Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 3, (March 2015), pp. 473–487
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On generalized fuzzy ordered AG-groupoids

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Received 27 May 2014; Revised 9 July 2014; Accepted 21 September 2014

Abstract. We introduce and use the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals to study the structural properties of a non-associative algebraic ordered structure. We characterize an intra-regular ordered $\mathcal{AG}$-groupoid by these generalized fuzzy ideals.

2010 AMS Classification: 06F05, 20M12, 08A72

Keywords: Ordered $\mathcal{AG}$-groupoid, Left invertive law, Medial law, Paramedial law and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh in his classic paper [24] of 1965, provides a natural framework for generalizing some of the basic notions of algebra. Kuroki [6] introduced the notion of fuzzy bi-ideals in semigroups. A new type of fuzzy subgroup, that is (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1] by using the notions of "belongingness and quasi-coincidence" of fuzzy points and fuzzy sets. The concepts of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroups [14]. It is now natural to investigate similar type of generalizations of existing fuzzy sub-systems of other algebraic structures. The concept of an $(\in, \in \vee q)$-fuzzy sub-near rings of a near ring introduced by Davvaz in [2]. In [7] Kazanchi and Yamak studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [15] Shabir et. al. characterized regular semigroups by the properties of $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals. In [7] Kazanchi and Yamak defined $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals in semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra. Generalizing the concept of $x_t q f$ Shabir and Jun [16], defined $x_t q_k f$ as $f(x) + t + k > 1$, where $k \in [0, 1)$. In [16], semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

Yousafzai and Khan have introduced the concept of an ordered $\mathcal{AG}$-groupoid and provided the basic theory for an ordered $\mathcal{AG}$ -groupoid in terms of fuzzy subsets [9].

The generalization of an ordered $\mathcal{AG}$-groupoid was also given by Yousafzai et. al. and they introduced the notion of an ordered Γ-$\mathcal{AG}^{**}$-groupoid [23].

The concepts of "$\in_\gamma$" and "q_δ" of fuzzy points and fuzzy sets were first introduced in [18] for studying fuzzy filters of BL-algebras, and then continued in [19, 20, 21, 22]. Besides, in ordered semigroups the concept of intuitionistic fuzzy interior ideal is introduced by Khan et. al [10]. In addition, ordered semigroups are further characterised interms of interval-valued fuzzy filters and fuzzy generalised bi-ideals in [3] and [11].

In this paper we have introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in an ordered $\mathcal{AG}$-groupoid and introduced some new results. We have characterized an intra-regular ordered $\mathcal{AG}$-groupoid by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

The concept of a left almost semigroup ($\mathcal{LA}$-semigroup)[8] was first introduced by Kazim and Naseeruddin in 1972. In [4], the same structure was called a left invertive groupoid. Protic and Stevanovic [13] called it an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid). An $\mathcal{AG}$-groupoid is a groupoid $\mathcal{S}$ whose elements satisfy the left invertive law $(ab)c = (cb)a$, $\forall\ a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid, the medial law [8] $(ab)(cd) = (ac)(bd)$, $\forall\ a, b, c, d \in \mathcal{S}$ holds. An $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of an $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in an $\mathcal{AG}$-groupoid. If an $\mathcal{AG}$-groupoid contains a left identity, then it is unique [12]. In an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$, $\forall\ a, b, c, d \in \mathcal{S}$ holds. If an $\mathcal{AG}$-groupoid contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, $\forall\ a, b, c \in \mathcal{S}$. If an $\mathcal{AG}$-groupoid $\mathcal{S}$ satisfy $a(bc) = b(ac)$, $\forall\ a, b, c \in \mathcal{S}$ without left identity, then $\mathcal{S}$ is called an $\mathcal{AG}^{**}$-groupoid. Several examples and interesting properties of $\mathcal{AG}$-groupoids can be found in [12] and [17].

2. Preliminaries

An ordered $\mathcal{AG}$-groupoid ($\mathfrak{po}$-$\mathcal{AG}$-groupoid) is a structure $(G, \cdot, \leq)$ in which the following conditions hold [9]:

(i) $(G, \cdot)$ *is an* $\mathcal{AG}$*-groupoid.*
(ii) $(G, \leq)$ *is a poset.*
(iii) $\forall\ a, b, x \in G$, $a \leq b \Rightarrow ax \leq bx$ $(xa \leq xb)$.

Example 2.1. Define a new binary operation "$\circ_e$" (e-sandwich operation) on an ordered $\mathcal{AG}$-groupoid $(\mathcal{S}, \cdot, \leq)$ with left identity e as follows:

$$a \circ_e b = (ae)b\ \forall\ a, b \in \mathcal{S}.$$

Then $(\mathcal{S}, \circ_e, \leq)$ *becomes an ordered semigroup.*

Example 2.2. *Let* $G = \{a, b, c\}$ *be an ordered* $\mathcal{AG}$*-groupoid with the following multiplication table and two differnet orders below*:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

$$(1) \qquad \leq := \{(a, a), (b, b), (c, c), (c, a), (c, b)\}$$

$$(2) \qquad \leq := \{(a, a), (b, b), (c, c), (a, c), (a, b)\}$$

An ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup. If an ordered $\mathcal{AG}$-groupoid has a right identity, then it becomes an ordered semigroup.

Let A be a non-empty subset an of ordered $\mathcal{AG}$-groupoid G, then

$$(A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we usually written as $(a]$.

