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Fixed and periodic point results in fuzzy cone metric spaces

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Abstract. In this paper, some fixed and periodic point theorems are established in fuzzy cone metric space for generalized contraction mappings and the main theorem is justified by a counter example.

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1. Introduction

In last few years different types of generalized metric spaces have been developed by different authors in different approach. Some generalized metric spaces are D-metric space [6], D^*-metric space [3, 19], Cone metric space [14] etc. Guang & Xian [14] generalized the notion of metric space by considering a real Banach space as the range set of the metric space which is known as Cone metric space. Many fixed point results of different types of contraction mappings have been established in generalized metric spaces specially in cone metric spaces (for reference please see [1, 2, 5, 8, 9, 10, 11, 13, 15, 17, 18, 20, 21]).

Recently the idea of fuzzy cone metric space has been introduced by present author [4] and some basic properties and fixed point theorems for different types of contraction mappings have been developed in fuzzy cone metric spaces.

In this paper, some fixed and periodic point theorems for generalized contraction mappings are established in fuzzy cone metric space by using normal fuzzy cone and the main theorem is justified by an example. It is to be noted that here fuzzy normal cone is used to simplify the method for solving the problem. On then other hand, if the results exist without using fuzzy normal cone then study will be more general.

The organization of the paper is as follows:

Section 1, comprises some preliminary results which are used in this paper.

In Section 2, a fixed point theorem for generalized contraction mapping is established in fuzzy cone metric space. Some periodic point results are studied in fuzzy cone metric spaces in Section 3.

2. Preliminaries

A fuzzy real number is a mapping $x: R \to [0\ ,\ 1]$ over the set R of all reals.
A fuzzy real number x is convex if $x(t) \geq min\ (x(s)\ ,\ x(r))$ where $s \leq t \leq r$.
α-level set of a fuzzy real number x is defined by $\{t \in R : x(t) \geq \alpha\}$ where $\alpha \in (0, 1]$.
If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha\ ,\ b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha\ ,\ b_\alpha]$ means the interval $(-\infty\ ,\ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy real number x is called non-negative if $x(t) = 0$, $\forall t < 0$.
Each real number r is considered as a fuzzy real number denoted by $\bar{r}$ and defined by
$\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
Kaleva [12] (Felbin [7]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0\ ,\ 1]$ where $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0\ ,\ 1]$.

According to Mizumoto and Tanaka [16], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

$$\begin{aligned}(x \oplus y)(t) &= Sup_{s \in R} min\ \{x(s)\ ,\ y(t-s)\}, \quad t \in R\\ (x \ominus y)(t) &= Sup_{s \in R} min\ \{x(s)\ ,\ y(s-t)\}, \quad t \in R\\ (x \odot y)(t) &= Sup_{s \in R, s \neq \underline{0}} min\ \{x(s)\ ,\ y(\tfrac{t}{s})\}, \quad t \in R\end{aligned}$$

Proposition 2.1 ([12]). *Let $\eta\ ,\ \delta \in E(R(I))$ and $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$, $\alpha \in (0\ ,\ 1]$. Then*

$$\begin{aligned}[\eta \bigoplus \delta]_\alpha &= [a^1_\alpha + a^2_\alpha\ ,\ b^1_\alpha + b^2_\alpha]\\ [\eta \ominus \delta]_\alpha &= [a^1_\alpha - b^2_\alpha\ ,\ b^1_\alpha - a^2_\alpha]\\ [\eta \odot \delta]_\alpha &= [a^1_\alpha a^2_\alpha\ ,\ b^1_\alpha b^2_\alpha]\end{aligned}$$

Definition 2.2 ([12]). A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim\limits_{n\to\infty} \eta_n = \eta$ if $\lim\limits_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim\limits_{n\to\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.

