z \in X$. Then there exists $\gamma$ such that

$$\mu_{R^*}(x,y) < \gamma < min\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\}.$$

Now

$$\gamma < \mu_{R^*}(x,z) = \sup_{\substack{\alpha \le t \\ \beta \ge k}}\{\alpha : (x,z) \in < R_{\alpha,\beta} >\}.$$

This implies there exists $\alpha_1 \le t$ and $\beta_1 \ge k$ such that $(x,z) \in < R_{\alpha_1,\beta_1} >$ and $\gamma < \alpha_1$ which gives $< R_{\alpha_1,\beta_1} > \subseteq < R_{\gamma,\beta_1} >$. Therefore $(x,z) \in < R_{\gamma,\beta_1} >$. Similarly $(z,y) \in < R_{\gamma,\beta_2} >$ for some $\beta_2 \ge k$. Thus $(x,z),(z,y) \in < R_{\gamma,\beta} >$, where $\beta = \beta_1 \vee \beta_2$. Therefore $(x,y) \in < R_{\gamma,\beta} >$ and thus $\mu_{R^*}(x,y) \ge \gamma$, which is a contradiction. Hence

$$\mu_{R^*}(x,y) \ge min\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\} \text{ for all } x,y,z \in X.$$

Thus for all $x,y \in X$

$$\mu_{R^*}(x,y) \ge \sup_{z \in X}\{\mu_{R^*}(x,z), \mu_{R^*}(z,y)\} = \mu_{R^* o R^*}(x,y).$$

Similarly $\nu_{R^* o R^*} \supseteq \nu_{R^*}$. So $R^* o R^* \subseteq R^*$. Therefore $R^*$ is IF $(t,k)$ equivalence relation on $X$. Moreover it can be easily verified that $R \subseteq R^*$. Finally to prove that $R^*$ is the smallest intuitionistic fuzzy $(t,k)$ equivalence relation containing $R$, let $T$ be an intuitionistic fuzzy $(t,k)$ equivalence relation such that $R \subseteq T$. It is required to show $R^* \subseteq T$. Let $x,y \in X$ and $\alpha \le t$ and $\beta \ge k$ be such that $(x,y) \in < R_{\alpha,\beta} >$. Now $R_{\alpha,\beta} \subseteq T_{\alpha,\beta}$ implies $< R_{\alpha,\beta} > \subseteq < T_{\alpha,\beta} > = T_{\alpha,\beta}$, by Lemma 3.12. So $(x,y) \in T_{\alpha,\beta}$ and therefore $\mu_T(x,y) \ge \alpha$ and $\nu_T(x,y) \le \beta$. Thus

$$\mu_T(x,y) \ge \sup_{\substack{\alpha \le t \\ \beta \ge k}}\{\alpha : (x,y) \in < R_{\alpha,\beta} >\} = \mu_{R^*}(x,y)$$

and

$$\nu_T(x,y) \le \inf_{\substack{\alpha \le t \\ \beta \ge k}}\{\beta : (x,y) \in < R_{\alpha,\beta} >\} = \nu_{R^*}(x,y),$$

for all $x,y \in X$. So $R^* \subseteq T$ □

**Theorem 3.14.** *Let $R$ be an intuitionistic fuzzy relation on a group $G$. Let $R^*$ be an intuitionistic fuzzy relation on a group $G$ defined by*

$$\mu_{R^*}(x,y) = \sup_{\substack{\alpha \le t \\ \beta \ge k}}\{\alpha : (x,y) \in < R_{\alpha,\beta} >\};$$

$$\nu_{R^*}(x,y) = \inf_{\substack{\alpha \le t \\ \beta \ge k}}\{\beta : (x,y) \in < R_{\alpha,\beta} >\}$$

*where $t = \sup_{x,y \in G}\{\mu_R(x,y)\}, k = \inf_{x,y \in G}\{\nu_R(x,y)\}$ and $< R_{\alpha,\beta} >$ denotes the smallest congruence containing $R_{\alpha,\beta}$ on $G$. Then $R^*$ is the intuitionistic fuzzy congruence on $G$ generated by $R$, that is, $R^* =< R >$.*