Let G be an ordered $\mathcal{AG}$-groupoid. By a left (right) ideal of G, we mean a non-empty subset A of G such that $(GA] \subseteq A$ $((AG] \subseteq A)$. By two-sided ideal or simply ideal, we mean a non-empty subset A of G which is both a left and a right ideal of G.

An $\mathcal{AG}$-subgroupoid A of G is called a bi-ideal of G if $((AG)A] \subseteq A$.

A non-empty subset A of G is called a generalized bi-ideal of G if $((AG)A] \subseteq A$.

A non-empty subset A of G is called an interior-ideal of G if $((GA)G] \subseteq A$.

An element a of an ordered $\mathcal{AG}$-groupoid G is called intra-regular element of G if there exists $x \in G$ such that $a \leq (xa^2)y$ and G is called an intra-regular, if every element of G is intra-regular or equivalently, $A \subseteq ((GA^2)G]$, $\forall\ A \subseteq G$ [9].

A fuzzy subset f of a given set G is described as an arbitrary function $f : G \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of G, $f \subseteq g$ means that, $f(x) \leq g(x)$, $\forall\ x \in G$.

Let f and g be any fuzzy subsets of an ordered $\mathcal{AG}$-groupoid G, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \lim_{a \leq bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in G, \text{ such that } a \leq bc \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy ordered $\mathcal{AG}$-subgroupoid of G if $f(xy) \geq f(x) \wedge f(y)$, $\forall\ x, y \in G$.

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy left (right) ideal of G if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$, $\forall\ x, y \in G$.

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy ideal of G if it is both fuzzy left and fuzzy right ideal of G.

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy generalized bi-ideal of G if $f((xy)z) \geq f(x) \wedge f(z)$, $\forall\ x, y$ and $z \in G$.

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy interior-ideal of G if $f((xy)z) \geq f(y)$, $\forall\ x, y, z \in G$.

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy quasi-ideal of G if $f \circ G \cap G \circ f \subseteq f$.

Let $\mathcal{F}(G)$ denotes the collection of all fuzzy subsets of an ordered $\mathcal{AG}$-groupoid G. Then $(\mathcal{F}(G), \circ)$ becomes an ordered $\mathcal{AG}$-groupoid [9].

The characteristic function χ_A for a non-empty set A of an ordered $\mathcal{AG}$-groupoid G is defined as follow:

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G which is defined as follow:

$$f(y) = \begin{cases} r(\neq 0), & if\ y \leq x \\ 0, & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0, 1]$.

In what follows let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X^{\delta}_{\gamma B}$ be the fuzzy subset of X by $X^{\delta}_{\gamma B}(x) \geq \delta$ if $x \in B$ and $X^{\delta}_{\gamma B}(x) \leq \gamma$, $\forall\ x \notin B$. Otherwise, clearly $X^{\delta}_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G, we say that:

(i) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(ii) $x_r q_\delta f$ if $f(x) + r > 2\delta$.

(iii) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(G)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows:

For any $f, g \in \mathcal{F}(G)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$, we mean that $x_r \in_\gamma f \Longrightarrow x_r \in_\gamma \vee q_\delta g$, $\forall\ x \in G$ and $r \in (\gamma, 1]$.

Moreover f and g are said to be (γ, δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

Lemma 2.3 ([5]). *In an ordered $\mathcal{AG}$-groupoid G, the following are true.*

(i) $A \subseteq (A], \forall\ A \subseteq G$.

(ii) $A \subseteq B \subseteq G \Longrightarrow (A] \subseteq (B], \forall\ A, B \subseteq G$.

(iii) $(A](B] \subseteq (AB], \forall\ A, B \subseteq G$.

(iv) $(A] = ((A]], \forall\ A \subseteq G$.

(vi) $((A](B]] = (AB], \forall\ A, B \subseteq G$.

Lemma 2.4 ([5]). *A non-empty subset A of an ordered $\mathcal{AG}$-groupoid G with left identity is a left ideal of $G \Longleftrightarrow$ it is a right ideal of G.*

Definition 2.5. A non-empty subset A of an ordered $\mathcal{AG}$-groupoid G is called semiprime if $a^2 \in A \Longrightarrow a \in A$. A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called semiprime if $\max f(a) \geq \min f(a^2), \forall\ a \in G$.

Lemma 2.6 ([5]). *Every right ideal of an intra-regular ordered $\mathcal{AG}$-groupoid G with left identity is semiprime.*

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy ideals of ordered $\mathcal{AG}$-groupoids

Lemma 3.1. *Let $f, g, h \subseteq \mathcal{F}(G)$ and $\gamma, \delta \in [0, 1]$. Then*

(i) $f \subseteq \vee q_{(\gamma,\delta)} g$ ($f \supseteq \vee q_{(\gamma,\delta)} g$) $\Leftrightarrow \max\{f(x), \gamma\} \leq \min\{g(x), \delta\}$ ($\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$), $\forall\ x \in G$.

(ii) If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.

Proof. It is simple. □

Corollary 3.2. *Let $\mathcal{F}(G)$ denote the set of all fuzzy sub sets of G. Then " $=_{(\gamma,\delta)}$ " is an equivalence relation on $\mathcal{F}(G)$.*

By Lemma 3.1, it is also notified that $f = \vee q_{(\gamma,\delta)} g \Leftrightarrow \max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$, $\forall\, x \in G$, where $\gamma, \delta \in [0, 1]$.