Note 2.3 ([12]). If $\eta,\ \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.4 ([12]). For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.5 ([7]). Let X be a vector space over R.
Let $|| \ || \ : X \to R^*(I)$ and let the mappings
$L, U \ : [0 \ , \ 1] \times [0 \ , \ 1] \to [0 \ , \ 1]$ be symmetric, nondecreasing in both arguments and satisfy
$L(0 \ , \ 0) = \ 0$ and $U(1 \ , \ 1) = \ 1$.
Write
$[||x||]_\alpha \ = \ [||x||^1_\alpha \ , \ ||x||^2_\alpha]$ for $x \in X, \ 0 \ < \alpha \ \leq 1$ and suppose for all $x \in X, \ x \neq \underline{0}$, there exists $\alpha_0 \ \in (0 \ , \ 1]$ independent of x such that for all $\alpha \ \leq \ \alpha_0$,

(A) $||x||^2_\alpha \ < \ \infty$
(B) $\inf ||x||^1_\alpha \ > \ 0$.

The quadruple $(X \ , \ || \ ||, \ L \ , \ U)$ is called a fuzzy normed linear space and $|| \ ||$ is a fuzzy norm if

(i) $||x|| \ = \ \bar{0}$ if and only if $x \ = \ \underline{0}$;
(ii) $||rx|| \ = \ |r|||x||, \ x \in X, \ r \in R$;
(iii) for all $x, y \ \in X$,
(a) whenever $s \ \leq \ ||x||^1_1, \ t \leq ||y||^1_1$ and $s + t \ \leq ||x + y||^1_1$,
$||x + y||(s + t) \ \geq \ L(||x||(s) \ , \ ||y||(t))$,
(b) whenever $s \ \geq \ ||x||^1_1, \ t \geq ||y||^1_1$ and $s + t \ \geq ||x + y||^1_1$,
$||x + y||(s + t) \ \leq \ U(||x||(s) \ , \ ||y||(t))$

Remark 2.6 ([7]). Felbin proved that,
if $L \ = \bigwedge(\text{Min})$ and $U = \bigvee(\text{Max})$ then the triangle inequality (iii) in the Definition 1.1 is equivalent to
$||x + y|| \ \preceq ||x|| \ \bigoplus ||y||$.
Further $|| \ ||^i_\alpha; \ i = 1, 2$ are crisp norms on X for each $\alpha \in (0 \ , \ 1]$.

Definition 2.7 ([4]). Let $(E, || \ ||)$ be a fuzzy real Banach space where $|| \ || : E \to R^*(I)$.
Denote the range of $|| \ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.8 ([4]). A member $\eta \in A \subset R^*(I)$ is said to be an interior point if $\exists \ r \ > 0$ such that
$S(\eta, r) = \{\delta \in R^*(I) \ : \eta \ominus \delta \prec \bar{r}\} \subset A$.
Set of all interior points of A is called interior of A.

Definition 2.9 ([4]). A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim\limits_{n \to \infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.10 ([4]). A subset P of $E^*(I)$ is called a fuzzy cone if
(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;
(ii) $a, b \in R, \ a, b \geq 0, \ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.

Note 2.11. If $\eta \in P$ then $\ominus\eta \in P \ \ \Rightarrow \eta = \bar{0}$.
For, suppose $[\eta]_\alpha = [\eta^1_\alpha \ , \ \eta^2_\alpha], \ \alpha \in (0, 1]$.
Since $\eta \in P \subset E^*(I)$, we have $\eta^1_\alpha, \ \eta^2_\alpha \geq 0 \ \forall \alpha \in (0, 1]$.
Now $[\ominus\eta]_\alpha = [-\eta^2_\alpha \ , \ -\eta^1_\alpha], \ \alpha \in (0, 1]$.
If $\eta \neq \bar{0}$, then $\eta^1_\alpha, \ \eta^2_\alpha > 0 \ \forall \alpha \in (0, 1]$.
i.e. $-\eta^2_\alpha \ \leq \ -\eta^1_\alpha < 0 \ \forall \alpha \in (0, 1]$.
This implies that $\ominus\eta$ does not belong to P. Hence $\eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta \ll \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
with $\bar{0} \leq ||x|| \leq ||y||$ implies $||x|| \preceq K||y||$. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that $||x_1|| \leq ||x_2|| \leq .... \leq ||x_n|| \leq .... \leq ||y||$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to \bar{0}$ as $n \to \infty$.
Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent.