*Proof.* It follows from Theorem 3.7 and Theorem 3.13. □

**Theorem 3.15.** *Let $R$ be an intuitionistic fuzzy relation in a set $X$. Let $t = \sup_{x,y \in X}\{\mu_R(x,y)\}$ and $k = \inf_{x,y \in X}\{\nu_R(x,y)\}$. Then $< R^{>}_{p,q} > \subseteq < R >^{>}_{p,q}$, $(p,q) \in [0,t[\times]k,1]$ and equality holds if $p+q \ge 1$.*
*Here $< R >$ denotes intuitionistic fuzzy $(t,k)$ equivalence relation.*

*Proof.* $(i)$ Let $(p,q) \in [0,t[\times]k,1]$ be such that $p+q \geq 1$ and $(x,y) \in < R >^{>}_{p,q}$. Then

$$\mu_{<R>}(x,y) > p$$

implies

$$\sup_{\substack{\alpha \leq t \\ \beta \geq k}} \{\alpha : (x,y) \in < R_{\alpha,\beta} >\} > p.$$

So there exists $\alpha_1 \leq t, \beta_1 \geq k$ such that $\alpha_1 > p$ and $(x,y) \in < R_{\alpha_1,\beta_1} >$. Now $R_{\alpha_1,\beta_1} \subseteq R_{\alpha_1,1-\alpha_1} \subseteq R^{>}_{p,q}$, as $p+q \geq 1$. So $< R_{\alpha_1,\beta_1} > \subseteq < R^{>}_{p,q} >$. Thus $(x,y) \in < R^{>}_{p,q} >$. Hence result follows. $\square$

Above result also holds for intuitionistic fuzzy $(t,k)$ congruence.

Following example shows $< R^{>}_{p,q} >$ may not be equal to $< R >^{>}_{p,q}$, if $p+q < 1$.

**Example 3.16.** Let $X = \{a,b\}$. Define intuitionistic fuzzy relation $R$ on $X$ as

$$\mu_R(a,a) = 0.1, \mu_R(a,b) = 0.2, \mu_R(b,a) = 0.3, \mu_R(b,b) = 0.4$$

and

$$\nu_R(a,a) = 0.4, \nu_R(a,b) = 0.2, \nu_(b,a) = 0.2, \nu_R(b,b) = 0.3$$

Then intuitionistic fuzzy $(t,k)$ equivalence relation $< R >$ generated by $R$ is given by

$$\mu_{<R>}(a,a) = \mu_{<R>}(b,b) = 0.4$$

$$\mu_{<R>}(a,b) = \mu_{<R>}(b,a) = 0.3$$

and

$$\nu_{<R>}(x,y) = 0.2, \text{ for all } x,y \in X.$$

So for $p = 1/3$ and $q = 0.25$, $< R^{>}_{p,q} >= \{(a,a),(b,b)\}$, whereas $< R >^{>}_{p,q} = X \times X$.

In [14], Thomas and Nair stated that for any intuitionistic fuzzy relation $R$ on a lattice $L$ and for any $\alpha,\beta \in [0,1]$, congruence relation on $L$ generated by strong $(\alpha,\beta)$-cut of $R$ is equal to the strong $(\alpha,\beta)$-cut of the intuitionistic fuzzy congruence on $L$, that is $< R^{>}_{\alpha,\beta} >=< R_{\alpha,\beta} >^{>}$. This assertion is not correct and it can be easily verified as done above that only one sided containment holds, that is, $< R^{>}_{\alpha,\beta} > \subseteq < R >^{>}_{\alpha,\beta}$ and equality holds if $\alpha+\beta \geq 1$. This can be verified by constructing the following example of an IFR $R$ on a lattice $L$ of subsets of $\{a\}$ as follows:

$$\mu_R(\varnothing,\varnothing) = 0.1, \mu_R(\varnothing,\{a\}) = 0.2, \mu_R(\{a\},\varnothing) = 0.3, \mu_R(\{a\},\{a\}) = 0.4$$

and

$$\nu_R(\varnothing,\varnothing) = 0.4, \nu_R(\varnothing,\{a\}) = 0.2, \nu_R(\{a\},\varnothing) = 0.2, \nu_R(\{a\},\{a\}) = 0.3.$$

## 4. Conclusion

In this paper, we have given a new construction of intuitionistic fuzzy subgroup generated by intuitionistic fuzzy set (Theorem 3.4, Theorem 3.7). A similar construction also works as for intuitionistic fuzzy equivalences and intuitionistic fuzzy congruences generated by intuitionistic fuzzy relation in a group (Theorem 3.13, Theorem 3.14).

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