Lemma 3.3. *Let A and B be any subsets of an ordered $\mathcal{AG}$-groupoid G, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$. Then*
(1) $A \subseteq B \Leftrightarrow \chi^{\delta}_{\gamma(A)} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma(B)}$.
(2) $\chi^{\delta}_{\gamma(A)} \cap \chi^{\delta}_{\gamma(B)} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma(A\cap B)}$.
(3) $\chi^{\delta}_{\gamma(A)} \circ \chi^{\delta}_{\gamma(B)} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma(AB]}$.

Proof. (1) : Assume that A and B are any subsets of an ordered $\mathcal{AG}$-groupoid G. Let for any $x \in G$ such that $x \in A \subseteq B$. Then $\chi^{\delta}_{\gamma(B)} \geq \delta \rightarrow (i)$. Let $x_r \in_{\gamma} \chi^{\delta}_{\gamma(A)}$, it follows $\chi^{\delta}_{\gamma(A)}(x) \geq r > \gamma$. Now either $\delta \geq r$ or $\delta < r$, and by using (i), we have $\chi^{\delta}_{\gamma(A)} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma(B)}$.

Conversely, let $\chi^{\delta}_{\gamma(A)} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma(B)}$ and $x \in A$, it follows $\chi^{\delta}_{\gamma(A)} \geq \delta$. Let $x_r \in_{\gamma} \chi^{\delta}_{\gamma(A)} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma(B)}$, where $\chi^{\delta}_{\gamma(A)}$ and $\chi^{\delta}_{\gamma(B)}$ are any fuzzy subsets of G. Thus $x_r \in_{\gamma} \chi^{\delta}_{\gamma(A)}$, $x_r \in_{\gamma} \chi^{\delta}_{\gamma(B)}$ or $x_r q_{\delta} \chi^{\delta}_{\gamma(B)}$. As $x_r \in_{\gamma} \chi^{\delta}_{\gamma(A)} \Longrightarrow \chi^{\delta}_{\gamma(A)}(x) \geq r > \gamma$ and $\chi^{\delta}_{\gamma(B)}(x) \geq r > \gamma$ or $\chi^{\delta}_{\gamma(B)}(x) + \delta > 2\delta \rightarrow (ii)$. We have to discuss two cases for (ii).

Case (a) : if $r < \delta$, then

$$(ii) \Rightarrow \chi^{\delta}_{\gamma(B)}(x) \geq 2\delta - r > \delta \Longrightarrow \chi^{\delta}_{\gamma(B)}(x) > \delta \Longrightarrow x_r \in_{\gamma} \chi^{\delta}_{\gamma(B)}.$$

Case(b) : if $r \geq \delta$, then

$$(ii) \Rightarrow \chi^{\delta}_{\gamma(B)}(x) \geq r \geq \delta \Longrightarrow \chi^{\delta}_{\gamma(B)}(x) \geq \delta \Longrightarrow x \in B.$$

Hence, $A \subseteq B$.
(2) : It is simple.
(3) : It is simple. □

Corollary 3.4. *Let G be an ordered $\mathcal{AG}$-groupoid and γ, γ_1, δ, $\delta_1 \in [0, 1]$ such that $\gamma < \delta$, $\gamma_1 < \delta_1$, $\gamma < \gamma_1$ and $\delta_1 < \delta$. Then any $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal of G is an $(\in_{\gamma_1}, \in_{\gamma_1} \vee q_{\delta_1})$-fuzzy left ideal over G.*

Definition 3.5. A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy $\mathcal{AG}$-subgroupoid of G if for all $a, b \in G$ and $s, t \in (\gamma, 1]$, the following conditions hold:
(i) If $a \leq b$ and $b_t \in_{\gamma} f \Longrightarrow a_t \in_{\gamma} \vee q_{\delta} f$.
(ii) If $a_t \in_{\gamma} f$ and $b_t \in_{\gamma} f \Longrightarrow (ab)_{\min\{t,s\}} \in_{\gamma} \vee q_{\delta} f$.

Let us consider an example 2.2 of an ordered $\mathcal{AG}$-groupoid with order (1). Let $\gamma = 0.45$ and $\delta = 0.49$. Define a fuzzy subset $f : G \rightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = a \\ 0.99 \text{ for } x = b \\ 0.84 \text{ for } x = c \end{cases}.$$

(1) *Let us consider all the possible cases for* $t \in (0.45, 1]$ *as follows*:

(*i*) *When* $t \in (0.45, 0.84]$, *it follows* $x_t \in_\gamma f$. *It is easy to see that* $x_t \in_\gamma f$ *and* $y \leq x \Longrightarrow y_t \in_\gamma f$ *for all* $x \in G$.

(*ii*) *When* $t \in (0.84, 0.9]$, *it follows* $c_t \bar{\in}_\gamma f$ *while* $a_t \in_\gamma f$ *and* $b_t \in_\gamma f$. *Now* $c \leq a$ *and* $a_t \in_\gamma f \Longrightarrow f(a) \geq t > \gamma$. *Consider* $f(c) + t > 0.84 + 0.84 = 1.64 > 2\delta$. *Hence* $c \leq a$ *and* $a_t \in_\gamma f \Longrightarrow c_t q_\delta f$. *Similarly* $b_t \in_\gamma f$ *and* $c \leq b \Longrightarrow c_t q_\delta f$.