In the following we always assume that E is a fuzzy real Banach space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.12 ([4])**.** Let X be a nonempty set. Suppose the mapping
$d: \ X \times X \to E^*(I)$ satisfies
(Fd1) $\bar{0} \leq d(x,y)$ $\forall x, y \in X$ and $d(x,y) = \bar{0}$ iff $x = y$;
(Fd2) $d(x,y) = d(y,x)$ $\ \forall x, y \in X$;
(Fd3) $d(x,y) \leq d(x,z) \oplus d(z,y)$ $\ \forall x, y, z \in X$.
Then d is called a fuzzy cone metric and (X,d) is called a fuzzy cone metric space.

Definition 2.13 ([4])**.** Let (X,d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} \ll ||c||$ there is a positive integer N such that for all $n > N$, $d(x_n, x) \ll ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim\limits_{n\to\infty} x_n = x$.

Definition 2.14 ([4])**.** Let (X,d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} \ll ||c||$, there exists a natural number N such that $\forall m, n > N$, $d(x_n, x_m) \ll ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.15 ([4])**.** Let (X,d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Proposition 2.16 ([4])**.** *Let (X,d) be a fuzzy cone metric space with fuzzy normal cone and $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ *converges to* x *iff* $d(x_n, x) \to \bar{0}$ *as* $n \to \infty$.
(ii) $\{x_n\}$ *is a Cauchy sequence iff* $d(x_n, x_m) \to \bar{0}$ *as* $m, n \to \infty$.

3. Fixed point theorem in fuzzy cone metric spaces

In this Section a fixed point theorem is established for generalized contraction mapping.