(*iii*) *When* $t \in (0.9, 0.99]$, *it follows* $b_t \in_\gamma f$ *while* $a_t \bar{\in}_\gamma f$ *and* $c_t \bar{\in}_\gamma f$.*It is easy to verify that* $b_t \in_\gamma f$ *and* $c \leq b \Longrightarrow c_t q_\delta f$.

(*iv*) *When* $t \in (0.99, 1]$, *it follows* $x_t \bar{\in}_\gamma f$ *for all* $x \in G$. *Hence nothing to show in this case.*

(2) *Now let us consider some basic comparisons as follows*:

(*i*) $a_t \in_\gamma f$, $a_s \in_\gamma f \Longrightarrow (aa)_{\min\{s,t\}} \in_\gamma f$.

(*ii*) $a_t \in_\gamma f$, $b_s \in_\gamma f \Longrightarrow (ab)_{\min\{s,t\}} \in_\gamma f$.

(*iii*) $a_t \in_\gamma f$, $c_s \in_\gamma f \Longrightarrow (ac)_{\min\{s,t\}} q_\delta f$.

(*iv*) $ba = ab$ *in the table.*

(*v*) $b_t \in_\gamma f$, $b_s \in_\gamma f \Longrightarrow (bb)_{\min\{s,t\}} \in_\gamma f$.

(*vi*) $b_t \in_\gamma f$, $c_s \in_\gamma f \Longrightarrow (bc)_{\min\{s,t\}} \in_\gamma f$.

(*vii*) $ca = ac$ *in the table.*

(*viii*) $c_t \in_\gamma f$, $b_s \in_\gamma f \Longrightarrow (cb)_{\min\{s,t\}} \in_\gamma f$.

(*ix*) $c_t \in_\gamma f$, $c_s \in_\gamma f \Longrightarrow (cc)_{\min\{s,t\}}. \in_\gamma f$.

Hence, f *is* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* $\mathcal{AG}$*-subgroupoid of* G.

Theorem 3.6. *A fuzzy subset* f *of an ordered* $\mathcal{AG}$*-groupoid* G *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* $\mathcal{AG}$*-subgroupoid if for all* $a, b \in G$ *and* $t \in (\gamma, 1]$, *the following conditions hold:*

(1) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with* $a \leq b$.

(2) $\max\{f(ab), \gamma\} \geq \min\{f(a), f(b), \delta\}$.

Proof. It is simple. □

Definition 3.7. A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if for all $a, b \in G$ and $t \in (\gamma, 1]$, the following conditions hold:

(*i*) If $a \leq b$ and $b_t \in_\gamma f$, it follows $a_t \in_\gamma \vee q_\delta f$.

(*ii*) If $b_t \in_\gamma f$, it follows $(ab)_t \in_\gamma \vee q_\delta f$ $(a_t \in_\gamma f \Longrightarrow (ab)_t \in_\gamma \vee q_\delta f)$.

Let us consider an example 2.2 *of an ordered* $\mathcal{AG}$*-groupoid with order* (2). *Let* $\gamma = 0.4$ *and* $\delta = 0.5$. *Define a fuzzy subset* $f : G \to [0, 1]$ *as follows*:

$$f(x) = \begin{cases} 0.7 \text{ for } x = a \\ 0.8 \text{ for } x = b \\ 0.9 \text{ for } x = c \end{cases} .$$

(1) *Let us consider all the possible cases for* $t \in (0.4, 1]$ *as follows*:

(*i*) *When* $t \in (0.4, 0.7]$, *it follows* $x_t \in_\gamma f$ *for all* $x \in G$. *It is easy to see that* $x_t \in_\gamma f$ *and* $y \leq x \Longrightarrow y_t \in_\gamma f$ *for all* $x \in G$.

(*ii*) *When* $t \in (0.7, 0.8]$, *it follows* $a_t \bar{\in}_\gamma f$ *while* $c_t \in_\gamma f$ *and* $b_t \in_\gamma f$. *Now* $a \leq c$ *and* $c_t \in_\gamma f \Longrightarrow f(a) \geq t > \gamma$. *Proceeding in the same way as in above example we get* $a_t q_\delta f$, *and Similar solution for* $a \leq b$.

(*iii*) *When* $t \in (0.8, 0.9]$, *it follows* $c_t \in_\gamma f$ *while* $a_t \bar{\in}_\gamma f$ *and* $b_t \bar{\in}_\gamma f$.*It is easy to verify that* $c_t \in_\gamma f$ *and* $a \leq c \Longrightarrow a_t q_\delta f$.

(iv) When $t \in (0.9, 1]$, it follows $\bar{x}_t \in_\gamma f$ for all $x \in G$. Nothing to show in this case.

(2) Again considering all possible cases for $t \in (0.4, 1]$

(i) When $t \in (0.4, 0.7]$, it follows $x_t \in_\gamma f$ for all $x \in G$. It is easy see that $(xy)_t \in_\gamma f$ for all $x \in G$ in this case.

(ii) When $t \in (0.7, 0.8]$, it follows $a_t \bar{\in}_\gamma f$ while $c_t \in_\gamma f$ and $b_t \in_\gamma f$. Now $b_t \in_\gamma f \Longrightarrow (ab)_t q_\delta f$, $(bb)_t q_\delta f$ and $(bc)_t q_\delta f$. Similarly $c_t \in f \Longrightarrow (ac)_t q_\delta f$, $(bc)_t \in_\gamma f$ and $(cc)_t q_\delta f$.

(iii) When $t \in (0.8, 0.9]$, it follows $c_t \in_\gamma f$ while $a_t \bar{\in}_\gamma f$ and $b_t \bar{\in}_\gamma f$. Now $c_t \in f \Longrightarrow (ac)_t q_\delta f$, $(bc)_t \in_\gamma f$ and $(cc)_t q_\delta f$.

(iv) When $t \in (0.9, 1]$, it follows $\bar{x}_t \in_\gamma f$ for all $x \in G$. Again nothing to solve in this case.

Hence, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Theorem 3.8. *A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if for all $a, b \in G$ and $\gamma, \delta \in [0, 1]$, the following conditions hold:*

(1) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with* $a \leq b$.

(2) $\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}$.

Proof. $(i) \Leftrightarrow (1)$: It is same as in Theorem 3.6.

$(ii) \Rightarrow (2)$: If there exists $a, b \in G$ such that

$$\max\{f(ab), \gamma\} < \min\{f(b), \delta\}.$$

Then

$$\max\{f(ab), \gamma\} < t \leq \min\{f(b), \delta\} \text{ for some } t \in (\gamma, 1].$$

It follows that $b_t \in_\gamma f$ but $(ab)_t \overline{\in}_\gamma f$ and $(ab)_t \overline{q_\delta} f$, a contradiction and hence $\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}$ for all $a, b \in G$.

$(2) \Rightarrow (ii)$: Assume that $a, b \in G$ and $t, s \in (\gamma, 1]$ such that $b_t \in_\gamma f$, then by definition we can write $f(b) \geq t > \gamma$, therefore

$$\max\{f(ab), \delta\} \geq \min\{f(b), \delta\} \geq \min\{t, \delta\}.$$

We have to consider two cases here:

Case(a): If $t \leq \delta$, then

$$f(ab) \geq t > \gamma \Longrightarrow (ab)_t \in_\gamma f.$$

Case(b): If $t > \delta$, then

$$f(ab) + t > 2\delta \Longrightarrow (ab)_t q_\delta f.$$

From both cases, we have $(ab)_t \in_\gamma \vee q_\delta f$, $\forall\ a, b \in G$. □

Lemma 3.9. *Let f be a fuzzy subset of an ordered $\mathcal{AG}$-groupoid G and $\gamma, \delta \in [0, 1]$. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if and only if f satisfies the following conditions.*

(i) $x \leq y \Rightarrow \max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$, $\forall\ x, y \in G$.

(ii) $S \circ f \subseteq \vee q_{(\gamma,\delta)} f$ and $f \circ S \subseteq \vee q_{(\gamma,\delta)} f$ $(S \circ f \subseteq \vee q_{(\gamma,\delta)} f$ and $f \circ S \subseteq \vee q_{(\gamma,\delta)} f)$.

Proof. It is simple. □

Definition 3.10. A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if for all $x, y, z \in G$ and $s, t \in (\gamma, 1]$, the following conditions hold:

(i) $a \leq b$ and $b_t \in_\gamma f \Longrightarrow a_t \in_\gamma \vee q_\delta f$.
(ii) $x_t \in_\gamma f$ and $y_s \in_\gamma f \Longrightarrow (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.
(iii) $x_t \in_\gamma f$ and $z_s \in_\gamma f \Longrightarrow ((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

Theorem 3.11. *A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if for all $x, y, z \in G$, $s, t \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$, the following conditions hold:*

(1) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with* $a \leq b$.
(2) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.
(3) $\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof. It is simple. □

Lemma 3.12. *A non-empty subset B of an ordered $\mathcal{AG}$-groupoid G is a bi-ideal of $G \Leftrightarrow \chi^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G, where $\gamma, \delta \in [0, 1]$.*

Proof. Let B be a bi-ideal of G and assume that $x, y \in B$. Then for any $a \in G$, we have $(xa)y \in B$, thus $\chi^\delta_{\gamma B}((xa)y) \geq \delta > \gamma$ and therefore $\chi^\delta_{\gamma B}(x) \geq \delta, \chi^\delta_{\gamma B}(y) \geq \delta$, this shows that $\chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \geq \delta$. Thus

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}((xa)y)\chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B} \wedge \delta = \delta.$$

Hence, $\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta$.

Let $x \in B$, $y \notin B$.Then

$$(xa)y \notin B, \forall\ a \in G \Longrightarrow \chi^\delta_{\gamma B}((xa)y) \leq \gamma < \delta,\ \chi^\delta_{\gamma B}(x) \geq \delta > \gamma \text{ and } \chi^\delta_{\gamma B}(y) < \gamma < \delta.$$

Therefore

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \gamma \text{ and } \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta = \chi^\delta_{\gamma B}(y).$$

Hence, $\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta$.