Theorem 3.1. *Let* $(X\ ,\ d)$ *be a complete fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Suppose the mappings* $f, g: \ X \to X$ *satisfying*
$$d(fx\ ,\ gy) \leq pd(x,y) \oplus q[d(x,fx) \oplus d(y,gy)] \oplus r[d(x,gy) \oplus d(y,fx)] \qquad (3.1.1)$$
$\forall x, y \in X$ *where* $p, q, r \geq 0$ *and* $p + 2q + 2r < \ 1$.
Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\}$ in X by
$x_{2n+1} = fx_{2n},\ x_{2n+2} = gx_{2n+1},\ n = 0, 1, 2, ...$
Now,
$d(x_{2n+1}\ ,\ x_{2n+2}) = d(fx_{2n}\ ,\ gx_{2n+1})$
$\leq pd(x_{2n}\ ,\ x_{2n+1}) \oplus q[d(x_{2n}\ ,\ fx_{2n}) \oplus d(x_{2n+1}\ ,\ gx_{2n+1})] \oplus r[d(x_{2n}\ ,\ gx_{2n+1}) \oplus d(x_{2n+1}\ ,\ fx_{2n})]$ by (3.1.1).
i.e. $d(x_{2n+1}\ ,\ x_{2n+2}) \leq (p + q + r)d(x_{2n}\ ,\ fx_{2n+1}) \oplus (q + r)d(x_{2n+1}\ ,\ x_{2n+2})$.
$\Rightarrow d(x_{2n+1}\ ,\ x_{2n+2}) \leq \delta d(x_{2n}\ ,\ x_{2n+1})$ where $\delta = \frac{p+q+r}{1-(q+r)} < 1$.
Similarly it can be shown that,
$d(x_{2n+3}\ ,\ x_{2n+2}) \leq \delta d(x_{2n+2}\ ,\ x_{2n+1})$.
Thus for all n,
$d(x_{n+1}\ ,\ x_{n+2}) \leq \delta d(x_n\ ,\ x_{n+1}) \leq ...... \leq \delta^{n+1} d(x_0\ ,\ x_1)$.
Now for any $m > n$,
$d(x_m\ ,\ x_n) \leq d(x_n\ ,\ x_{n+1}) \oplus d(x_{n+1}\ ,\ x_{n+2}) \oplus ..... \oplus d(x_{m-1}\ ,\ x_m)$.
i.e. $d(x_m\ ,\ x_n) \leq (\delta^n + \delta^{n+1} + ... + \delta^{m-1} d(x_1\ ,\ x_0)$.
i.e. $d(x_m\ ,\ x_n) \leq \frac{\delta^n}{1-\delta} d(x_1\ ,\ x_0)$.
Since P is a fuzzy normal cone we have,
$d(x_n\ ,\ x_m) \preceq \frac{\delta^n}{1-\delta} K d(x_1\ ,\ x_0)$
$\Rightarrow d^i_\alpha(x_n\ ,\ x_m) \leq \frac{\delta^n}{1-\delta} K d^i_\alpha(x_1\ ,\ x_0)\ \ \forall \alpha \in (0, 1]$ and $i = 1, 2$.
Letting $m, n \to \infty$ and since $\delta < 1$ we get,
$\lim\limits_{m,n\to\infty} d^i_\alpha(x_n\ ,\ x_m) = 0\ \ \forall \alpha \in (0, 1]$ and $i = 1, 2$.
$\Rightarrow \lim\limits_{m,n\to\infty} d(x_n\ ,\ x_m) = \bar{0}$.
Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\exists x \in X$ such that
$\lim\limits_{m,n\to\infty} d(x_n\ ,\ x) = \bar{0}$.
Now from (2.1.1), we have
$d(x, gx) \leq d(x, x_{2n+1}) \oplus d(x_{2n+1}, gx)$ (Triangle inequality)
i.e. $d(x, gx) \leq d(x, x_{2n+1}) \oplus d(fx_{2n}, gx)$.
i.e. $d(x, gx) \leq d(x, x_{2n+1}) \oplus pd(x_{2n}\ ,\ x) \oplus q[d(x_{2n}\ ,\ x_{2n+1}) \oplus d(x\ ,\ gx)] \oplus r[d(x_{2n}\ ,\ gx) \oplus d(x\ ,\ x_{2n+1})]$.
i.e. $d(x, gx) \leq d(x, x_{2n+1}) \oplus pd(x_{2n}\ ,\ x) \oplus q[d(x_{2n}\ ,\ x_{2n+1}) \oplus d(x\ ,\ gx)] \oplus r[d(x_{2n}\ ,\ x) \oplus d(x\ ,\ gx) \oplus d(x\ ,\ x_{2n+1})]$.
i.e. $d(x, gx) \leq \frac{1}{1-q-r}[d(x, x_{2n+1}) \oplus pd(x_{2n}\ ,\ x) \oplus qd(x_{2n}\ ,\ x_{2n+1}) \oplus r\{d(x_{2n}\ ,\ x) \oplus d(x, x_{2n+1})\}]$.
i.e. $d(x, gx) \leq \frac{1}{1-q-r}[d(x, x_{n+1}) \oplus d(x_{n+1}, x_{2n+1}) \oplus pd(x_{2n}\ ,\ x_{n+1}) \oplus pd(x_{n+1}\ ,\ x) \oplus qd(x_{2n}\ ,\ x_{2n+1}) \oplus r\{d(x_{2n}\ ,\ x_n) \oplus d(x_n\ ,\ x) \oplus d(x_n\ ,\ x) \oplus d(x_n\ ,\ x_{2n+1})\}]$.
Since P is normal we have,
$d(x, gx) \preceq \frac{1}{1-q-r} K[d(x, x_{n+1}) \oplus d(x_{n+1}, x_{2n+1}) \oplus pd(x_{2n}\ ,\ x_{n+1}) \oplus pd(x_{n+1}\ ,\ x) \oplus qd(x_{2n}\ ,\ x_{2n+1}) \oplus r\{d(x_{2n}\ ,\ x_n) \oplus d(x_n\ ,\ x) \oplus d(x_n\ ,\ x) \oplus d(x_n\ ,\ x_{2n+1})\}]$.
Thus,
$d^i_\alpha(x, gx) \leq \frac{1}{1-q-r} K[d^i_\alpha(x, x_{n+1}) + d^i_\alpha(x_{n+1}, x_{2n+1}) + pd^i_\alpha(x_{2n}\ ,\ x_{n+1}) + pd^i_\alpha(x_{n+1}\ ,\ x) + qd^i_\alpha(x_{2n}\ ,\ x_{2n+1}) + r\{d^i_\alpha(x_{2n}\ ,\ x_n) + d^i_\alpha(x_n\ ,\ x) + d^i_\alpha(x_n\ ,\ x) + d^i_\alpha(x_n\ ,\ x_{2n+1})\}]\ \ \forall \alpha \in (0, 1]$ and $i = 1, 2$.
If we take limit as $n \to \infty$, then right hand side tends to zero (as $\{x_n\}$ is Cauchy).
So $d^i_\alpha(x, gx) = 0\ \ \forall \alpha \in (0, 1]$ and $i = 1, 2$.