Let $x \notin B$, $y \in B$.Then

$$\begin{aligned} (xa)y &\notin B, \forall\ a \in G \Longrightarrow \chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \delta > \gamma, \\ \chi^\delta_{\gamma B}(x) &< \delta, \chi^\delta_{\gamma B}(y) \geq \delta \text{ and } \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta = \chi^\delta_{\gamma B}(x). \end{aligned}$$

Therefore

$$\chi^\delta_{\gamma B}((xa)y) \vee \delta \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta.$$

Let $x, y \notin B$.Then

$$(xa)y \notin B, \forall\ a \in G \Longrightarrow \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \leq \gamma \text{ and } \chi^\delta_{\gamma B}((xa)y) \leq \gamma.$$

Thus

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma = \gamma \text{ and } \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta \leq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \leq \gamma.$$

Hence, $\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta$. Converse is simple. □

Lemma 3.13. *Let A be a non-empty set of an ordered $\mathcal{AG}$-groupoid G. Then A is a left (right, two-sided) ideal of $G \Leftrightarrow \chi^\delta_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, two-sided) ideal of G, where $\gamma, \delta \in [0, 1]$.*

Proof. It is simple. □

Example 3.14. *Let $S = \{0,1,2,3\}$ be an ordered $\mathcal{AG}$-groupoid. Define the following multiplication table and ordered below.*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$$\leq := \{(0,0),(1,1),(2,2),(3,3),(0,1)\}$$

Define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.75 \text{ for } x = 0 \\ 0.65 \text{ for } x = 1 \\ 0.7 \text{ for } x = 2 \\ 0.5 \text{ for } x = 3 \end{cases}$$

Then clearly f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy left ideal of S.

$$\leq := \{(a,b),(a,c),(a,d)\}$$

Again define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = 0 \\ 0.7 \text{ for } x = 1 \\ 0.6 \text{ for } x = 2 \\ 0.5 \text{ for } x = 3 \end{cases}$$

Then f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.5})$ -fuzzy bi-ideal.

4. Characterizations of intra-regular ordered AG-groupoids in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

Lemma 4.1. *Let G be an ordered $\mathcal{AG}$-groupoid, then the following are true.*

(i) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideal of an intra-regular G with left identity is semiprime.

(ii) A non-empty subset R of G is a right ideal of $G \Longleftrightarrow \chi_R$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of G.

Proof. It is simple. □

Theorem 4.2. *The following conditions are equivalent for an ordered $\mathcal{AG}$-groupoid G with left identity.*

(i) G is intra-regular.

(ii) $R \cap L = (RL]$, where R is any right ideal and L is any left ideal of G such that R is semiprime.

(iii) $f \cap g = \vee q_{(\gamma,\delta)} f \circ g$, where f is any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideal and g is any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G such that f is semiprime.

Proof. $(i) \Longrightarrow (iii)$: Let f be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G with left identity. Now for $a \in G$ there exist $b, c \in G$ such that $a \leq (ba^2)c$. Therefore

$$\begin{aligned} a &\leq (b(aa))c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c)))a = (c((ba^2)(bc)))a \\ &= (c((cb)(a^2b)))a = (c(a^2((cb)b)))a = (a^2(c((cb)b)))a. \end{aligned}$$

Thus $(a^2(c((cb)b)), a) \in A_a$, since $A_a \neq \varnothing$, therefore

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left[\lim_{(a^2(c((cb)b)),a)\in A_a} \{f(a^2(c((cb)b))) \wedge g(a)\}, \gamma\right] \\ &\geq \max\left[\min\{f(a^2(c((cb)b))), g(a)\}, \gamma\right] \\ &= \min\left[\max\{f(a^2(c((cb)b))), \gamma\}, \max\{g(a), \gamma\}\right] \\ &\geq \min\left[\min\{f(a), \delta\}, \min\{g(a)\}, \delta\right] \\ &= \min\{(f \cap g)(a), \delta\}, \end{aligned}$$

This shows that $f \circ g \supseteq \vee q_{(\gamma,\delta)} f \cap g$. Now by using Lemma 3.9, $f \circ g \subseteq \vee q_{(\gamma,\delta)} f \cap g$, and by using Lemma 4.1, f is semiprime.

$(iii) \Longrightarrow (ii)$: Let R be any right ideal and L be any left ideal of G, then by Lemma 3.13, $C^\delta_{\gamma R}$ and $C^\delta_{\gamma L}$ are the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of G respectively such that $C^\delta_{\gamma R}$ is semiprime. $(RL] \subseteq R \cap L$ is obvious [9]. Let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 3.3 and given assumption, we have

$$C^\delta_{\gamma_{(RL]}}(a) =_{(\gamma,\delta)} (C^\delta_{\gamma R} \circ C^\delta_{\gamma L})(a) =_{(\gamma,\delta)} (C^\delta_{\gamma R} \cap C^\delta_{\gamma L})(a) =_{(\gamma,\delta)} C^\delta_{\gamma R}(a) \wedge C^\delta_{\gamma L}(a) = 1,$$

This implies that $a \in (RL]$ and therefore $R \cap L = (RL]$. Now by using Lemma 2.6, R is semiprime.

$(ii) \Longrightarrow (i)$: Clearly $(Ga]$ and $(a^2G]$ are the left and right ideals of G with left identity [9] such that $a \in (Ga]$ and $a^2 \in (a^2G]$. Since by assumption, $(a^2G]$ is semiprime, therefore $a \in (a^2G]$. Now by using Lemma 2.3, we have

$$\begin{aligned} a &\in (a^2G] \cap (Ga] = ((a^2G](Ga]] \subseteq ((a^2G)(Ga)] = ((aG)(Ga^2)] \\ &= (((Ga^2)G)a] = (((Ga^2)(eG))a] \subseteq (((Ga^2)(GG))a] \\ &= (((GG)(a^2G))a] = ((a^2((GG)G))a] \subseteq ((a^2G)G] \\ &= ((GG)(aa)] = ((aa)(GG)] \subseteq ((aa)G] = ((Ga)a] \\ &\subseteq ((Ga)(a^2G)] = (((a^2G)a)G] = (((aG)a^2)G] \subseteq ((Ga^2)G], \end{aligned}$$

This shows that G is intra-regular. □

Theorem 4.3. *The following conditions are equivalent for an ordered $\mathcal{AG}$ -groupoid G with left identity.*

(i) G is intra-regular.

(ii) $f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f$, where both f and g are any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of G.

(iii) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G is idempotent.

(iv) Every bi-ideal of G is idempotent.

Proof. $(i) \implies (ii)$: Let f and g be both $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of an intra-regular ordered $\mathcal{AG}$-groupoid G with left identity. Then for every $a \in G$, there exist $b, c \in G$ such that $a \leq (ba^2)c$.

$$\begin{aligned} a &\leq (ba^2)c = (b(aa))c = (a(ba))c = (c(ba))a \\ &\leq (c(b((ba^2)c)))a = (c(b((b(aa))c)))a \\ &= (c(b((a(ba))c)))a = (c(a((ba))(bc)))a \\ &= ((a(ba))(c(bc)))a = (((c(bc))(ba))a)a \\ &\leq (((c(bc))(b((ba^2)c)))a)a = (((c(bc))((ba^2)(bc)))a)a \\ &= (((ba^2)((c(bc))(bc)))a)a = ((((bc)(c(bc)))(a^2b))a)a \\ &= ((a^2(((bc)(c(bc)))b))a)a = (((aa)(((bc)(c(bc)))b))a)a \\ &= (((b((bc)(c(bc))))(aa))a)a = ((a((b((bc)(c(bc))))a))a)a. \end{aligned}$$

Thus $((a((b((bc)(c(bc))))a))a, a) = (v, a) \in A_a$. Since $A_a \neq \varnothing$,

$$\begin{aligned} \max\{(g \circ f)(a), \gamma\} &= \max\left[\lim_{(v,a)\in A_a} \{g(v) \wedge f(a)\}, \gamma\right] \\ &\geq \max\left[\min\{g(v), f(a)\}, \gamma\right] \\ &\geq \min\left[\max\{g(v, \gamma\}, \max\{f(a), \gamma\}\right] \\ &\geq \min\left[\min\{g(a), \delta\}, \min\{f(a)\}, \delta\right] \\ &= \min\{(g \cap f)(a), \delta\}, \end{aligned}$$

thus

$$g \circ f \supseteq \vee q_{(\gamma,\delta)} g \cap f = \vee q_{(\gamma,\delta)} f \cap g \Rightarrow f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f.$$

$(ii) \implies (iii)$: Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G,

$$f \cap f \subseteq \vee q_{(\gamma,\delta)} f \circ f \subseteq \vee q_{(\gamma,\delta)} f \implies f = \vee q_{(\gamma,\delta)} f \circ f.$$

This implies that f is idempotent.

$(iii) \implies (iv)$: Let B be a bi-ideal of G such that $b \in B$. Then by using Lemma 3.12, $C^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G. Now by using Lemma 3.3, we have

$$1 = C^{\delta}_{\gamma B}(b) = \vee q_{(\gamma,\delta)}(C_B \circ C_B)(b) = \vee q_{(\gamma,\delta)} C_{(B^2]}(b),$$

This shows that $b \in (B^2]$, therefore $B \subseteq (B^2]$ and $(B^2] \subseteq B$ is obvious. Hence $B = (B^2]$.

$(iv) \implies (i)$: Clearly $(Ga]$ is a bi-ideal of G with left identity, therefore by using given assumption and Lemma 2.3, we have

$$\begin{aligned} a &\in (Ga] = ((Ga](Ga]] = ((Ga)(Ga)] = ((aG)(aG)] \\ &= ((aa)(GG)] = ((ea^2)(GG)] \subseteq ((Ga^2)G]. \end{aligned}$$

Therefore, G is intra-regular. □

Theorem 4.4. *The following conditions are equivalent for an ordered $\mathcal{AG}$-groupoid G with left identity.*

(i) G is intra-regular.

(ii) $A \cap B \subseteq (BA]$, where both A and B are any left ideals of G.

(iii) $f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f$, *where both* f *and* g *are any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideals of* G.

Proof. $(i) \implies (iii)$: Let f and g be both $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of an intra-regular ordered $\mathcal{AG}$-groupoid G with left identity. Now for any $a \in G$, there exist $b,c \in G$ such that $a \leq (ba^2)c$, then

$$a \leq (ba^2)c = (b(aa))c = (a(ba))c = (c(ba))a.$$

Thus $(c(ba), a) \in A_a$. Since $A_a \neq \varnothing$,

$$\begin{aligned} \max\{(g \circ f)(a), \gamma\} &= \max\left[\lim_{(c(ba),a)\in A_a} \{g(c(ba)) \wedge f(a)\}, \gamma\right] \\ &\geq \max\left[\min\{g(c(ba)), f(a)\}, \gamma\right] \\ &= \min\left[\max\{g(c(ba)), \gamma\}, \max\{f(a), \gamma\}\right] \\ &\geq \min\left[\min\{g(a), \delta\}, \min\{f(a)\}, \delta\right] \\ &= \min\{(f \cap g)(a), \delta\}, \end{aligned}$$

This shows that $g \circ f \subset \vee q_{(\gamma,\delta)} f \cap g$.