Hence $d(x, gx) = \bar{0}$. So $x = gx$.
Now,
$d(fx, x) = d(fx, gx) \leq pd(x, x) \oplus q[d(x, fx) \oplus d(x, gx)] \oplus r[d(x, gx) \oplus d(x, fx)]$.
i.e. $d(fx, x) \leq (q + r)d(x, fx)$
$\Rightarrow ((q+r) - 1)d(x, fx) \in P$ (3.1.2).
Since $q + r < 1$, so $q + r - 1 < 0$. From properties of P, from (3.1.2), it follows that $d(fx, x) = \bar{0}$. So, $fx = x$.
For uniqueness, suppose that y is another fixed point of f and g.
Then $d(x, y) = d(fx, gy) \leq pd(x, y) \oplus q[d(x, fx) \oplus d(y, gy)] \oplus r[d(x, gy) \oplus d(y, fx)]$
$\Rightarrow d(x, y) \leq (p + 2r)d(x, y)$
$\Rightarrow d(x, y) = \bar{0}$ (since $p + 2r < 1$).
Hence $x = y$. This completes the proof. □

The above Theorem 3.1 is justified by the following Example.

Example 3.2. Let us consider the metric space $X = [0, 4]$ with the usual metric $\rho(x, y) = |x - y| \ \ \forall x, y \in X$.
Clearly it is a complete metric space.
We choose $E = R$ and define $|| \, || : \ R \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t \geq |x| \\ 0 & \text{if } t < |x| \ . \end{cases}$$

Then $[||x||]_\alpha = \ [|x| \ , \ |x|] \ \ \forall \alpha \in (0, 1]$.
It can be verified that $|| \, ||$ satisfies $(N1) - (N3)$. So $(E \ , \ || \, ||)$ is a fuzzy normed linear space (Felbin's sense). Again since $(R \ , \ | \, |)$ is complete, thus $(R \ , \ || \, ||)$ is a complete fuzzy normed linear space.
Now define $d : \ X \times X \to E^*(I)$ by

$$d(x, y)(t) = \begin{cases} \frac{\rho(x,y)}{t} & \text{if } t \geq \rho(x, y) \\ 0 & \text{if } t < \rho(x, y). \end{cases}$$

Now $d(x, y)(t) \geq \alpha \ \Rightarrow \frac{\rho(x,y)}{t} \geq \alpha \ \Rightarrow t \leq \frac{\rho(x,y)}{\alpha}$.
So $[d(x, y)]_\alpha = [\rho(x, y) \ , \ \frac{\rho(x,y)}{t}] \ \ \forall \alpha \in (0, 1]$ (3.2.1).
If we choose the ordering $\leq$ as $\preceq$ and define $P = \{\eta \in E^*(I) : \ \eta \succeq \bar{0}\}$,
then P is a fuzzy cone (please see []) and $(X \ , \ d)$ is a fuzzy cone metric space.
Define $f : \ X \to X$ by $f(x) = \frac{x}{3}$ and $g : \ X \to X$ by $g(x) = \frac{x}{7}$.
We choose $p = \frac{1}{2}, \ q = r = \frac{1}{16}$.
Then $p + 2(q + r) = \ \frac{1}{2} + 2(\frac{1}{16} + \frac{1}{16}) = \frac{3}{4} \ < 1$.
Now we verify that
$d(fx \ , \ gy) \preceq pd(x, y) \oplus q[d(x, fx) \oplus d(y, gy)] \oplus r[d(x, gy) \oplus d(y, fx)]$ holds $\forall x, y \in X$.
We have,
$d^1_\alpha(fx \ , \ gy) = d^1_\alpha(\frac{x}{3} \ , \ \frac{y}{7}) = \rho(\frac{x}{3} \ , \ \frac{y}{7}) = |\frac{x}{3} - \frac{y}{7}|$ by (3.2.1) (3.2.2).
Now,
$pd^1_\alpha(x, y) + q[d^1_\alpha(x, fx) + d^1_\alpha(y, gy)] + r[d^1_\alpha(x, gy) + d^1_\alpha(y, fx)]$
$= \frac{1}{2}\rho(x, y) + \frac{1}{16}[\rho(x, fx) + \rho(y, gy) + \rho(x, gy) + \rho(y, fx)]$
$= \frac{1}{2}|x - y| + \frac{1}{16}[|x - \frac{x}{3}| + |y - \frac{y}{7}| + |x - \frac{y}{7}| + |y - \frac{x}{3}|]$
$= \frac{1}{4}|x - y| + \frac{1}{16}[|4x - 4y| + |\frac{2x}{3}| + |\frac{6y}{7}| + |x - \frac{y}{7}| + |y - \frac{x}{3}|]$
$\geq \frac{1}{4}|x - y| + \frac{1}{16}|4x - 4y + \frac{2x}{3} + \frac{6y}{7} + x - \frac{y}{7} + y - \frac{x}{3}|$