$(iii) \implies (ii)$: Let A and B be any left ideals of G. Then by Lemma 3.13, $C^\delta_{\gamma A}$ and $C^\delta_{\gamma B}$ are any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of G. Let $x \in A \cap B$, then by using Lemma 3.3, we have

$$1 = C^\delta_{\gamma A \cap B}(x) = \vee q_{(\gamma,\delta)} (C^\delta_{\gamma A} \cap C_B)(x) \leq (C_B \circ C^\delta_{\gamma A})(x) = \vee q_{(\gamma,\delta)} C^\delta_{\gamma (BA]}(x),$$

This implies that $a \in (BA]$ and therefore $A \cap B \subseteq (BA]$.

$(ii) \implies (i)$: Since $(Ga]$ is a left ideal of G with left identity [9] such that $a \in (Ga]$, by using given assumption and Lemma 2.3, we have

$$\begin{aligned} a &\in (Ga] \cap (Ga] \subseteq ((Ga](Ga]] = ((Ga)(Ga)] = ((aG)(aG)] \\ &= ((aa)(GG)] = ((ea^2)(GG)] \subseteq ((Ga^2)G] \end{aligned}$$

Hence, G is intra-regular. □

Theorem 4.5. *The following conditions are equivalent for an ordered* $\mathcal{AG}$*-groupoid* G *with left identity.*

(i) G *is intra-regular.*

(ii) $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)} (f \circ g) \circ (f \circ h)$, *where* f *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal,* h *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideal and* g *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal of* G.

(iii) $f \cap g \subseteq \vee q_{(\gamma,\delta)} (f \circ g) \circ f$, *where* f *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal and* g *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal of* G.

(iv) $L \cap B \subseteq ((LB]L]$, *where* L *is a left ideal and* B *is a bi-ideal of* G.

Proof. $(i) \implies (ii)$: Let f, g and h be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left, right and bi-ideals of an intra-regular G with left identity respectively. Now for any $a \in G$, there exist $b, c \in G$ such that $a \leq (ba^2)c$, therefore

$$\begin{aligned} a &\leq (ba^2)c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c))a = (c(b(a(ba))c))a \\ &= (c((a(ba))(bc)))a = ((a(ba))(c(bc)))a = (((c(bc))(ba))a)a \\ &\leq (((c(bc))(ba))a)((ba^2)c) = (((c(bc))(ba))a)((b(aa))c) \\ &= (((c(bc))(ba))a)((a(ba))c) = (((c(bc))(ba))a)((c(ba))a), \end{aligned}$$

This shows that $(((c(bc))(ba))a, (c(ba))a) = (ua, va) \in A_a$. Since $A_a \neq \varnothing$,

$$\begin{aligned}
\max\{(f \circ g) \circ (f \circ h)(a), \gamma\} &= \lim_{(ua,va)\in A_a} \{(f \circ g)(ua) \wedge (f \circ h)(va)\} \\
&\geq \max\left[\min\{f(u), g(a), f(v), h(a), \gamma\}\right] \\
&= \min[\max\{f(u), \gamma\}, \max\{g(a), \gamma\}, \max\{f(v), \gamma\}, \\
&\quad \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(u), \delta\}, \min\{g(a), \delta\}, \min\{f(v), \delta\}, \\
&\quad \min\{h(a), \delta\}] \\
&= \min\{(f \cap g \cap h)(a), \delta\},
\end{aligned}$$

This shows that $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ (f \circ h)$.

$(ii) \implies (iii)$: Since G is a fuzzy right ideal of itself,

$$f \cap g = \vee q_{(\gamma,\delta)} f \cap g \cap G \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ (f \circ G) \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ f$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ f$.

$(iii) \implies (iv)$ is simple.

$(iv) \implies (i)$: Since $(Ga]$ is both left and bi-ideal of G containing a, therefore by using given assumption and Lemma 2.3, we have

$$\begin{aligned}
a &\in (Ga] \cap (Ga] = (((Ga](Ga]](Ga]] = (((Ga)(Ga)](Ga]] \\
&= (((GG)(aa)](Ga]] \subseteq ((Ga^2)G].
\end{aligned}$$

Therefore, G is intra-regular. □

5. Conclusions

Order theory is a branch of Mathematics which investigates our intuitive notion of order using binary relations. It provides a formal framework for describing statements such as "this is less than that" or "this precedes that". The study of an algebraic structure using the order theory plays a prominent role in Mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages.

Since we know that an ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup [9], therefore in this regard, we have applied the order theory on the structure of an $\mathcal{AG}$-groupoid and generalized the concept of an ordered semigroup in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

The following topics may be considered for further study of ordered $\mathcal{AG}$-groupoids in more generalized form:

To obtain similar and more generalized results in the structure of ordered Γ-$\mathcal{AG}^{**}$-groupoids (see [5]).

To characterize ordered hyper-$\mathcal{AG}$-groupoids by introducing the concept of $(\in, \in \vee q)$, $(\in, \in \vee q_k)$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals by using pure left (right) identity.

Acknowledgements. Authors are deeply grateful to an unknown referee for valuable comments and suggestions.

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