$= \frac{1}{4}|x-y| + \frac{1}{16}|4x + \frac{2x}{3} + x - \frac{x}{3} - 4y + \frac{6y}{7} - \frac{y}{7} + y|$
$= \frac{1}{4}|x-y| + \frac{1}{16}|\frac{12x+2x+3x-x}{3} - \frac{28y-6y+y-7y}{7}|$
$= \frac{1}{4}|x-y| + \frac{1}{16}|\frac{16x}{3} - \frac{16y}{7}| = \frac{1}{4}|x-y| + |\frac{x}{3} - \frac{y}{7}|$
$\geq |\frac{x}{3} - \frac{y}{7}| = d^1_\alpha(fx, gy)$. by (3.2.2)
Thus,
$pd^1_\alpha(x,y) + q[d^1_\alpha(x,fx) + d^1_\alpha(y,gy)] + r[d^1_\alpha(x,gy) + d^1_\alpha(y,fx)]$
$\geq d^1_\alpha(fx,gy)\ \forall \alpha \in (0,1]$ (3.2.3).
Similarly we get,
$pd^2_\alpha(x,y) + q[d^2_\alpha(x,fx) + d^2_\alpha(y,gy)] + r[d^2_\alpha(x,gy) + d^2_\alpha(y,fx)]$
$\geq d^2_\alpha(fx,gy)\ \forall \alpha \in (0,1]$ (3.2.4).
From (3.2.3) and (3.2.4) we get,
$d(fx\ ,\ gy) \preceq pd(x,y) \oplus q[d(x,fx) \oplus d(y,gy)] \oplus r[d(x,gy) \oplus d(y,fx)]\ \forall x,y \in X.$
Thus f and g satisfy all the conditions of Theorem 3.1.
From definition of f and g, it follows that $x = 0$ is the unique common fixed point of f and g.

4. Periodic Point Theorems

It is obvious that, if f is a mapping which has a fixed point p, then it is a fixed point of f^n for every natural number n. However the converse may not be true. For example consider $X = [0,1]$ and f defined by $fx = 1-x$. Then f has a unique fixed point at $\frac{1}{2}$, but every iterate of f is the identity mapping, which has every point of $[0,1]$ is a fixed point. On the other hand, if $X = [0,\pi]$, $fx = \cos x$, then iterate of f has the same fixed point as f [8, 11, 20]. If a map satisfies $F(f) = F(f^n)$ for each $n \in N$, where $F(f)$ denotes the set of all fixed points of f, then it is said to have property S(instead of P[11]). We shall say that f and g have property Q [11] if $F(f) \bigcap F(g) = F(f^n) \bigcap F(g^n)$.
In this Section, we establish some fixed point theorems in fuzzy cone metric spaces which satisfy the properties S and Q.

Theorem 4.1. *Let f be a self-map of a fuzzy cone metric space $(X\ ,\ d)$ and P be a fuzzy normal cone with normal constant K satisfying,*
(i) $d(fx\ ,\ f^2x) \leq \lambda d((x\ ,\ fx)\ \ \forall x \in X$ where $0 \leq \lambda < 1$.
or
(ii) $d(fx\ ,\ f^2x)\ < d((x\ ,\ fx)\ \ \forall x \in X,\ x \neq fx$.
If $F(f) \neq \phi$, then f has the property S.

Proof. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial.
Let $u \in F(f^n)$. Suppose f satisfies (i). Then
$d(u\ ,\ fu) = d(f(f^{n-1}u)\ ,\ f^2(f^{n-1}u)) \leq \lambda d(f^{n-1}u\ ,\ f^n u) \leq \lambda^2 d(f^{n-2}u\ ,\ f^{n-1}u) \leq \ldots \leq \lambda^n d(u\ ,\ fu)$.
Since P is a fuzzy normal cone with normal constant K, we get
$d(u,fu) \preceq K\lambda^n d(u,fu)$
$\Rightarrow d^i_\alpha(u,fu) \leq K\lambda^n d^i_\alpha(u,fu)$ for $i = 1,2$ and $\alpha \in (0,1]$
$\Rightarrow \lim\limits_{n\to\infty} d^i_\alpha(u,fu) = 0$ for $i = 1,2$ and $\alpha \in (0,1]$ (since $\lambda < 1$)
$\Rightarrow d(u\ ,\ fu) = \bar{0}$.
So $fu = u$.

Next suppose that f satisfies (ii). If $fu = u$, then the theorem is clear.
Suppose if possible that $fu \neq u$. Then from the argument of (i), we have $d(u, fu) < d(u, fu)$ which is a contradiction.
Thus in all cases $fu = u$ and hence $F(f^n) = F(f)$. □

Theorem 4.2. *Let $(X\ ,\ d)$ be a complete fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Suppose the mappings $f, g :\ X \to X$ satisfy (3.1.1).*
Then f and g have the property Q.

Proof. From Theorem 4.1, it follows that f and g have a common fixed point in X.
Let $u \in F(f^n) \bigcap F(g^n)$.
Now,
$d(u\ ,\ gu) = d(f(f^{n-1}u)\ ,\ g(g^{n-1}u)) \leq pd(f^{n-1}u\ ,\ f^n u) \oplus q[d(f^{n-1}u\ ,\ f^n u) \oplus d(f^n u\ ,\ g^{n+1}u)] \oplus r[d(f^{n-1}u\ ,\ g^{n+1}u) \oplus d(g^n u\ ,\ f^n u)]$
i.e. $d(u\ ,\ gu) \leq pd(f^{n-1}u\ ,\ u) \oplus q[d(f^{n-1}u\ ,\ u) \oplus d(u\ ,\ gu)] \oplus r[d(f^{n-1}u\ ,\ u) \oplus d(u\ ,\ gu)]$
$\Rightarrow d(u\ ,\ gu) \leq \delta d(f^{n-1}u\ ,\ u)$ where $\delta = \frac{p+q+r}{1-q-r} < 1$.
We have,
$d(u\ ,\ gu) = d(f^n u\ ,\ g^{n+1}u) \leq \delta d(f^{n-1}u\ ,\ u) \leq .... \leq \delta^n d(u\ ,\ fu)$.
Since P is a fuzzy normal cone with normal constant K we have,
$d(u\ ,\ gu) \preceq \delta^n K d(u\ ,\ fu)$
$\Rightarrow d^i_\alpha(u\ ,\ gu) \leq \delta^n K d^i_\alpha(u\ ,\ fu)$ for $i = 1, 2$ and $\alpha \in (0, 1)$
$\Rightarrow \lim\limits_{n\to\infty} d^i_\alpha(u\ ,\ gu) = 0$ for $i = 1, 2$ and $\alpha \in (0, 1)$ (since $\delta < 1$)
$\Rightarrow d(u\ ,\ gu) = \bar{0}$.
So $gu = u$. Hence $gu = fu = u$ by Theorem 4.1.
This implies that $F(f) \bigcap F(g) = F(f^n) \bigcap F(g^n)$.
Thus f and g have the property Q. □

5. Conclusion

In last few years, different types of generalized metric spaces have been developed by many authors. Cone metric space is one such development. Many fixed point results for contraction mappings have been established.
Recently idea of fuzzy cone metric space is introduced some basic properties. In this paper, some fixed and periodic point theorems are established in fuzzy cone metric spaces.
I think that this paper could be of interest to the researchers to study fixed point theory for different types of contraction mappings in fuzzy cone metric spaces and specially fixed point results can be applied to find out the existence and uniqueness of the solutions of fuzzy integral equations.

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